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THE CONVERGENCE OF SOME SEQUENCES CONNECTED TO HADAMARD'S INEQUALITY

1. Introduction

Let $f : I \to R$ be a convex mapping on the interval of real numbers I, and $a, b, \in \overset{\circ}{I}$ with $a < b$ ($\overset{\circ}{I}$ is the interior of I). The following integral inequality is well known in literature as Hadamard's inequality (see also [4]),

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

For some recent refinements, counterparts and generalizations of this classic fact see the papers [1–7] where further references are given.

In [6], S. S. Dragomir, J. E. Pecărić and J. Săndor proved the following refinement of (1):

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) dx_1\,dx_2 \ldots dx_n \\ &\le \frac{1}{(b-a)^{n-1}} \int_a^b \left(\frac{x_1+x_2+\ldots+x_n}{n-1}\right) dx_1\,dx_2 \ldots dx_{n-1} \\ &\le \ldots \le \frac{1}{(b-a)} \int_a^b f(x)\,dx \end{aligned} \tag{2}$$

where $n \in N$ and $n \ge 2$. Some applications for Γ-functions with interesting connections in Number Theory are also given.

In paper [5], among other results, the authors proved the following refinement of Hadamard's inequality for weighted means:

$$(3)\quad f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^n} \int_a^b f\left(\frac{p_1x_1+\ldots+p_nx_n}{P_n}\right) dx_1\, dx_2 \ldots dx_n \le \frac{1}{(b-a)} \int_a^b f(x)\, dx$$

where $p_i \ge 0$ $(i = 1, \ldots, n)$ with $P_n = p_1 + p_2 + \ldots + p_n > 0$.

In this paper we will study the convergence of some sequence which can be associated in a natural way to Hadamard's inequality.

2. The main results

We will start with the following generalization of the inequality (3) with a different proof (compare with [5] in which Jensen's inequality for multiple integral was used):

THEOREM 1. *Let I be an interval with $a, b \in \overset{\circ}{I}$ and $a < b$. If $f : I \to R$ is a convex function on I and $q_i(m) \ge 0$ for all $i, m \in N^*$ ($N^* = \{1, 2, \ldots\}$) then we have the inequality*

$$(4)\quad f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^m} \int_a^b \cdots \int_a^b f\left(\frac{q_1(m)x_1+\ldots+q_m(m)x_m}{Q_m}\right) dx_1\, dx_2 \ldots dx_m \le \frac{1}{(b-a)} \int_a^b f(x)\, dx$$

where $Q_m := q_1(m) + \ldots + q_m(m) > 0$, $m \in N^*$.

To prove this fact we will use the following lemma:

LEMMA 1. *Suppose that $g : (a, b) \to R$, $a, b \in R$, $a < b$, is a convex function on (a, b). Then there exist two sequences $(a_n), (b_n)$ of real numbers such that*

$$(5)\quad g(t) = \sup_{n\ge 1}(a_n t + b_n) \quad \text{for all } t \in (a, b).$$

P r o o f of the Lemma. Since g is convex on (a, b), hence

$$g(t) \ge g(x) + (t - x)g'_+(x) \quad \text{for all } t, x \in (a', b).$$

From that we get

$$g(t) \ge \sup\{g(x) + (t - x)g'_+(x) : x \in (a, b) \cap \mathbf{Q}\} \quad \text{for all } t \in (a, b).$$

Fix t in (a,b) and let $x_n \in \mathbf{Q} \cap (a,b)$ such that $x_n \to t$. Since g is continuous in t we find that

$$g(t) = \lim_{n\to\infty} [g(x_n) + (t - x_n)g'_+(x_n)], \; t \in (a,b)$$

because $(g'_+(x_n))$ is bounded in R. Consequently

$$(6) \quad g(t) = \sup\{g(x) + t - x)g'_+(x) : x \in (a,b) \cap \mathbf{Q}\} \quad \text{for all } t \in (a,b).$$

If $h : N^* \to \mathbf{Q} \cap (a,b)$ is one of bijections of the set N^* onto $\mathbf{Q} \cap (a,b)$, then we can choose $a_n = g'_+(h(n))$ and $b_n = g(h(n)) - h(n)a_n$, and by (6) we obtain (5).

Proof of Theorem. By the above lemma there exist two sequences (a_n^f) and (b_n^f) such that

$$f(t) = \sup_{n\geq 1}(a_n^f t + b_n^f), \quad t \in (a,b)$$

and thus

$$f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) \geq a_n^f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) + b_n^f$$

for all $n \geq 1$ and $x_i \in (a,b)$.

Integrating on $[a,b]^m$ we get

$$\int_a^b \cdots \int_a^b f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) dx_1 \ldots dx_m \geq \left(a_n^f \frac{a+b}{2} + b_n^f\right)(b-a)^m$$

for all $n \geq 1$. Passing to the supremum in the above inequality, we deduce the first inequality in (4).

Now, let observe by the convexity of f that

$$f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) \leq \frac{q_1(m)f(x_1) + \ldots + q_m(m)f(x_m)}{Q_m}$$

which gives by integration on $[a,b]^m$ the second part of (4).

The following result for convergence holds:

THEOREM 2. *Let $f : I \subset R \to R$ be a convex function on $I, a, b, \in \mathring{I}$, with $a < b$ and $q_i(m) \geq 0$ for all $i, m \in N^*$. If $Q_m = q_1(m) + \ldots + q_m(m) > 0$ and*

$$\lim_{m\to\infty} \frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2} = 0$$

then

$$\lim_{m\to\infty} \frac{1}{(b-a)^m} \int_a^b \cdots \int_a^b f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) dx_1 \ldots dx_m = f\left(\frac{a+b}{2}\right).$$

We will prove the following lemma which is interesting by itselt.

LEMMA 2. *Let $f : [a,b] \to R$ be a Lebesgue integrable function on $[a,b]$ and assume that it is continuous at $\frac{a+b}{2}$. If $q_i(m) \geq 0$ $(i, m \in N)$ are such that $Q_m > 0$ and*

$$\lim_{m\to\infty} \frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2} = 0 \tag{7}$$

then

$$\lim_{m\to\infty} \frac{1}{(b-a)^m} \times \int_{[a,b]^m} f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) dx_1 \ldots dx_m = f\left(\frac{a+b}{2}\right) \tag{8}$$

where the above integral is understood in the Lebesgue sense.

Proof of Lemma. It is clear that the above integral exists. Choosing $x_i = a + t_i(b-a)$, $i = 1,1$, we get

$$\frac{1}{(b-a)^m} \int_{[a,b]^m} f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m^2}\right) dx_1 \ldots dx_m$$
$$= \int_{[0,1]^m} f\left(a + (b-a)\frac{q_1(m)t_1 + \ldots + q_m(m)t_m}{Q_m}\right) dt_1 \ldots dt_m.$$

Since f is continuous at $x_0 = \frac{a+b}{2}$, therefore for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|x - \frac{a+b}{2}\right| < \delta$ implies $\left|f(x) - f\left(\frac{a+b}{2}\right)\right| < \frac{\varepsilon}{2}$.

Consider the mappings $\varphi_m : [0,1]^m \to [a,b]$ given by

$$\varphi_m(t) = a + \frac{b-a}{Q_m}[q_1(m)t_1 + \ldots + q_m(m)t_m]$$

where $t = (t_1, \ldots, t_m)$, and define the sets

$$A_\delta(\varphi_m) := \left\{t \in [0,1]^m : \left|\varphi_m(t) - \int_{[0,1]^m} \varphi_m(s)\, ds\right| \geq \delta\right\}.$$

Note that the sets $A_\delta(\varphi_m)$ are Lebesgue measurable in $[0,1]^m$.

If $A_\delta(\varphi_m) \neq \emptyset$, then

$$\left|\varphi_m(t) - \int_{[0,1]^m} \varphi_m(s)\, ds\right|^2 \geq \delta^2 \quad \text{for all } t \in A_\delta(\varphi_m) \text{ and thus} \tag{9}$$

$$\int_{[0,1]^m} \varphi_m(t) - \int_{[0,1]^m} \varphi_m(s)\, ds)^2\, dt \geq$$

$$\int\limits_{A_\delta(\varphi_m)} \Big(\varphi_m(t) - \int\limits_{[0,1]^m} \varphi_m(s)\,ds\Big)^2 dt \geq \delta^2 \operatorname{mes}(A_\delta(\varphi_m)), \text{ i.e.,}$$

$$\int\limits_{[0,1]^m} \varphi_m^2(t)\,dt - \Big(\int\limits_{[0,1]^m} \varphi_m(t)\,dt\Big)^2 \geq \delta^2 \operatorname{mes}(A_\delta(\varphi_m)).$$

But a simple calculation shows that

$$\begin{aligned}\int\limits_{[0,1]^m} \varphi_m^2(t)\,dt = &\int\limits_{[0,1]^m} \Big[a^2 + 2a\frac{b-a}{Q_m}(q_1t_1 + \ldots + q_mt_m) \\ &+ \frac{(b-a)^2}{Q_m^2}(q_1t_1 + \ldots + q_mt_m)^2\Big]\,dt \\ &= a^2 + a(b-a) + \frac{(b-a)^2}{Q_m^2}\Big[\frac{1}{3}\sum_{i=1}^m q_i^2 + \frac{1}{2}\sum_{1\leq i\leq j\leq m} q_i(m)q_j(n)\Big]\end{aligned}$$

and

$$\Big[\int\limits_{[0,1]^m} \varphi_m(t)\,dt\Big]^2 = a^2 + a(b-a) + \frac{(b-a)^2}{4}.$$

By the inequality (9) we get

$$\begin{aligned}&\int\limits_{[0,1]^m} \varphi_m^2(t)\,dt - \Big(\int\limits_{[0,1]^m} \varphi_m(t)\,dt\Big)^2 \\ &\qquad = \frac{b-a)^2}{12}\,\frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2} \geq \delta^2 \operatorname{mes}(A_\delta(\varphi_m))\end{aligned}$$

Now we have succesively

$$\begin{aligned}\Big|\int\limits_{[0,1]^m} f(\varphi_m(t))\,dt - f\Big(\frac{a+b}{2}\Big)\Big| &\leq \int\limits_{[0,1]^m} \Big|f(\varphi_m(t))\,dt - f\Big(\frac{a+b}{2}\Big)\Big|\,dt \\ &= \int\limits_{[0,1]^m \setminus A_\delta(\varphi_m)} \Big|f(\varphi_m(t)) - f\Big(\frac{a+b}{2}\Big)\Big|\,dt \\ &\quad + \int\limits_{A_\delta(\varphi_m)} \Big|f(\varphi_m(t)) - f\Big(\frac{a+b}{2}\Big)\Big|\,dt \\ &\leq \frac{\varepsilon}{2} + 2M \operatorname{mes}(A_\delta(\varphi_m)) \\ &\leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2}\,\frac{(b-a)^2}{12}\,\frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2},\end{aligned}$$

where $M = \sup\{|f(x)| : x \in [a,b]\}$. Since

$$\frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2} \to 0$$

hence there exists $m_\varepsilon \in N$ such that

$$\frac{2M}{\delta^2} \frac{(b-a)^2}{12} \frac{q_1^2(m) + \ldots + q_m^2(m)}{Q_m^2} < \frac{\varepsilon}{2}$$

for all $m \geq m_\varepsilon$, and thus we have

$$\left| \int_{[0,1]^m} f(\varphi_m(t))\, dt - f\left(\frac{a+b}{2}\right) \right| \leq \varepsilon \quad \text{i.e. the limit (8).}$$

Proof of the Theorem. Since $f : I \to R$ is convex on I and $a, b \in \overset{\circ}{I}$, f is continuous on $[a,b]$. Now, applying Lemma 2 and observing that

$$\int_{[a,b]^m} f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) dx_1 \ldots dx_m$$

$$= \int_a^b \cdots \int_a^b f\left(\frac{q_1(m)x_1 + \ldots + q_m(m)x_m}{Q_m}\right) dx_1 \ldots dx_m,$$

where the last integral is considered in Riemann's sense, the proof of the theorem is finished.

COROLLARY. *Let $f : I \subset R \to R$ be a convex function on I, $a, b \in \overset{\circ}{I}$ with $a < b$. Then*

$$\lim_{m\to\infty} \frac{1}{(b-a)^m} \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \ldots + x_m}{m}\right) dx_1 \ldots dx_m =$$

$$\inf_{m \geq 1} \left[\frac{1}{(b-a)^m} \int_a^b \cdots \int_a^b f\left(\frac{x_1 + \ldots + x_m}{m}\right) dx_1 \ldots dx_m\right] = f\left(\frac{a+b}{2}\right).$$

Proof. Follows by Theorem 2 for $q_1(m) = \ldots = q_m(m) = \frac{1}{m}$ $(m \in N^*)$.

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