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SIGNED MEASURES AND HAHN–JORDAN DECOMPOSITIONS OF FUZZY MEASURABLE SPACES

1. Introduction

The notion of signed measure m of fuzzy measurable space (X, M) was introduced by the author in [1]. We have proved there that the Hahn decomposition of any fuzzy measurable space (X, M) with respect to m always exists. In the present paper, we give an another proof of this theorem. This proof is based on the method of Boolean representation which enable us to prove many interesting results of measure theory, as, for example, the Kolmogorov–Sinaj theorem on generators for the fuzzy case [2].

2. Basic definitions and facts

Let X be a nonempty set. The system M of fuzzy subsets of X such that

(i) the constant zero function belongs to M;
(ii) if $f \in M$, then $f' := 1 - f \in M$;
(iii) if $f_n \in M, n = 1, 2, \ldots$, then $\bigvee_{n=1}^{\infty} f_n \in M$;
(iv) the constant function $1/2$ does not belong to M

is called a soft fuzzy σ-algebra [3].

The symbol $\vee_n f_n$ denotes here the Zadeh fuzzy union, i.e. the pointwise supremum of functions f_n. It is known that any soft fuzzy σ-algebra M is a distributive σ-lattice with the complementation $' : f \to f'$ for which the de Morgan laws hold: $(\bigvee_{n=1}^{\infty} f_n)' = \bigwedge_{n=1}^{\infty} f_n'$ and $(\bigwedge_{n=1}^{\infty} f_n)' = \bigvee_{n=1}^{\infty} f_n'$ for any sequence $\{f_n\}_{n=1}^{\infty} \subset M$. Of course, $\wedge_n f_n$ is the Zadeh fuzzy intersection, i.e. the pointwise infimum of functions f_n.

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The above described couple (X, M) is said to be a fuzzy measurable space. We say that two fuzzy sets $f, g \in M$ are orthogonal (and we write $f \perp g$) if $f \leq g'$ (pointwisely).

The results of Piasecki [3] motivated us to define a signed measure on a soft fuzzy σ-algebra M.

DEFINITION 2.1 [1]. Let (X, M) be a fuzzy measurable space. A mapping $m : M \to R$ such that

(i) $m(f \vee f') = m(1)$ for every $f \in M$;

(ii) if $\{f_n\}_{n=1}^{\infty}$ is a sequence of pairwise orthogonal fuzzy subsets from M, i.e. $f_i \perp f_j$ whenever $i \neq j$, then $m(\bigvee_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$

is said to be a *signed measure.*

A nonnegative signed measure m of (X, M) is called a measure of (X, M). In particular, a measure m with the property $m(1) = 1$ is a so-called fuzzy probability measure, in Piasecki's terminology m is a fuzzy P-measure [3] The triplet (X, M, m), where m is a fuzzy P-measure, is a fuzzy probability space. This structure was studied e.g. in [4]–[6].

A signed measure m of (X, M) has the following properties ([6]):

(2.1) $m(f') = m(1) - m(f)$ for every $f \in M$,

(2.2) $m(f) = 0$ for every $f \in W(M)$,

(2.3) if $f, g \in M$, $f \leq g$, then $m(g) = m(f) + m(f' \wedge g)$,

(2.4) $m(f) = m(f \wedge g)$ for every $f, g \in M$ such that $g' \in W(M)$,

(2.5) $m(f) = m(f \vee g)$ for any $f \in M$ and any $g \in W(M)$.

The symbol $W(M)$ denotes the system all W-empty sets from M. We recall that $f \in M$ is a W-empty set (W-universum) if $f \leq f'$ ($f' \leq f$) ([3]). From the property (2.3), it is evident that any measure of (X, M) is a nondecreasing function. We note that for signed measure the monotonicity does not hold, in general.

3. The Hahn and Jordan decomposition

Let m be any signed measure defined on a soft fuzzy σ-algebra M of fuzzy subsets of X. A fuzzy set $a \in M$ is called positive (negative) with respect to a signed measure m if $m(a \wedge f) \geq 0$ ($m(a \wedge f) \leq 0$) for every $f \in M$. An ordered couple (a, b), where a is positive and b is negative with respect to m such that $b = a'$, is called a Hahn decomposition of (X, M) with respect to m. The following theorem has been proved by the author in [1], but here we present a new proof using a Boolean representation.

THEOREM 3.1. *Let m be any signed measure defined on a soft fuzzy σ-algebra M. Then a Hahn decomposition of a fuzzy measurable space (X, M) with respect to m exists.*

Proof. Let us define in the set M the relation $\sim$ as follows: for every $f, g \in M$, $f \sim g$ iff $m(f' \wedge g) = m(f \wedge g') = 0$. The system $[M] = \{[f]; f \in M\}$, where $[f] = \{g \in M;\ f \sim g\}$, is a Boolean σ-algebra for which the following equalities hold: (i) $\bigvee_{n=1}^{\infty} [f_n] = [\bigvee_{n=1}^{\infty} f_n]$; (ii) $\bigwedge_{n=1}^{\infty} [f_n] = [\bigwedge_{n=1}^{\infty} f_n]$; (iii) $[f]' = [f']$ whenever $\{[f_n]\}_{n=1}^{\infty} \subset [M]$ and $[f] \in [M]$. The proofs of the above statements can be found in [2].

If we define the mapping $\mu : [M] \to R$ by the equality $\mu([f]) := m(f)$ for every $[f] \in [M]$, then μ is a signed measure on the Boolean σ-algebra $[M]$.

From the Loomis-Sikorski theorem [7] applied to the Boollean σ-algebra $[M]$ it follows that there exist a measurable space $(\Omega, \mathcal{S})$ ($\mathcal{S}$ is a σ-algebra of subsets of Ω) and a σ-homomorphism h from $\mathcal{S}$ onto $[M]$. Let us define the mapping $\bar{\mu} : \mathcal{S} \to R$ in the following way: $\bar{\mu}(A) = \mu(h(A)), A \in \mathcal{S}$. It is easy to verify that $\bar{\mu}$ is a signed measure on $\mathcal{S}$. By Theorem 10.7 of [8] there exists a Hahn decomposition (A, B) of the measurable space $(\Omega, \mathcal{S})$ with respect to $\bar{\mu}$ such that $A, B \in \mathcal{S}$. If we denote $h(A) = [a]$, $b = a'$, then $a, b \in M$. Let us prove that a is positive, i.e. that $m(a \wedge f) \geq 0$ for every $f \in M$. Let $f \in M$. Then $[f] \in [M]$ and, since h is an epimorphism, there exists $G \in \mathcal{S}$ such that $h(G) = [f]$. Calculate:

$$m(a \wedge f) = \mu([a \wedge f]) = \mu([a] \wedge [f]) = \mu(h(A) \wedge h(G)) = \\ = \mu(h(A \cap G)) = \bar{\mu}(A \cap G) \geq 0.$$

Analogously,

$$m(b \wedge f) = \mu([a' \wedge f]) = \mu([a]' \wedge [f]) == \mu((h(A))' \wedge h(G)) = \\ = \mu(h(A^c) \wedge h(G)) = \mu(h(A^c \cap G)) = \bar{\mu}(A^c \cap G) = \bar{\mu}(B \cap G) \leq 0.$$

The symbol A^c denotes a complement of A. This means that the couple (a, b) is a Hahn decomposition of (X, M) with respect to m. ■

Let m be any signed measure on M and (a, b) be any Hahn decomposition of (X, M) with respect to m. Let us define the mappings m^+ and m^- by the equalities

$$(3.1) \qquad m^+(f) = m(f \wedge a),$$

$$(3.2) \qquad m^-(f) = -m(f \wedge b),$$

for every $f \in M$. It is easy to verify that the mappings m^+ and m^- are measures on M and they are independent of given Hahn decomposition.

Moreover, for any $f \in M$, it holds

$$m(f) = m^+(f) - m^-(f). \tag{3.3}$$

DEFINITION 3.1. The formula (3.3) is called a *Jordan decomposition of a signed measure* m. The measure m^+ (m^-) is said to be a positive (negative) part of m. The mapping $|m|$ defined for any $f \in M$ by

$$|m|(f) = m^+(f) + m^-(f) \tag{3.4}$$

is called a *total variation of a signed measure* m.

Let $(X, \mathcal{S}, P)$ be a probability space and $\xi : X \to R$ be a random variable in the sense of classical probability theory [9]. It is known that the integral $\nu(A) = \int\limits_A \xi dP$, $A \in \mathcal{S}$ is (under the conventional assumptions on the integrability) a signed measure on $\mathcal{S}$. In the following, we formulate this result for fuzzy probability spaces. To this, we recall the notions of observable x of (X, M) and integral of x. A fuzzy analogy of a random variable is an observable.

An observable of a fuzzy measurable space (X, M) is a mapping $x : \mathcal{B}(R) \to M$ satisfying the following conditions:

(i) $x(E^c) = 1 - x(E)$, $E \in \mathcal{B}(R)$.
(ii) If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{B}(R)$, $E_i \cap E_j = \emptyset$ for $i \neq j$, then $x(\bigcup\limits_{n=1}^{\infty} E_n) = \bigvee\limits_{n=1}^{\infty} x(E_n)$.

(The symbol $\mathcal{B}(R)$ denotes a Borel σ-algebra of the real line R.)

If f is a fuzzy set from M, then the mapping x_f defined via

$$x_f(E) = \begin{cases} f \wedge f' & \text{if } 0, 1 \notin E, \\ f' & \text{if } 0 \in E,\ 1 \notin E, \\ f & \text{if } 0 \notin E,\ 1 \in E, \\ f \vee f' & \text{if } 0, 1 \in E, \end{cases}$$

for any $E \in \mathcal{B}(R)$, is an observable of (X, M) and it plays the role of the indicator of the fuzzy set $f \in M$.

If m is a measure on a soft fuzzy σ-algebra M, then a mapping $m_x : \mathcal{B}(R) \to R$ defined by $m_x(E) = m(x(E))$, $E \in \mathcal{B}(R)$, is a measure on $\mathcal{B}(R)$. By an integral of x with respect to a measure m we understand the expression

$$\int x \, dm := \int\limits_R t \, dm_x(t)$$

(if the integral on right-hand exists and it is finite).

If $\psi : R \to R$ be a Borel measurable mapping, then $\psi(x) : E \to x(\psi^{-1}(E))$, $E \in \mathcal{B}(R)$, is an observable of (X, M), too. In particular, if $\psi(t) = t^2$, $t \in R$, we put $x^2 := \psi(x)$, etc.

Further, we put

$$\int \psi(x)\, dm := \int_R \psi(t)\, dm_x(t)$$

under the conventional assumptions on the integrability.

Let x, y be two observables of (X, M). By the sum of x and y (see [10]) we mean an observable z such that

$$z((-\infty, t)) = \bigvee_{r \in Q} x((-\infty, r)) \wedge y((-\infty, t - r)), \qquad t \in R$$

where Q is the set of all rationals in the real line R. We write $z = x + y$. In the paper [10], it has been proved that the sum of any pair observables always exists and it is unique. The product $x.y$ of two observables x, y of (X, M) is defined as follows:

$$x.y = ((x + y)^2 - x^2 - y^2)/2.$$

Let m be a fuzzy P-measure of (X, M). An integral of an observable x of (X, M) over a fuzzy set $f \in M$ is defined via

$$\nu(f) = \int_f x\, dm := \int x.x_f\, dm, \tag{3.5}$$

where x_f is the indicator of the fuzzy set f.

This integral has been defined in [11] and in another way in [12]. It is not hard to show their coincidency.

EXAMPLE 3.1. Let m be a fuzzy P-measure on M and x be an observable of (X, M) such that $\int_R |t|\, dm_x(t) < \infty$. Then the mapping ν defined by the formula (3.5) is a signed measure on M. The σ-additivity of ν has been proved in [12] and in another way in [6]. In [6], it has been proved that $\nu(f \vee f') = \nu(1)$ for every $f \in M$.

If we put $\psi^+(t) = max(t, 0)$, $\psi^-(t) = -min(t, 0)$, $t \in R$, then $x = x^+ - x^-$, where

$$x^+ := \psi^+(x), x^- := \psi^-(x).$$

The mappings $\nu^+ : M \to R$, $\nu^- : M \to R$ defined for every $f \in M$ by $\nu^+(f) = \int_f x^+\, dm$, $\nu^-(f) = \int_f x^-\, dm$, respectively, are a positive and a negative part of ν, respectively. So, $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

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