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CR-SUBMANIFOLDS OF A NEARLY r-COSYMPLECTIC MANIFOLD

CR-submanifolds have been defined and studied by Professor A. Bejancu ([1], [2]) and others. In this paper, we have defined and studied CR-submanifolds of a nearly r-cosymplectic manifold. Certain interesting results have been stated and proved in this paper.

1. Preliminaries

Let $\overline{M}$ be $(2n + r)$-dimensional differentiable manifold of class C^∞. Suppose there exists on $\overline{M}$, a tensor field ϕ of type $(1,1)$, $r(C^\infty)$ contravariant vectorfields ξ_p and $r(C^\infty)$ 1-forms η^p (r some finite integer and $p = 1, 2, \ldots, r$) satisfying

$$\phi^2 = -I + \sum_{p=1}^{r} \eta^p \otimes \xi_p, \tag{1.1}$$

where

$$\begin{cases} \text{(i)} & \phi\xi_p = 0, \\ \text{(ii)} & \eta^p \circ \phi = 0, \\ \text{(iii)} & \eta^p(\xi_q) = \delta^p_q, \end{cases} \tag{1.2}$$

where $p, q = 1, 2, \ldots, r$ and δ^p_q denotes Kronecker delta.

Thus in view of the equations (1.1) and (1.2) the manifold $\overline{M}$ will be said to possess an almost r-contact structure [5].

Suppose further that the manifold $\overline{M}$ is endowed with a Riemannian metric g satisfying

$$g(\overline{X}, \overline{Y}) = g(X, Y) - \sum_{p=1}^{r} \eta^p(X)\eta^p(Y) \tag{1.3}$$

and

$$g(\xi_p, X) = \eta^p(X). \tag{1.4}$$

Then we say that in view of the equations (1.1) to (1.4) the manifold $\overline{M}$ admits an almost r-contact metric structure.

Let us call such a manifold as nearly r-cosymplectic manifold if ϕ is killing, i.e.

$$(\tilde{\nabla}_X \phi)(Y) + (\tilde{\nabla}_Y \phi)(X) = 0 \tag{1.5}$$

for any the vectorfields X and Y on $\overline{M}$; $\tilde{\nabla}$ denotes the Riemannian connection for the metric tensor g on $\overline{M}$.

On such a nearly r-cosymplectic manifold $\overline{M}$, the vectorfields ξ_p are killing i.e.

$$g(\tilde{\nabla}_X \xi_p, Y) + g(X, \tilde{\nabla}_Y \xi_p) = 0 \tag{1.6}$$

for $p = 1, 2, \ldots, r$ and X, Y are arbitrary vectorfields on $\overline{M}$.

Let M be a submanifold of $\overline{M}$ such that the vectorfields ξ_p are tangets to M. Let us denote the r-dimensional distribution formed by the vectorfields ξ_p by $\{\xi_p\}$. We say that M is CR-submanifold of $\overline{M}$ if there exist differentiable distributions D and $D^\perp$ on M such that

$$TM = \{D\} \oplus \{D^\perp\} \oplus \{\xi_p\}, \tag{i}$$

where $D, D^\perp$ are mutually orthogonal and TM denotes the tangent bundle of M;

(ii) The distribution D is invariant by ϕ, i.e.

$$\phi(D_x) = D_x \quad \text{for every } x \text{ in } M, \text{ and}$$

(iii) The distribution $D^\perp$ is anti-invariant by ϕ, i.e. $\phi(D_x^\perp) \subset T_x(M^\perp)$, where $T_x(M^\perp)$ denotes the normal space of M at $x \in M$.

Let us call such a submanifold M of almost r-contact metric manifold $\overline{M}$ as semi r-invariant CR-submanifold. We denote by P, Q the projection morphisms of TM to D and $D^\perp$ respectively so that we have [2]

$$X = PX + QX + \sum_{p=1}^{r} \eta^p(X)\xi_p \tag{1.7}$$

for all $X \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the module of differentiable sections of the tangent bundle TM. Also

$$\phi V = BV + CV \tag{1.8}$$

for all $V \in \Gamma(TM^\perp)$, where BV denotes the tangent part of ϕV and CV its normal part. Let ∇ be the Levi-Civita connection and $\nabla^\perp$ the normal connection in M induced by the Riemannian connection $\tilde{\nabla}$ on $\overline{M}$. Then Gauss and Weingarten equations are respectively as

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.9}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{1.10}$$

where A_V is the fundamental tensor of Weingarten with respect to the vectorfield V in the normal bundle and 'h' is the second fundamental form of M. The operator A_V satisfies

$$g(A_V X, Y) = g(h(X,Y), V) \tag{1.11}$$

for X, Y tangents to M and V normal to M.

2. Some results

In this section we shall establish some propositions on CR-submanifold M of a nearly r-cosymplectic manifold $\overline{M}$.

PROPOSITION 2.1. *Let M be a CR-submanifold of a nearly r-cosymplectic manifold $\overline{M}$. Then we have*

$$\begin{aligned} 2(\tilde{\nabla}_X \phi)(Y) = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) \\ - h(Y,\phi X) - \phi[X,Y] \end{aligned} \tag{2.1}$$

for all $X, Y \in \Gamma(D)$.

Proof. We have

$$(\tilde{\nabla}_X \phi)(Y) = \tilde{\nabla}_X(\phi Y) - \phi\tilde{\nabla}_X Y. \tag{2.2}$$

In view of the equation (1.9), the above equation (2.2) takes the form

$$(\tilde{\nabla}_X \phi)(Y) = \nabla_X(\phi Y) + h(X,\phi Y) - \phi\tilde{\nabla}_X Y. \tag{2.3}$$

Interchanging X, Y in the above equation (2.3) we get

$$(\tilde{\nabla}_Y \phi)(X) = \nabla_Y(\phi Y) + h(Y,\phi X) - \phi\tilde{\nabla}_Y X. \tag{2.4}$$

Since the structure tensor ϕ is killing, in view of the equation (1.5), the above equation (2.4) takes the form:

$$-(\tilde{\nabla}_X \phi)(Y) = \nabla_Y(\phi X) + h(Y,\phi X) - \phi\tilde{\nabla}_Y X. \tag{2.5}$$

Subtraction of the equation (2.5) from (2.3) yields

$$2(\tilde{\nabla}_X \phi)(Y) = \nabla_X(\phi Y) - \nabla_Y(\phi X) + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y].$$

Since $\tilde{\nabla}$ is a Riemannian connection on the enveloping manifold $\overline{M}$, the following proposition can be proved:

PROPOSITION 2.2. *We have for all* $X, Y \in \Gamma(D^{\perp})$

$$2(\tilde{\nabla}_X\phi)(Y) = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X(\phi Y) - \nabla^{\perp}_Y(\phi X) - \phi[X,Y].$$

Proof. We have from the equation (2.2)

$$(\tilde{\nabla}_X\phi)(Y) = \tilde{\nabla}_X(\phi Y) - \phi\tilde{\nabla}_X Y. \tag{2.6}$$

In view of the equation (1.10), the above equation takes the form

$$(\tilde{\nabla}_X\phi)(Y) = -A_{\phi Y}X + \nabla^{\perp}_X(\phi Y) - \phi\tilde{\nabla}_X Y. \tag{2.7}$$

Interchanging X and Y in the above equation and using the fact that ϕ is killing, we obtain

$$-(\tilde{\nabla}_X\phi)(Y) = -A_{\phi X}Y + \tilde{\nabla}_Y(\phi X) - \phi\tilde{\nabla}_Y X. \tag{2.8}$$

Subtracting (2.8) from (2.7) and using the fact that $\tilde{\nabla}$ is Riemannian connection on $\overline{M}$, we get the required result.

PROPOSITION 2.3. *We have for any* $X \in \Gamma(D)$ *and* $Y \in \Gamma(D^{\perp})$

$$2(\tilde{\nabla}_X\phi)(Y) = \nabla^{\perp}_X(\phi Y) - \nabla_Y(\phi X) - A_{\phi Y}X - h(\phi X, Y) + \phi[X,Y]. \tag{2.9}$$

Proof. By virtue of equations (2.5) and (2.7), the above proposition follows in a straightforward manner.

PROPOSITION 2.4. *Let M be a CR-submanifold of a nearly r-cosymplectic manifold* $\overline{M}$. *Then we have*

$$2(\tilde{\nabla}_X\phi)(\xi_p) = \phi[\xi_p, X] - \nabla_{\xi_p}(\phi X) - h(\phi X, \xi_p), \tag{2.10}$$

where $p = 1, 2, 3, \ldots, r$ *and for any* $X \in \Gamma(D)$.

Proof. We can write

$$(\tilde{\nabla}_X\phi)(\xi_p) = \tilde{\nabla}_X(\phi\xi_p) - \phi\tilde{\nabla}_X\xi_p. \tag{2.11}$$

By virtue of the equation (1.2)(i), the above equation takes the form

$$(\tilde{\nabla}_X\phi)(\xi_p) = -\phi\tilde{\nabla}_X\xi_p. \tag{2.12}$$

Also

$$-(\tilde{\nabla}_{\xi_p}\phi)(X) = -\{\tilde{\nabla}_{\xi_p}(\phi X) - \phi\tilde{\nabla}_{\xi_p}X\}$$

or

$$-(\tilde{\nabla}_{\xi_p}\phi)(X) = -\{\nabla_{\xi_p}(\phi X) + h(\phi X, \xi_p)\} + \phi\tilde{\nabla}_{\xi_p}X. \tag{2.13}$$

In view of the equation (1.5), we can write the above equation in the form

$$(\tilde{\nabla}_X\phi)(\xi_p) = -\nabla_{\xi_p}(\phi X) - h(\phi X, \xi_p) + \phi\tilde{\nabla}_{\xi_p}X. \tag{2.14}$$

Addition of (2.12) and (2.14) yields

$$2(\tilde{\nabla}_X\phi)(\xi_p) = \phi[\xi_p, X] - \nabla_{\xi_p}(\phi X) - h(\phi X, \xi_p)$$

for $\phi = 1, 2, \ldots, r$ and $X \in \Gamma(D)$.

PROPOSITION 2.5. *For any $X \in \Gamma(D^{\perp})$ we have*

$$2(\tilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \phi\nabla_{\xi_p}X - \phi\nabla_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X). \tag{2.15}$$

Proof. From the equation (2.12) we have also

$$\begin{aligned}-(\tilde{\nabla}_{\xi_p}\phi)(X) &= -\{\tilde{\nabla}_{\xi_p}(\phi X) - \phi\tilde{\nabla}_{\xi_p}X\}\\ &= -\{-A_{\phi X}\xi_p + \nabla^{\perp}_{\xi_p}(\phi X)\} + \phi\tilde{\nabla}_{\xi_p}X.\end{aligned}$$

Since the structure tensor ϕ is killing, the above equation becomes

$$(\tilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p - \nabla^{\perp}_{\xi_p}(\phi X) + \phi\tilde{\nabla}_{\xi_p}X. \tag{2.16}$$

Adding the equations (2.12) and (2.16), we get

$$2(\tilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \phi\tilde{\nabla}_{\xi_p}X - \phi\tilde{\nabla}_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X). \tag{2.17}$$

By virtue of the equation (1.9), the above equation (2.17) takes the form

$$2(\tilde{\nabla}_X\phi)(\xi_p) = A_{\phi X}\xi_p + \phi\nabla_{\xi_p}X - \phi\nabla_X\xi_p - \nabla^{\perp}_{\xi_p}(\phi X)$$

for $p = 1, 2, \ldots, r$. This proves the proposition.

3. Totally r-contact umbilical CR-submanifold of a nearly r-cosymplectic manifold

We say that CR-submanifold M of the nearly r-cosymplectic manifold $\overline{M}$ as totally r-contact umbilical submanifold if there exists a normal vectorfield H such that

$$h(X,Y) = g(\phi X, \phi Y)H + \frac{1}{r}\sum_{p=1}^{r}\{\eta(X)h(Y,\xi_p) - \eta(Y)h(X,\xi_p)\} \tag{3.1}$$

for any $X, Y \in \Gamma(TM)$. If $H = 0$, we say that M is totally r-contact geodesic submanifold of $\overline{M}$. One can easily verify the following lemma.

LEMMA 3.1. *On a nearly r-cosymplectic manifold we have*

$$(\tilde{\nabla}_X\phi)(\phi X) = \sum_{p=1} g(X,\xi_p)\tilde{\nabla}_X\xi_p \tag{3.2}$$

for any $X \in \Gamma(T\overline{M})$.

THEOREM 3.2. *Let M be a proper CR-submanifold of a nearly r-cosymplectic manifold $\overline{M}$. If M is totally r-contact umbilical it is also totally r-contact geodesic.*

Proof. For any $X \in \Gamma(D)$, we have from Lemma 3.2

$$g((\tilde{\nabla}_X\phi)\phi X, H) = 0.$$

Making use of the equation (1.1), (1.3), (1.9) and (1.10), we obtain

$$(3.3)\qquad \begin{aligned} g((\tilde{\nabla}_X\phi)\phi X, H) &= g(\tilde{\nabla}_X\phi X, \phi H) - g(\tilde{\nabla}_X X, H)\\ &= g(X, \tilde{\nabla}_X H) - g(\phi X, \tilde{\nabla}_X \phi H). \end{aligned}$$

Making use of the equation (3.1), we get

$$(3.4)\qquad g(\phi X, A_{\phi H}X) = g(h(X,\phi X),\phi H) = g(X,\phi X)g(H,\phi H) = 0.$$

Thus from (3.2), (3.3), and (3.4), it follows that

$$(3.5)\qquad g(X,X)g(H,H) = 0 \quad \text{for any } X \in \Gamma(D).$$

Since M is proper CR-submanifold, from (3.5) it follows that $H = 0$. Hence M is totally r-contact geodesic.

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