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GLOBAL EXISTENCE RESULTS FOR CERTAIN INTEGRODIFFERENTIAL EQUATIONS

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $|\cdot|$ be a norm on $\mathbb{R}^n$. For a fixed $r > 0$ we define $C = C([-r,0],\mathbb{R}^n)$ to be the Banach space of all continuous functions $x : [-r,0] \to \mathbb{R}^n$, endowed with the sup-norm

$$\|x\| = \sup\{|x(t)| : t \in [-r,0]\}.$$

For any continuous function $x : [-r,T] \to \mathbb{R}^n$, $T > 0$ and every $t \in [0,T]$, we denote by x_t the element of C defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-r,0],$$

for details, see [1], [4].

This paper is concerned with the global existence of solutions for initial value problems for functional integrodifferential equations of the forms

$$x'(t) = f(t, x_t, \int_0^t k(t,s)h(s,x_s)\,ds), \quad t \in [0,T], \tag{1}$$

$$x_0 = \phi, \tag{2}$$

and

$$[x'(t) - g(t,x_t)]' = f(t, x_t, \int_0^t k(t,s)h(s,x_s)\,ds), \quad t \in [0,T], \tag{3}$$

$$x_0 = \phi, \quad x'(0) = \alpha, \tag{4}$$

where k is measurable real valued function for $t \ge s \ge 0$, $\phi \in C$, α is a real constant, $h, g : [0,T] \times C \to \mathbb{R}^n$ and $f : [0,T] \times C \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

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The main tool employed in our analysis is based on a simple and classical application of the topological transversality theorem of Granas [3], known as Leray–Schauder alternative. Recently, in [7], [8] the authors used this method to study the global existence of solutions of certain special versions of the above equations. An interesting feature of this method is that this yields simultaneously the existence of a solution and the maximal interval of existence. For further applications of this method to study the global existence of solutions of initial value problems for various types of differential, functional differential and differential delay equations, see [2], [5], [6] and the references given therein. Our results given here are further extensions of the results given in [7], [8] to more general functional integrodifferential equations.

2. Statement of results

Our existence theorems are based on the following theorem, which is a version of the topological transversality theorem given by A. Granas in ([3], p. 61).

THEOREM G. *Let B be a convex subset of a normal linear space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) = \{x \in B : x = \lambda F x \quad \text{for some } 0 < \lambda < 1\}.$$

Then either $U(F)$ is unbounded or F has a fixed point.

We list the following hypotheses used in our discussion.

(A_1) There exists a continuous function $p : [0,T] \to \mathbb{R}_+ = [0,\infty)$ such that

$$|f(t,\psi,u)| \leq p(t)(\|\psi\| + |u|),$$

for every $t \in [0,T]$, $\psi \in C$, $u \in \mathbb{R}^n$.

(A_2) There exists a continuous function $q : [0,T] \to \mathbb{R}_+$ such that

$$|h(t,\psi)| \leq q(t)H(\|\psi\|),$$

for every $t \in [0,T]$, $\psi \in C$, where $H : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(A_3) There exists a constant $L \geq 0$ such that

$$|k(t,s)| \leq L, \quad t \geq s \geq 0.$$

(A_4) There exist nonnegative constants c_1, c_2 such that

$$|g(t,\psi)| \leq c_1\|\psi\| + c_2,$$

for every $t \in [0,T]$ and $\psi \in C$.

Our main results are given in the following theorems.

THEOREM 1. *Suppose that the hypotheses* (A_1)–(A_4) *are satisfied. Then the initial value problem* (1)–(2) *has a solution* x *defined on* $[-r,T]$ *provided* T *satisfies*

$$\int_0^T M(s)\,ds < \int_c^\infty \frac{ds}{s+H(s)}, \tag{5}$$

where

$$c = \|\phi\| \quad \text{and} \quad M(t) = \max\{p(t), Lq(t)\}, \quad t \in [0,T]. \tag{5}$$

THEOREM 2. *Suppose that the hypotheses* (A_1)–(A_4) *are satisfied. Then the initial value problem* (3)–(4) *has a solution* x *defined on* $[-r,T]$ *provided* T *satisfies*

$$\int_0^T N(s)\,ds < \int_0^\infty \frac{ds}{2s+H\left(\frac{1}{2}s\right)}, \tag{6}$$

where

$$c = (1+c_1T)\|\phi\| + (|\alpha| + 2c_2)T, \tag{7}$$

and

$$N(t) = \max\{1, c_1, p(t), Lq(t)\}, \quad t \in [0,T]. \tag{8}$$

Remark 1. We note that, in a recent paper [8], the topological transversality method is used to study the global existence of solutions of the special versions of equations (1)–(2) and (3)–(4) when the function f is of the form

$$A(t)x(t) + \int_0^t k(t,s)h(s,x_s)\,ds \quad \text{and} \quad f_1(t,x_t) + f_2(t,x_t)$$

respectively, where $A(t)$ is a continuous $n \times n$ matrix for $t \in [0,T]$ and $f_1, f_2 : [0,T] \times C \to \mathbb{R}^n$ are continuous functions satisfying some suitable assumptions. It is easy to observe that Theorem 1 cannot be obtained from Theorem 2 by integrating equation (3). Our results given here are influenced by the recent results obtained by various authors in [2], [5]–[8] by using topological arguments based on Leray–Schauder degree theory and a very interesting result given by Wintner in [10].

3. Proofs of Theorems 1 and 2

Since the proofs of Theorems 1 and 2 resemble one another, we give the details for Theorem 2 only, the proof of Theorem 1 can be completed by closely looking at the proof of Theorem 2.

To prove the existence of a solution of (3)–(4) we apply Theorem G. First we establish the bounds for the initial value problem $(3)_\lambda - (4)$, $\lambda \in (0,1)$, where

$$(3)_\lambda \qquad [x'(t) - \lambda g(t,x_t)]' = \lambda f\Big(t, x_t, \int_0^t k(t,s)h(s,x_s)\,ds\Big), \qquad t \in [0,T].$$

Let $x(t)$ be a solution of $(3)_\lambda$–(4). Then it satisfies the equivalent integral equation

$$(9) \qquad x(t) = \phi(0) + [\alpha - \lambda g(0,\phi)]t + \lambda \int_0^t g(s,x_s)\,ds$$
$$+\lambda \int_0^t \int_0^s f\Big(\tau, x_\tau, \int_0^\tau k(\tau,\eta)h(\eta,x_\eta)\,d\eta\Big)\,d\tau\,ds, \qquad t \in [0,T].$$

From (9) and using the hypotheses (A_1)–(A_4) we have

$$(10) \qquad |x(t)| \le c + \int_0^t c_1 \|x_s\|\,ds + \int_0^t \int_0^s p(\tau)\|x_\tau\|\,d\tau\,ds$$
$$+ \int_0^t \int_0^s p(\tau)\Big(\int_0^\tau Lq(\eta)H(\|x_\eta\|)\,d\eta\Big)\,d\tau\,ds,$$

where c is defined by (7). Consider the function $z : [-r,T] \to \mathbf{R}$ given by

$$z(t) = \sup\{|x(s)| : -r \le s \le t\}, \qquad 0 \le t \le T,$$

and let $t^* \in [-r,t]$ be such that $z(t) = |x(t^*)|$. If $t^* \in [0,t]$, by (10), we have

$$(11) \qquad z(t) = |x(t^*)| \le c + \int_0^{t^*} c_1\|x_s\|\,ds + \int_0^{t^*}\int_0^s p(\tau)\|x_\tau\|\,d\tau\,ds$$
$$+ \int_0^{t^*}\int_0^s p(\tau)\Big(\int_0^\tau Lq(\eta)H(\|x_\eta\|)\,d\eta\Big)\,d\tau\,ds$$
$$\le c + \int_0^t c_1 z(s)\,ds + \int_0^t\int_0^s p(\tau)z(\tau)\,d\tau\,ds$$
$$+ \int_0^t\int_0^s p(\tau)\Big(\int_0^\tau Lq(\eta)H(z(\eta))\,d\eta\Big)\,d\tau\,ds$$
$$\le c + \int_0^t N(s)z(s)\,ds + \int_0^t\int_0^s N(\tau)z(\tau)\,d\tau\,ds$$

$$+\int_0^t \int_0^s N(\tau)\Big(\int_0^\tau N(\eta)H(z(\eta))\,d\eta\Big)\,d\tau\,ds,$$

where $N(t)$ is defined by (8). If $t^* \in [-r,0]$, then $z(t) = \|\phi\| \le c$, by (7) and (11) holds. Denoting by $u(t)$ the right hand side of (11) we have

$$z(t) \le u(t), \quad t \in [0,T], \quad u(0) = c,$$

and

$$u'(t) \le N(t)\Big[u(t) + \int_0^t N(\tau)u(\tau)\,d\tau + \int_0^t N(\tau)\Big(\int_0^\tau N(\eta)H(u(\eta))\,d\eta\Big)\,d\tau\Big], \quad t \in [0,T].$$

Denote by $v(t)$ the expression in the brackets above, then, since the components with integrals are nonnegative

$$u(t) \le v(t), \quad t \in [0,T], \quad v(0) = u(0) = c,$$

and

$$v'(t) \le N(t)\Big[2v(t) + \int_0^t N(\eta)H(v(\eta))\,d\eta\Big], \quad t \in [0,T].$$

Denote by $w(t)$ the expression in the brackets above, then since the components with integrals are nonnegative

$$v(t) \le \frac{1}{2}w(t), \quad t \in [0,T], \quad w(0) = v(0) = c,$$

and

$$w'(t) \le 2N(t)w(t) + N(t)H\Big(\frac{1}{2}w(t)\Big), \quad t \in [0,T],$$

i.e.

$$\frac{w'(t)}{2w(t) + H\left(\frac{1}{2}w(t)\right)} \le N(t), \quad t \in [0,T]. \tag{12}$$

Integrating (12) from 0 to t and using (6) we obtain

$$\int_c^{w(t)} \frac{ds}{2s + H\left(\frac{1}{2}s\right)} \le \int_0^t N(s)\,ds \le \int_0^T N(s)\,ds < \int_0^\infty \frac{ds}{2s + H\left(\frac{1}{2}s\right)}, \quad t \in [0,T]. \tag{13}$$

From (13) we conclude by the mean value theorem that there is a constant Q independent of $\lambda \in (0,1)$ such that $w(t) \le Q$, for $t \in [0,T]$. Then we

have succesively $v(t) \leq Q$, $u(t) \leq Q$, $z(t) \leq Q$ for $t \in [0,T]$. Since for every $t \in [0,T]$, $\|x_t\| \leq z(t)$, we have $\|x\| \leq Q$.

In the next step, we rewrite the initial value problem (3)–(4) as follows, see [4]. For $\phi \in C$ define $\overline{\phi} \in B$, $B = C([-r,T],\mathbf{R}^n)$ by

$$\overline{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ \phi(0), & 0 < t \leq T. \end{cases}$$

If $x(t) = y(t)+\overline{\phi}(t)$, $t \in [-r,T]$, then, provided that x satisfies the equivalent integral equation to (3)–(4), it is easy to see that y satisfies

$$y_0 = 0,$$

$$y(t) = [\alpha - g(0,\phi)]t + \int_0^t g(s, y_s + \overline{\phi}_s)\, ds$$

$$+ \int_0^t \int_0^s f\Big(\tau, y_\tau + \overline{\phi}_\tau, \int_0^T k(\tau,\eta)h(\eta, y_\eta + \overline{\phi}_\eta)\, d\eta\Big)\, d\tau\, ds, \qquad 0 \leq t \leq T.$$

Define $F : B_0 \to B_0$, $B_0 = \{y \in B : y_0 = 0\}$ by

$$(14)\quad Fy(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ [\alpha - g(0,\phi)]t + \int_0^t g(s, y_s + \overline{\phi}_s)\, ds \\ \quad + \int_0^t \int_0^s f(\tau, y_\tau + \overline{\phi}_\tau, \int_0^T k(\tau,\eta)h(\eta, y_\eta + \overline{\phi}_\eta)\, d\eta)\, d\tau\, ds, & 0 \leq t \leq T. \end{cases}$$

Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_m\}$ be a bounded sequence in B_0, i.e. $\|w_m\| \leq b$, for all m, where b is a positive constant. We obviously have $\|w_{mt}\| \leq b$, $t \in [0,T]$, for all m. For (14) and using the hypotheses (A_1)–(A_4) and letting $\beta = \sup\{N(t) : 0 \leq t \leq T\}$, we see that $\{Fw_m\}$ is uniformly bounded.

Now we shall show that the sequence $\{Fw_m\}$ is equicontinuous. To prove this we consider the following cases:

(i) $0 \leq t_1 \leq t_2$. Then

$$|Fw_m(t_2) - Fw_m(t_1)|$$

$$\leq \int_{t_1}^{t_2} |[\alpha - g(0,\phi)]|\, ds + \int_{t_1}^{t_2} |g(s, y_s + \overline{\phi}_s|\, ds$$

$$+\int_{t_1}^{t_2}\int_0^s \left|f\Big(\tau, y_\tau+\overline{\phi}_\tau, \int_0^\tau k(\tau,\eta)h(\eta, y_\eta+\overline{\phi}_\eta)\,d\eta\Big)\right| d\tau\, ds$$

$$\le \int_{t_1}^{t_2} (|\alpha| + c_1\|\phi\| + c_2)\,ds + \int_{t_1}^{t_2} [c_1(b+\|\phi\|)+c_2]\,ds$$

$$+\int_{t_1}^{t_2}\int_0^s \beta\Big(b+\|\phi\| + \int_0^\tau \beta H(b+\|\phi\|)\,d\eta\Big)d\tau\, ds.$$

(ii) $t_1 \le 0 \le t_2$. Then we get the estimation as above but with the integrals over $[t_1,t_2]$ replaced by those over $[0,t_2]$.

(iii) $t_1 \le t_2 \le 0$. Then

$$|Fw_m(t_2) - Fw_m(t_1)| = 0.$$

From (i)–(iii), since they imply $\|Fw_m(t_2)-Fw_m(t_1)\| \le \gamma|t_2-t_1|$, with constant $\gamma > 0$, for every $t_1, t_2 \in [-r,T]$, we conclude that $\{Fw_m\}$ is equicontinuous and hence by the Arzela–Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F) = \{y \in B_0 : y = \lambda F y,\ \lambda \in (0,1)\}$ is bounded, since for every solution y in $U(F)$ the function $x = y + \overline{\phi}$ is a solution of $(3)_\lambda$–(4), for which we have proved $\|x\| \le Q$ and hence $\|y\| \le Q + \|\phi\|$. Now in virtue of Theorem G, the operator F has a fixed point in B_0. This means that the initial value problem (3)–(4) has a solution. The proof is completed.

Remark 2. We note that one can easily use the ideas of this paper to obtain a result similar to that of given in our Theorem 1, for the integrodifferential equation with delayed arguments of the form (see, also [9], [10], [11])

$$(15)\quad x'(t) = \eta(t)f\Big(t, x(\sigma_1(t)),\dots,x(\sigma_n(t)), \int_0^t k(t,s)h(s, x(\omega_1(s)),\dots,x(\omega_m(s)))\,ds\Big), \qquad t\in[0,T],$$

$$(16)\quad x(t) = \phi(t), \qquad -r \le t \le 0,$$

under some suitable conditions on the functions involved in (15)–(16). For such result on the special version of (15)–(16), (see, [6], p. 351). It is possible to combine the ideas of this paper together with the ideas of [2], [5]–[8] to extend the foregaing existence results to integrodifferential equations with advanced arguments as well as with advanced and delayed arguments. The various results on such equations will be reported elsewhere.

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