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AN INTEGRAL ANALOGUE OF A THEOREM OF LEINDLER

Dedicated to Professor Janina Wolska-Bochenek

1. Preliminaries

Let L_{loc} be the set of all measurable (complex-valued) functions Lebesgue-integrable on every finite subinterval of $R = (-\infty, \infty)$. Denote by L^∞ the set of those functions $f \in L_{loc}$ for which

$$\|f\| = \operatorname{ess\,sup}_{-\infty < t < \infty} |f(t)| < \infty.$$

Write $f \in W^\alpha$, $1 \leq \alpha < \infty$, if $f \in L_{loc}$ and

$$\int_{-\infty}^{\infty} \left(\frac{|f(t)|}{1+|t|} \right)^\alpha dt < \infty.$$

If $f \in L_{loc}$ and if

$$\|f\|_2 = \left\{ \int_{-\infty}^{\infty} |f(t)|^2 dt \right\}^{1/2} < \infty,$$

the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(v) = \operatorname{l.i.m.}_{\varrho \rightarrow \infty}^{(2)} \frac{1}{\sqrt{2\pi}} \int_{-\varrho}^{\varrho} f(u) e^{-iuv} du \quad \text{for a.e. } v \in R.$$

In this case, the Parseval formula

$$(1) \quad \|\hat{f}\|_2 = \|f\|_2$$

holds (see [1, p. 209]).

Let E_σ , $0 \leq \sigma < \infty$, be the class of all entire functions of exponential type, of order σ at most. Considering a function $f \in L_{loc}$, introduce the quantity

$$A_\sigma(f) = \inf_{F \in E_\sigma} \|f - F\|$$

called the best approximation of f by entire functions of class E_σ . Obviously, $A_\sigma(f)$ is non-negative and, sometimes, equals infinity. Moreover, $A_\sigma(f) \geq A_\tau(f)$ for $\tau > \sigma \geq 0$.

Take into account the entire function D_λ , $0 \leq \lambda < \infty$, given by

$$D_\lambda(z) = \frac{\sin \lambda z}{z} \quad \text{for } z \neq 0.$$

Suppose that $f \in L_{\text{loc}}$ and write

$$S_\lambda[f](z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(u) D_\lambda(z-u) du,$$

whenever the improper Lebesgue integral on the right-hand side exists. This is the so-called Dirichlet (singular) integral of f at a point $z = x + iy$ ($x, y \in R$).

Put

$$V_{n,m}^{(p)}[f](x) = \left\{ \frac{1}{m} \int_{n-m}^n |S_\lambda[f](x) - f(x)|^p d\lambda \right\}^{1/p}$$

for positive numbers m, n, p ($m \leq n$) and $x \in R$. Let, by convention,

$$V_{n,m}(x) = V_{n,m}[f](x) = V_{n,m}^{(1)}[f](x)$$

and

$$\tau_{n,m}(x) = \tau_{n,m}[f](x) = m\{V_{n+m,m}(x) - V_{n,m}(x)\}.$$

It was proved in [3, Section 4] that under the assumptions $f \in W^1 \cap W^2$, $A_\sigma(f) < \infty$ for $\sigma > 0$ and $0 < m < n < \infty$, the estimate

$$(2) \quad \|V_{n,m}^{(p)}[f]\| \leq K(p) A_{n-m}(f) \log \frac{2n}{m}$$

holds with a certain constant $K(p)$ depending only on p . Now, in case $p = 1$, an other result will be presented. It corresponds to the theorem on strong approximation by Fourier series formulated in [2, p. 29].

The symbols K, K_1, K_2 used below will mean positive absolute constants not necessarily the same at each occurrence.

2. Basic estimates

Consider a function f from $W^1 \cap W^2$, with $A_\sigma(f) < \infty$ for $\sigma > 0$.

LEMMA 1. Suppose that $0 < m < n < \infty$, $m \leq q < \infty$. Then $S_\lambda[f](x)$ for $x \in R$ is continuous in $\lambda \geq 0$ and

$$(3) \quad \sup_{x \in R} \left\{ \frac{1}{m} \int_{n-m}^n |S_{\lambda+q}[f](x) - S_{\lambda}[f](x)| d\lambda \right\} \leq K A_{n-m}(f) \log \frac{2q}{m}.$$

Proof. Since $f \in W^1$, the Dirichlet integral $S_{\lambda}[f](x)$ is continuous in $\lambda \geq 0$ (see Remark in [3, Sec. 2]).

Given any $\varepsilon > 0$, let F be a function of class E_{n-m} such that

$$\|f - F\| < A_{n-m}(f) + \varepsilon.$$

Evidently $F \in W^2$ and, by Lemma 1 of [3, Sec. 3],

$$(4) \quad S_{\lambda}[f - F](x) = S_{\lambda}[f](x) - F(x) \quad \text{for } \lambda > n - m \text{ and } x \in R.$$

In view of (4),

$$\begin{aligned} & \int_{n-m}^n |S_{\lambda+q}[f](x) - S_{\lambda}[f](x)| d\lambda \\ &= \int_{n-m}^n |S_{\lambda+q}[f - F](x) - S_{\lambda}[f - F](x)| d\lambda \\ &= \int_{n-m}^n \left| \frac{1}{\pi} \lim_{\varrho \rightarrow \infty} \int_{-\varrho}^{\varrho} \{f(x-t) - F(x-t)\} \{D_{\lambda+q}(t) - D_{\lambda}(t)\} dt \right| d\lambda \\ &= \int_{n-m}^n \left| \frac{1}{2\pi} \lim_{\varrho \rightarrow \infty} \int_{-\varrho}^{\varrho} \{f(x-t) - F(x-t) + f(x+t) - F(x+t)\} \right. \\ & \quad \left. \times \{D_{\lambda+q}(t) - D_{\lambda}(t)\} dt \right| d\lambda. \end{aligned}$$

Putting

$$g_x(t) = \begin{cases} \frac{f(x-t) - F(x-t) + f(x+t) - F(x+t)}{t} & \text{if } |t| \geq \frac{1}{m}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\begin{aligned} & \int_{n-m}^n |S_{\lambda+q}[f](x) - S_{\lambda}[f](x)| d\lambda \\ & \leq \int_{n-m}^n \left\{ \frac{1}{\pi} \left(\int_0^{1/q} + \int_{1/q}^{1/m} \right) |f(x-t) - F(x-t) + f(x+t) - F(x+t)| \right. \\ & \quad \times \left| \frac{\sin(\lambda+q)t - \sin \lambda t}{t} \right| dt \Big\} d\lambda \\ & \quad + \int_{n-m}^n \left| \frac{1}{2\pi} \lim_{\varrho \rightarrow \infty} \int_{-\varrho}^{\varrho} g_x(t) \{\sin(\lambda+q)t - \sin \lambda t\} dt \right| d\lambda \end{aligned}$$

$$=: I_1 + I_2 + I_3.$$

Clearly,

$$\begin{aligned} I_1 &\leq \frac{2}{\pi} \int_{n-m}^n \left\{ (A_{n-m}(f) + \varepsilon) \int_0^{1/q} \left| \frac{\sin(\lambda + q)t - \sin \lambda t}{t} \right| dt \right\} d\lambda \\ &\leq \frac{2}{\pi} (A_{n-m}(f) + \varepsilon) \int_{n-m}^n \left\{ \int_0^{1/q} q dt \right\} d\lambda = \frac{2}{\pi} (A_{n-m}(f) + \varepsilon) m. \end{aligned}$$

Analogously,

$$I_2 \leq \frac{2}{\pi} \int_{n-m}^n \left\{ (A_{n-m}(f) + \varepsilon) \int_{1/q}^{1/m} \frac{2}{t} dt \right\} d\lambda = \frac{4}{\pi} (A_{n-m}(f) + \varepsilon) m \log \frac{q}{m}.$$

By the Cauchy-Schwarz inequality and the Parseval formula (1),

$$\begin{aligned} \frac{1}{m} I_3 &\leq \left\{ \frac{1}{m} \int_{n-m}^n \left| \frac{1}{2\pi} \lim_{\varrho \rightarrow \infty} \int_{-\varrho}^{\varrho} g_x(t) (\sin(\lambda + q)t - \sin \lambda t) dt \right|^2 d\lambda \right\}^{1/2} \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{m} \int_{n-m}^n |\widehat{g}_x(\lambda + q) - \widehat{g}_x(\lambda)|^2 d\lambda \right\}^{1/2} \\ &\leq \sqrt{\frac{2}{\pi m}} \left\{ \int_{-\infty}^{\infty} |g_x(t)|^2 dt \right\}^{1/2} = \sqrt{2} \sqrt{\frac{2}{\pi m}} \left\{ \int_{1/m}^{\infty} |g_x(t)|^2 dt \right\}^{1/2} \\ &\leq 4 \sqrt{\frac{1}{\pi m}} (A_{n-m}(f) + \varepsilon) \left\{ \int_{1/m}^{\infty} \frac{1}{t^2} dt \right\}^{1/2}. \end{aligned}$$

Consequently,

$$I_3 \leq 4\sqrt{1/\pi} (A_{n-m}(f) + \varepsilon) m.$$

Summing up we get at once the desired assertion (3).

LEMMA 2. Let $0 < m \leq n < \infty$ and let $0 < 2\mu \leq m$. Then

$$(5) \quad \|\tau_{n,m}[f] - \tau_{n-\mu,m-\mu}[f]\| \leq K\mu A_{n-\mu}(f) \log \frac{m}{\mu}.$$

Proof. Clearly, for a.e. $x \in R$,

$$\begin{aligned} \tau_{n,m}(x) - \tau_{n-\mu,m-\mu}(x) &= \left(\int_n^{n+m} - \int_{n-\mu}^n - \int_{n-\mu}^{n+m-2\mu} \right) |S_\lambda[f](x) - f(x)| d\lambda \\ &= \left(\int_{n+m-2\mu}^{n+m} - \int_{n-\mu}^n - \int_{n-\mu}^n \right) |S_\lambda[f](x) - f(x)| d\lambda. \end{aligned}$$

By Lemma 1,

$$\begin{aligned}
 & \left| \left(\int_{n+m-2\mu}^{n+m-\mu} - \int_{n-\mu}^n \right) |S_\lambda[f](x) - f(x)| d\lambda \right| = \mu |V_{n+m-\mu,\mu}(x) - V_{n,\mu}(x)| \\
 &= \left| \int_{n-\mu}^n (|S_{\lambda+m-\mu}[f](x) - f(x)| - |S_\lambda[f](x) - f(x)|) d\lambda \right| \\
 &\leq \mu \cdot \frac{1}{\mu} \int_{n-\mu}^n |S_{\lambda+m-\mu}[f](x) - S_\lambda[f](x)| d\lambda \\
 &\leq \mu K A_{n-\mu}(f) \log \frac{2(m-\mu)}{\mu}, \\
 &\left| \left(\int_{n+m-\mu}^{n+m} - \int_{n-\mu}^n \right) |S_\lambda[f](x) - f(x)| d\lambda \right| \\
 &\leq \int_{n-\mu}^n |S_{\lambda+m}[f](x) - S_\lambda[f](x)| d\lambda \\
 &\leq \mu K A_{n-\mu}(f) \log \frac{2m}{\mu}.
 \end{aligned}$$

Hence

$$\|\tau_{n,m} - \tau_{n-\mu,m-\mu}\| \leq K \mu A_{n-\mu}(f) \left\{ \log \frac{2(m-\mu)}{\mu} + \log \frac{2m}{\mu} \right\},$$

and the estimate (5) is established.

PROPOSITION . If $0 < m < n < \infty$ then

$$(6) \quad \|\tau_{n,m}[f]\| \leq K \sum_{\nu=0}^{[m]} A_{n-m+\nu}(f).$$

Proof. In case $0 < m < 2$ Lemma 1 gives

$$\begin{aligned}
 \|\tau_{n,m}[f]\| &= m \|V_{n+m,m} - V_{n,m}\| \\
 &= \left\| \int_n^{n+m} |S_\lambda[f] - f| d\lambda - \int_{n-m}^n |S_\lambda[f] - f| d\lambda \right\| \\
 &\leq \left\| \int_{n-m}^n |S_{\lambda+m}[f] - S_\lambda[f]| d\lambda \right\| \\
 &\leq K m A_{n-m}(f) \log \frac{2m}{m} \leq K_1 A_{n-m}(f).
 \end{aligned}$$

Therefore, we may suppose that $m \geq 2$.

Following Stechkin ([2, p. 33]) take into account the positive sequence defined by

$$m_0 = m, \quad m_s = m_{s-1} - \{m_{s-1}/2\} \quad (s = 1, 2, \dots).$$

Evidently, there exists a smallest index $t \geq 1$ such that $1 \leq m_t < 2$ and

$$m = m_0 > m_1 > \dots > m_t \geq 1, \quad m_{t-1} - m_t = 1.$$

Since

$$\tau_{n,m} = \sum_{s=1}^t \{ \tau_{n-m+m_{s-1}, m_{s-1}} - \tau_{n-m+m_s, m_s} \} + \tau_{n-m+m_t, m_t},$$

we have

$$\|\tau_{n,m}\| \leq \sum_{s=1}^t \|\tau_{n-m+m_{s-1}, m_{s-1}} - \tau_{n-m+m_s, m_s}\| + \|\tau_{n-m+m_t, m_t}\|.$$

In view of Lemma 2,

$$\|\tau_{n-m+m_{s-1}, m_{s-1}} - \tau_{n-m+m_s, m_s}\| \leq K\mu A_{n-m+m_{s-1}-\mu}(f) \log \frac{m_{s-1}}{\mu}$$

where $\mu = m_{s-1} - m_s > m_{s-1}/4$ ($s = 1, \dots, t$).

By Lemma 1,

$$\begin{aligned} \|\tau_{n-m+m_t, m_t}\| &= m_t \|V_{n-m+2m_t, m_t} - V_{n-m+m_t, m_t}\| \\ &\leq 2 \left\| \frac{1}{m_t} \int_{n-m}^{n-m+m_t} |S_{\lambda+m_t}[f] - S_\lambda[f]| d\lambda \right\| \leq K A_{n-m}(f) \log 2. \end{aligned}$$

Hence

$$\begin{aligned} \|\tau_{n,m}\| &\leq K_1 \left\{ \sum_{s=1}^t (m_{s-1} - m_s) A_{n-m+m_s}(f) + A_{n-m}(f) \right\} \\ &\leq K_1 \left\{ 4 \sum_{s=1}^{t-1} (m_s - m_{s+1}) A_{n-m+m_s}(f) + A_{n-m+m_t} + A_{n-m}(f) \right\}, \end{aligned}$$

i.e.

$$\begin{aligned} \|\tau_{n,m}\| &\leq K_2 \left\{ \sum_{s=1}^{t-1} \int_{m_{s+1}}^{m_s} A_{n-m+\nu}(f) d\nu + A_{n-m}(f) \right\} \\ &= K_2 \left\{ \int_{m_t}^{m_1} A_{n-m+\nu}(f) d\nu + A_{n-m}(f) \right\}. \end{aligned}$$

Further,

$$\int_{m_1}^{m_2} A_{n-m+\nu}(f) d\nu \leq \int_1^{m-1} A_{n-m+\nu}(f) d\nu \leq \sum_{k=1}^{[m]-1} \int_k^{k+1} A_{n-m+\nu}(f) d\nu.$$

Consequently,

$$\|\tau_{n,m}\| \leq K_2 \sum_{k=0}^{[m]-1} A_{n-m+k}(f),$$

which leads to (6), immediately.

3. Main result

Suppose that $f \in W^1 \cap W^2$, $A_\sigma(f) < \infty$ for $\sigma > 0$, $\lim_{\sigma \rightarrow \infty} A_\sigma(f) = 0$.

THEOREM . If $0 < m < n < \infty$ then

$$(7) \quad \|V_{n,m}[f]\| \leq K \sum_{\nu=0}^{[n]} \frac{A_{n-m+\nu}(f)}{m+\nu}.$$

Proof. When $0 < n < 1$,

$$\left\| \int_{n-m}^n |S_\lambda[f] - f| d\lambda \right\| \leq K_1 A_{n-m}(f).$$

Indeed, given any $\varepsilon > 0$ there is an entire function $F \in E_{n-m}$ such that

$$\|f - F\| < A_{n-m}(f) + \varepsilon.$$

Thus, by Lemma 1 and Lemma 2' of [3, Sec. 3],

$$\begin{aligned} & \int_{n-m}^n |S_\lambda[f](x) - f(x)| d\lambda \\ &= \int_{n-m}^n |S_\lambda[f - F](x) + F(x) - f(x)| d\lambda \\ &< Km\{A_{n-m}(f) + \varepsilon\} \log \frac{2n}{m} \\ &\quad + m|F(x) - f(x)| \quad \text{for a.e. } x \in R. \end{aligned}$$

Consequently,

$$\left\| \int_{n-m}^n |S_\lambda[f] - f| d\lambda \right\| \leq Km A_{n-m}(f) \log \frac{2}{m} + A_{n-m}(f) \leq K_1 A_{n-m}(f),$$

and the estimate (7) follows.

In case $n \geq 1$ we construct, similarly to [2, p. 34], a finite sequence $\{n_k\}$ of positive numbers $n_0 = n, n_1 > n_0, \dots$. Namely, assuming that $n_s < 2n$, we define n_{s+1} as follows. Denote by ν_s the smallest non-negative integer such that

$$(8) \quad A_{n_s-m+\nu_s}(f) \leq \frac{1}{2} A_{n_s-m}(f)$$

and set

$$n_{s+1} = \begin{cases} n_s + m & \text{if } \nu_s \leq m, \\ n_s + \nu_s & \text{if } m < \nu_s < 2n + m - n_s, \\ 2n + m & \text{if } \nu_s \geq 2n + m - n_s. \end{cases}$$

When $n_{s+1} < 2n$ we continue the procedure; in case $n_{s+1} \geq 2n$ we stop the construction and define $t := s + 1$.

Clearly,

$$(9) \quad n = n_0 < n_1 < \dots < n_t, \quad 2n \leq n_t \leq 2n + m$$

and $n_{s+1} - n_s \geq m$ for $s = 0, 1, \dots, t-1$. By (8),

$$(10) \quad A_{n_{s+1}-m}(f) \leq \frac{1}{2} A_{n_s-m}(f) \quad \text{for } s = 0, 1, \dots, t-2.$$

Moreover, if $n_{s+1} - n_s > m$ ($0 \leq s \leq t-1$) then

$$(11) \quad \frac{1}{2} A_{n_s-m}(f) \leq A_{n_{s+1}-m-1}(f)$$

(see [2, p. 35]).

Since

$$V_{n,m} = \sum_{s=0}^{t-1} \{V_{n_s,m} - V_{n_{s+1},m}\} + V_{n_t,m},$$

we have

$$\|V_{n,m}\| \leq \sum_{s=0}^{t-1} \|V_{n_s,m} - V_{n_{s+1},m}\| + \|V_{n_t,m}\|.$$

In view of (2) and (9),

$$\begin{aligned} \|V_{n_t,m}\| &\leq K A_{n_t-m}(f) \log \frac{2n_t}{m} \leq K A_{2n-m}(f) \log \frac{4n+2m}{m} \\ &\leq 2K A_{2n-m}(f) \log \frac{2[n]+m+2}{m} \leq 2K A_{2n-m}(f) \sum_{\nu=0}^{2[n]+1} \frac{1}{m+\nu} \\ &\leq 4K A_{n-m+[n]}(f) \sum_{\nu=0}^{[n]} \frac{1}{m+\nu} \leq 4K \sum_{\nu=0}^{[n]} \frac{A_{n-m+\nu}(f)}{m+\nu}. \end{aligned}$$

Putting $\lambda_s = [n_{s+1} - n_s]$, we observe that

$$(12) \quad \|V_{n_{s+1},m} - V_{n_s,m}\| \leq K \sum_{\nu=0}^{\lambda_s} \frac{A_{n_s-m+\nu}(f)}{m+\nu} \quad \text{for } s = 0, 1, \dots, t-1.$$

Indeed, if $n_{s+1} - n_s = m$ then

$$V_{n_{s+1},m} - V_{n_s,m} = V_{n_s+m,m} - V_{n_s,m} = \frac{1}{m} \tau_{n_s,m}.$$

Hence, by Proposition,

$$\|V_{n_{s+1},m} - V_{n_s,m}\| \leq \frac{K}{m} \sum_{\nu=0}^{[m]} A_{n_s-m+\nu}(f) \leq 2K \sum_{\nu=0}^{[m]} \frac{A_{n_s-m+\nu}(f)}{m+\nu}.$$

In case $n_{s+1} - n_s > m$, Lemma 1 and (11) yield

$$\begin{aligned} \|V_{n_{s+1},m} - V_{n_s,m}\| &\leq K A_{n_s-m}(f) \log \frac{2(n_{s+1} - n_s)}{m} \\ &\leq K A_{n_s-m}(f) \frac{2}{m} \quad \text{if } \lambda_s = 0, \end{aligned}$$

and

$$\begin{aligned} \|V_{n_{s+1},m} - V_{n_s,m}\| &\leq 4K A_{n_{s+1}-m-1}(f) \sum_{\nu=0}^{\lambda_s} \frac{1}{m+\nu} \\ &\leq 8K \sum_{\nu=0}^{\lambda_s-1} \frac{A_{n_s-m+\nu}(f)}{m+\nu} \quad \text{if } \lambda_s \geq 1. \end{aligned}$$

Thus (12) holds.

In view of (12),

$$\sum_{s=0}^{t-1} \|V_{n_{s+1},m} - V_{n_s,m}\| \leq K \sum_{s=0}^{t-1} \sum_{\nu=0}^{\lambda_s} \frac{A_{n_s-m+\nu}(f)}{m+\nu} =: K \Sigma_2.$$

Since $\lambda_s \leq n_{s+1} - n_s \leq 2n + m - n = n + m$ for $s = 0, 1, \dots, t-1$, we get

$$\Sigma_2 \leq \sum_{\nu=0}^{[n+m]} \frac{1}{m+\nu} \sum_{s: \lambda_s \geq \nu} A_{n_s-m+\nu}(f).$$

Let $p = \min\{s : \lambda_s \geq \nu\}$. Then

$$\sum_{s: \lambda_s \geq \nu} A_{n_s-m+\nu}(f) = A_{n_p-m+\nu}(f) + \sum_{s \geq p+1: \lambda_s \geq \nu} A_{n_s-m+\nu}(f)$$

and, by (10),

$$\begin{aligned} \sum_{s \geq p+1: \lambda_s \geq \nu} A_{n_s-m+\nu}(f) &\leq \sum_{s \geq p+1} A_{n_s-m}(f) \leq 2A_{n_{p+1}-m}(f) \\ &\leq 2A_{n_p+\nu-m}(f) \leq 2A_{n-m+\nu}(f). \end{aligned}$$

Consequently,

$$\begin{aligned} \Sigma_2 &\leq \sum_{\nu=0}^{\lfloor n+m \rfloor} \frac{1}{m+\nu} \cdot 3A_{n-m+\nu}(f) \\ &\leq 3 \left(\sum_{\nu=0}^{\lfloor n \rfloor} + \sum_{\nu=\lfloor n \rfloor+1}^{\lfloor 2n \rfloor} \right) \frac{A_{n-m+\nu}(f)}{m+\nu} \\ &\leq 6 \sum_{\nu=0}^{\lfloor n \rfloor} \frac{A_{n-m+\nu}(f)}{m+\nu}. \end{aligned}$$

Thus

$$\|V_{n,m}\| \leq K_1 \sum_{\nu=0}^{\lfloor n \rfloor} \frac{A_{n-m+\nu}(f)}{m+\nu} + K_2 \sum_{\nu=0}^{\lfloor n \rfloor} \frac{A_{n-m+\nu}(f)}{m+\nu},$$

and the proof of (7) is complete.

Remark. In case $A_0(f) < \infty$, Theorem also holds when $0 < m = n < \infty$.

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