DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 4 1995

Christoph Schmoeger

SEMI-FREDHOLM OPERATORS AND LOCAL SPECTRAL THEORY

Dedicated to Professor Janina Wolska-Bochenek

In this paper we treat some problems which arose in [8]. We investigate a semi-Fredholm operator T acting on a complex Banach space X which satisfies $\bigcap_{n\geq 1} T^n(X) = \{0\}$ or $\overline{\bigcup_{n\geq 1} N(T^n)} = X$.

Throughout this paper let X denote an infinite-dimensional complex Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators on X. For $T \in \mathcal{L}(X)$ set $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} T(X)$, where $N(T)$ is the kernel and $T(X)$ the range of T. Define the *generalized kernel* $\mathcal{K}(T)$ and the *generalized range* $\mathcal{R}(T)$ to be the subspaces

$$\mathcal{K}(T) = \bigcup_{n\geq 1} N(T^n), \qquad \mathcal{R}(T) = \bigcap_{n\geq 1} T^n(X).$$

Write

$$\Phi_+(X) = \{T \in \mathcal{L}(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\},$$
$$\Phi_-(X) = \{T \in \mathcal{L}(X) : \beta(T) < \infty\}.$$

Observe that $T(X)$ is closed if $T \in \Phi_-(X)$ [3, Satz 55.4]. $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$ is the set of *semi-Fredholm operators* on X, while $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is the set of *Fredholm operators* in $\mathcal{L}(X)$. If $T \in \Phi_\pm(X)$, $\operatorname{ind}(T) = \alpha(T) - \beta(T)$, a finite or infinite integer is called the *index* of T.

Let $T \in \mathcal{L}(X)$ be arbitrary. The sequence $N(T), N(T^2), N(T^3), \ldots$ is increasing, while $T(X), T^2(X), T^3(X), \ldots$ is a decreasing sequence of subspaces. Define $p(T)$, the *ascent* of T, to be the smallest integer $n \geq 0$ such

AMS Classification: 47A11, 47A53.
Key words and phrases: semi-Fredholm operators, local spectrum.

that $N(T^n) = N(T^{n+1})$ or ∞ if no such n exists. Define $q(T)$, the *descent* of T, to be the smallest integer $m \geq 0$ with $T^m(X) = T^{m+1}(X)$ or ∞ if no such m exists.

PROPOSITION 1. *Let* $T \in \Phi_\pm(X)$. *Then there is* $m \in \mathbb{N}$ *such that*

(a) $N(T) \cap T^m(X) = N(T) \cap T^{m+k}(X)$ *for* $k = 0, 1, 2, \ldots$,
(b) $N(T^m) + T(X) = N(T^{m+k}) + T(X)$ *for* $k = 0, 1, 2, \ldots$.

Proof. (a) is contained in the proof of [3, Hilfssatz 72.7].

(b) Let m be the integer in (a). We prove by induction that (b) holds. If $k = 0$ we are done. Now suppose that $N(T^m) + T(X) = N(T^{m+k}) + T(X)$ for some $k \geq 0$. Let $x \in N(T^{m+k+1}) + T(X)$, thus $x = Ty + z$, $y \in X$, $z \in N(T^{m+k+1})$. This gives $T^{m+k}z \in N(T) \cap T^{m+k}(X) = N(T) \cap T^{m+k+1}(X)$. Hence $T^{m+k}z = T^{m+k+1}u$ for some $u \in X$. We derive $z - Tu \in N(T^{m+k})$, therefore $x = Ty + z = (z - Tu) + T(u + y) \in N(T^{m+k}) + T(X) = N(T^m) + T(X)$. ■

PROPOSITION 2. *If* T *is a semi-Fredholm operator on* X, *then*

(a) $T^n \in \Phi_\pm(X)$ *for each* $n \in \mathbb{N}$ *and* $\mathcal{R}(T)$ *is closed.*
(b) $T(\mathcal{R}(T)) = \mathcal{R}(T)$.

Proof. [3, Aufgabe 82.5, Hilfssatz 72.7]. ■

Let us review some classical concepts of local spectral theory. These concepts are due to N. Dunford [1].

An operator $T \in \mathcal{L}(X)$ is said to have the *single valued extension property* (SVEP) in $\lambda_0 \in \mathbb{C}$ if for any analytic function $f : D \to X$, D a neigbourhood of λ_0, with $(\lambda I - T)f(\lambda) \equiv 0$ on D, we have $f \equiv 0$. T is said to have the SVEP in $\mathbb{C}$, if T has the SVEP in each $\lambda_0 \in \mathbb{C}$. Let $T \in \mathcal{L}(X)$ be arbitrary and fix $x \in X$. The *local resolvent set* $\delta_T(x)$ is definied by

$$\begin{aligned}\delta_T(x) = \{\lambda \in \mathbb{C} :\ & \text{There is a neighbourhood U of } \lambda \text{ and} \\ & \text{an analytic function } f : U \to X \text{ such that} \\ & (\mu I - T)f(\mu) = x \text{ for each } \mu \in U\}.\end{aligned}$$

The *local spectrum* $\gamma_T(x)$ is given by $\gamma_T(x) = \mathbb{C} \setminus \delta_T(x)$. It is immediately seen that $\delta_T(x)$ is open, $\gamma_T(x)$ is closed, $\rho(T) \subseteq \delta_T(x)$ and $\gamma_T(x) \subseteq \sigma(T)$, where $\rho(T)$ denotes the resolvent set and $\sigma(T)$ denotes the spectrum of T. Observe that $\gamma_T(0) = \emptyset$. It follows from [1] that if T has the SVEP in $\mathbb{C}$, then

$$\gamma_T(x) \neq \emptyset \text{ for each } x \in X \setminus \{0\}$$

and

$$\sigma(T) = \bigcup_{x \in X} \gamma_T(x).$$

In [4] M. Mbekhta introduced the following concepts:
For $T \in \mathcal{L}(X)$ define

$$K(T) = \{x \in X : \text{ There exists } c > 0 \text{ and a sequence } (x_n)_{n\geq 1} \text{ in } X \text{ such that } Tx_1 = x,\ Tx_{n+1} = x_n \text{ and } \|x_n\| \leq c^n\|x\| \text{ for all } n \in \mathbb{N}\},$$

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Clearly we have $K(T) \subseteq \mathcal{R}(T)$ and $\mathcal{K}(T) \subseteq H_0(T)$. It follows from [4] that for $\lambda_0 \in \mathbb{C}$

$$K(\lambda_0 I - T) = \{x \in X : \lambda_0 \in \delta_T(x)\},$$
$$H_0(\lambda_0 I - T) \subseteq \{x \in X : \gamma_T(x) \subseteq \{\lambda_0\}\},$$
$$(\lambda_0 I - T)(K(\lambda_0 I - T)) = K(\lambda_0 I - T)$$

and

$$(\lambda_0 I - T)(H_0(\lambda_0 I - T)) \subseteq H_0(\lambda_0 I - T).$$

Furthermore, if T has the SEVP in $\mathbb{C}$, we have

$$H_0(\lambda_0 I - T) = \{x \in X : \gamma_T(x) \subseteq \{\lambda_0\}\}$$

and

$$x \in H_0(\lambda_0 I - T) \setminus \{0\} \Leftrightarrow \gamma_T(x) = \{\lambda_0\}.$$

PROPOSITION 3. *If $T \in \Phi_\pm(X)$ then $K(T) = \mathcal{R}(T)$.*

Proof. By Proposition 2, $\mathcal{R}(T)$ is closed and $T(\mathcal{R}(T)) = \mathcal{R}(T)$. From [7, Proposition 2] we derive $\mathcal{R}(T) \subseteq K(T)$. Since $K(T) \subseteq \mathcal{R}(T)$, we get $\mathcal{R}(T) = K(T)$. ∎

PROPOSITION 4. *Suppose that $T \in \Phi_\pm(X)$. The following assertions are equivalent*:

(a) *T has the SVEP in* 0.
(b) $p(T) < \infty$.
(c) $\mathcal{K}(T) \cap \mathcal{R}(T) = \{0\}$.
(d) $N(T) \cap \mathcal{R}(T) = \{0\}$.

If one — and thus each — of the above assertions is valid, we have

$$T \in \Phi_+(X), \quad \alpha(T) \leq \beta(T), \quad \mathcal{K}(T) = N(T^{p(T)})$$

and

$$p(T - \lambda I) = \alpha(T - \lambda I) = 0 \text{ in a deleted neighbourhood of } 0.$$

Proof. The equivalence of (a) and (b) follows from [2, Theorem 15]. If (a) (and thus (b)) holds, we have $\alpha(T) \leq \beta(T)$ by [2, Corollary 11] and therefore $T \in \Phi_+$. The equivalence of (b) and (c) follows now from [8, Proposition 2.6]. For (b)⇔(d) use [3, Satz 72.8]. To complete the proof, use again [8, Proposition 2.6]. ∎

Notation. X^* denotes the dual space of X and T^* the adjoint operator of $T \in \mathcal{L}(X)$.

PROPOSITION 5. *Suppose that $T \in \Phi_{\pm}(X)$. The following assertions are equivalent:*

(a) *T^* has the SVEP in* 0.

(b) $q(T) < \infty$.

(c) $\mathcal{K}(T) + \mathcal{R}(T) = X$.

(d) $\mathcal{K}(T) + T(X) = X$.

If one of these assertions is valid, we have

$$T \in \Phi_-(X), \quad \beta(T) \leq \alpha(T), \quad \mathcal{R}(T) = T^{q(T)}(X)$$

and

$$q(T - \lambda I) = \beta(T - \lambda I) = 0 \text{ in a deleted neighbourhood of } 0.$$

Proof. [2, Corollary 16] shows that (a) and (b) are equivalent. If (a) (and thus (b)) holds, [2, Corollary 12] gives $\beta(T) \leq \alpha(T)$, hence $T \in \Phi_-(X)$. Now use [8, Proposition 2.7] to derive the equivalence of (b) and (c). That (c) implies (d) is clear. Suppose that (d) holds. By Proposition 1, there is $m \in \mathbb{N}$ such that

$$N(T^m) + T(X) = N(T^{m+k}) + T(X) \text{ for } k = 0, 1, 2, \ldots$$

Since $\mathcal{K}(T) + T(X) = X$ this gives $X = N(T^m) + T(X)$. Let $y \in T^m(X)$, thus $y = T^m x$ for some $x \in X$. x has a decomposition $x = u + v$ with $T^m u = 0$ and $v \in T(X)$. Hence $y = T^m v \in T^{m+1}(X)$. Hence we have proved that $T^m(X) = T^{m+1}(X)$, therefore $q(T) \leq m < \infty$. Thus (d) implies (b). To complete the proof, use [8, Proposion 2.7]. ∎

Some more concepts are useful at this point. Let $T \in \mathcal{L}(X)$. The set

$$\Sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_{\pm}(X)\}$$

is called the *semi-Fredholm region* of T. Write

$$\Phi(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi(X)\}$$

for the *Fredholm region* of T and

$$\rho_W(T) = \{\lambda \in \Phi(T) : \operatorname{ind}(\lambda I - T) = 0\}.$$

The *Weyl spectrum* $\sigma_W(T)$ of T is defined by $\sigma_W(T) = \mathbb{C} \setminus \rho_W(T)$.

It is well known that $\Sigma(T)$, $\Phi(T)$, $\rho_W(T)$ are open and $\sigma_W(T) \subseteq \sigma(T)$.

THEOREM 1. *Let $T \in \mathcal{L}(X)$ be given and suppose that $\mathcal{R}(T) = \{0\}$. Then*

(a) $K(T) = \{0\}$.
(b) $N(\lambda I - T) = \{0\}$ *for all* $\lambda \neq 0$.
(c) T *has the SVEP in* $\mathbb{C}$.
(d) *For each* $x \neq 0$: $0 \in \gamma_T(x)$ *and* $\gamma_T(x)$ *is connected.*
(e) $H_0(\lambda I - T) = \{0\}$ *for each* $\lambda \neq 0$.
(f) $\sigma(T) = \sigma_W(T)$ *is connected.*
(g) $q(T - \lambda I) = \infty$ *for all* $\lambda \in \sigma(T) \setminus \{0\}$.
(h) T *is nilpotent if and only if* $q(T) < \infty$.

Proof. (a) is clear since $K(T) \subseteq \mathcal{R}(T)$.

(b) For $\lambda \neq 0$ we have $N(\lambda I - T) \subseteq \mathcal{R}(T)$.

(c) follows from (b).

(d) If $0 \in \delta_T(x)$ then $x \in K(T) = \{0\}$, thus $0 \in \gamma_T(x)$ for each $x \neq 0$. If $x \neq 0$, put $F = \gamma_T(x)$. Assume that $F = F_1 \cup F_2$ with F_1, F_2 closed, $F_1, F_2 \neq \emptyset$, $F_1 \cap F_2 = \emptyset$. The local Riesz decomposition theorem [6, Theorem 2.3] shows that

$$x = x_1 + x_2 \text{ with } \gamma_T(x_i) \subseteq F_i \quad (i = 1, 2).$$

We have $x_1 \neq 0$. Indeed, suppose that $x_1 = 0$, thus $x = x_2$. This gives $F = \gamma_T(x) = \gamma_T(x_2) \subseteq F_2$, hence $F_1 = \emptyset$, a contradiction. Similarly $x_2 \neq 0$. It follows that

$$0 \in \gamma_T(x_1) \cap \gamma_T(x_2) \subseteq F_1 \cap F_2 = \emptyset.$$

This contradiction shows that $\gamma_T(x)$ is connected.

(e) Let $\lambda \neq 0$ and let $x \in H_0(\lambda I - T)$. Assume that $x \neq 0$. Since T has the SVEP in $\mathbb{C}$, this yields $\gamma_T(x) = \{\lambda\}$. By (d) we derive $0 \in \{\lambda\}$, a contradiction.

(f) We first show that $\sigma(T)$ is connected. By (d), $0 \in \gamma_T(x)$ and $\gamma_T(x)$ is connected for each $x \neq 0$. Thus $\bigcap_{x \neq 0} \gamma_T(x) \neq \emptyset$, hence $\sigma(T) = \bigcup_{x \neq 0} \gamma_T(x)$ is connected. Next we show that $0 \in \sigma_W(T)$. To this end assume that $0 \in \rho_W(T)$. Denote by Ω the connected component of $\Phi(T)$ which contains 0. Apply [3, Satz 104.1] to obtain $\operatorname{ind}(\lambda I - T) = 0$ for all $\lambda \in \Omega$. (b) shows that $p(\lambda I - T) = 0$ for $\lambda \in \Omega \setminus \{0\}$. By [3, Satz 104.6] we get $\Omega \setminus \{0\} \subseteq \rho(T)$, hence 0 is an isolated point of $\sigma(T)$. This yields $\sigma(T) = \{0\}$, since $\sigma(T)$ is connected. But then we have $\mathbb{C} \setminus \{0\} \subseteq \rho(T) \subseteq \Phi(T)$ and $0 \in \rho_W(T) \subseteq \Phi(T)$, consequently $\mathbb{C} = \Phi(T)$. From [3, Satz 104.9] we get $\dim X < \infty$, a contradiction, hence $0 \in \sigma_W(T)$.

It remains to show that $\rho_W(T) \subseteq \rho(T)$. Let $\lambda \in \rho_W(T)$, therefore $\lambda \neq 0$. By (b), we conclude

$$0 = \operatorname{ind}(\lambda I - T) = \alpha(\lambda I - T) - \beta(\lambda I - T) = -\beta(\lambda I - T),$$

thus $\beta(\lambda I - T) = \alpha(\lambda I - T) = 0$, therefore $0 \in \rho(T)$.

(g) If $\mu \neq 0$ and $q(\mu I - T) < \infty$, then, making use of [3, Satz 72.3] and (b), we see that

$$0 = p(\mu I - T) = q(\mu I - T),$$

consequently $\mu \in \rho(T)$.

(h) If $q = q(T) < \infty$, then $\{0\} = \mathcal{R}(T) = T^q(X)$, thus $T^q = 0$. Conversely, if T is nilpotent, then it is clear that T has finite descent. ∎

Now we are in a position to state the main result of this paper.

THEOREM 2. *Let $T \in \Phi_{\pm}(X)$ and let Ω denote the connected component of $\Sigma(T)$ which contains* 0. *Then*

$$\mathcal{R}(T) = \{0\} \text{ if and only if } p = p(T) < \infty \text{ and } \Omega \subseteq \bigcap_{x \notin N(T^p)} \gamma_T(x).$$

In this case T has the properties (a) to (g) of Theorem 1, *and the following assertions are valid:*

(i) $q(\lambda I - T) = \infty$ *and* $\beta(T - \lambda I) > 0$ *for all* $\lambda \in \sigma(T)$.
(ii) $T \in \Phi_+(X)$ *and* $\operatorname{ind}(\lambda I - T) < 0$ *for all* $\lambda \in \Sigma(T) \cap \sigma(T)$.
(iii) T^* *does not have the SVEP in* $\mathbb{C}$.

Proof. Suppose that $\mathcal{R}(T) = \{0\}$. Proposition 4 shows that $p = p(T) < \infty$ and $\mathcal{K}(T) = N(T^p)$. [5, Theorem 4.2] gives

$$N(T^p) = \mathcal{K}(T) = \mathcal{K}(T) + \mathcal{R}(T) = \mathcal{K}(\lambda I - T) + \mathcal{R}(\lambda I - T)$$

for all $\lambda \in \Omega$. We get by Theorem 1(b) and Proposition 3

$$N(T^p) = \mathcal{R}(\lambda I - T) = K(\lambda I - T) = \{x \in X : \lambda \in \delta_T(x)\}$$

for each $\lambda \in \Omega \setminus \{0\}$. Thus, if $x \notin N(T^p)$, we have $\Omega \setminus \{0\} \subseteq \gamma_T(x)$. Since $0 \in \gamma_T(x)$ for each $x \neq 0$ (Theorem 1(d)), we therefore derive

$$(\star) \qquad \Omega \subseteq \bigcap_{x \notin N(T^p)} \gamma_T(x).$$

Conversely, suppose that $p = p(T) < \infty$ and that $(\star)$ holds. Thus $0 \in \gamma_T(x)$ for each $x \notin N(T^p)$. Let us assume that there exists $x \in \mathcal{R}(T)$ with $x \neq 0$. Since $p(T) < \infty$, Proposition 4 gives $x \notin N(T^p)$, thus $0 \in \gamma_T(x)$. But this is a contradictions, since

$$x \in \mathcal{R}(T) = K(T) = \{x \in X : 0 \in \delta_T(x)\}.$$

Hence $\mathcal{R}(T) = \{0\}$.

It remains to show that (i) to (iii) are valid. To prove (i), assume that $q(T) < \infty$. Theorem 1(h) gives $\sigma(T) = \{0\}$. Use [3, Satz 72.5] and Proposition 4 to conclude that $0 \in \Phi(T)$. Hence we have $\Phi(T) = \mathbb{C}$, a contradiction since $\dim X = \infty$, thus $q(T) = \infty$. Use Theorem 1(g) to complete the proof of (i).

(ii) $T \in \Phi_+(X)$ follows from Proposition 4. Since $N(\lambda I - T) = 0$ for $\lambda \neq 0$ and $\alpha(T) \leq \beta(T)$ (Proposition 4), we get $\mathrm{ind}(\lambda I - T) \leq 0$ for all $\lambda \in \Sigma(T) \cap \sigma(T)$. Assume that $\mathrm{ind}(\xi I - T) = 0$ for some $\xi \in \Sigma(T) \cap \sigma(T)$. This gives $\xi \in \rho_W(T) = \rho(T)$ (Theorem 1(f)), a contradiction. So we have $\mathrm{ind}(\lambda I - T) < 0$ for each $\lambda \in \Sigma(T) \cap \sigma(T)$.

(iii) Suppose that T^* has the SVEP in $\mathbb{C}$, then by [2, Corollary 13] we get

$$\Omega \subseteq \rho_W(T) = \rho(T),$$

a contradiction, since $\Omega \subseteq \bigcap_{x \notin N(T^p)} \gamma_T(x) \subseteq \sigma(T)$. ∎

If $T \in \mathcal{L}(X)$ it is well known that

$$T \in \Phi_+(X) \iff T^* \in \Phi_-(X),$$
$$T \in \Phi_-(X) \iff T^* \in \Phi_+(X)$$

and

$$T \in \Phi_\pm(X) \Rightarrow \alpha(T) = \beta(T^*), \quad \beta(T) = \alpha(T^*), \quad \mathrm{ind}(T) = -\,\mathrm{ind}(T^*)$$

(see [3, § 82]). If $T^n(X)$ is closed for each integer n, it is easy to check that

$${}^\perp\mathcal{R}(T^*) = \overline{\mathcal{K}(T)} \text{ and } {}^\perp\mathcal{K}(T^*) = \mathcal{R}(T),$$

where ${}^\perp\mathcal{R}(T^*)$ (resp. ${}^\perp\mathcal{K}(T^*)$) denotes the pre-annihilator of $\mathcal{R}(T^*)$ (resp. $\mathcal{K}(T^*)$) in X.

These results and the results of J.K. Finch [2] concerning semi-Fredholm operators allow us to deduce the dual statement to Theorem 2 omitting its proof.

THEOREM 3. *Let $T \in \Phi_\pm(X)$ and let Ω denote the connected component of $\Sigma(T)$ which contains 0. Then*

$$\overline{\mathcal{K}(T)} = X \text{ if and only if } q = q(T) < \infty \text{ and } \Omega \subseteq \bigcap_{x^* \notin N(T^{*q})} \gamma_{T^*}(x^*).$$

In this case T has the following properties:

(a) $N(T^* - \lambda I^*) = \{0\}$ *for all* $\lambda \neq 0$.
(b) T^* *has the SVEP in* $\mathbb{C}$.

(c) *T does not have the SVEP in $\mathbb{C}$.*
(d) *For each $x^* \in X^* \setminus \{0\}$: $0 \in \gamma_{T^*}(x^*)$ and $\gamma_{T^*}(x^*)$ is connected.*
(e) *$\sigma(T) = \sigma_W(T)$ is connected.*
(f) *$T \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - T) > 0$ for all $\lambda \in \Sigma(T) \cap \sigma(T)$.*

References

[1] N. Dunford *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. 64 (1958), 217–274.

[2] J. K. Finch, *The single valued extension property on a Banach space*, Pac. J. Math. 58 (1975), 61–69.

[3] H. Heuser, *Funktionalanalysis*, 3. ed. Teubner, 1991.

[4] M. Mbekhta, *Sur la théorie spectrale locale et limite des nilpotents*, Proc. Amer. Math. Soc. 110 (1990), 621–631.

[5] M. Ó Searcóid and T. T. West, *Continuity of the generalized kernel and range of semi-Fredholm operators*, Math. Proc. Cambridge Philos. Soc. 105 (1989), 513–522.

[6] M. Radjabalipour, *Decomposable operators*, Bull. Iranian Math. Soc. 9 (1978), 1–49.

[7] Ch. Schmoeger, *On isolated points of the spectrum of a bounded linear operator*, Proc. Amer. Math. Soc. 117 (1993), 715–719.

[8] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. R. Ir. Acad. 87A (1987), 137–146.

MATHEMATISCHES INSTITUT I
UNIVERSITÄT KARLSRUHE
Postfach 6980
D-76128 KARLSRUHE, GERMANY

Received July 5, 1994.