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Jörg Witte

INITIAL VALUE CONDITIONS FOR LINEAR PARTIAL DIFFERENTIAL OPERATORS IN $\mathbb{C}^n$ GIVEN ON n COMPLEX HYPERPLANES

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

In this paper differential operators of the form

$$(1) \qquad \partial_{z_1}^{m_1} \ldots \partial_{z_n}^{m_n} + P(\partial_{z_1}, \ldots, \partial_{z_n}),$$

are investigated, where $P(\partial_{z_1}, \ldots, \partial_{z_n})$ is a polynomial in $\partial_{z_1}, \ldots, \partial_{z_n}$ having holomorphic coefficients with degree lower than $m_1 + \ldots + m_n$ and the exponent of the power of the differential operator ∂_{z_j} occurring in $P(\partial_{z_1}, \ldots, \partial_{z_n})$ is at most m_j. On the complex hyperplanes $\{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i = w_i\}$ holomorphic functions are given. We look for a holomorphic solution F of of the partial differential equation

$$(2) \qquad \partial_{z_1}^{m_1} \ldots \partial_{z_n}^{m_n} + P(\partial_{z_1}, \ldots, \partial_{z_n}) f = u,$$

where u is holomorphic. To the solution the initial conditions

$$\partial_{z_j}^k F(z_1, \ldots, z_n) \,|_{z_j = w_j} = \varphi_j^k(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$$

are given. In a simply connected domain, existence and uniqueness will be proved, if these functions satisfy some compatibility conditions. In this case we will call these functions the Goursat conditions at the point $w = (w_1, \ldots, w_n)$ of the solution F. Existence and uniqueness seems to be reason enough to study Goursat problems. But also in many other contexts, Goursat problems play an important role. For example Vekua described in [1] general representations of solutions of elliptic differential equations in the plane by solving Goursat problems in $\mathbb{C}^2$. The author proved that the construction of fundamental solution of certain elliptic differential operators in the plane can be reduced to solve initial value problems of ordinary differential equations, which can be regarded as special Goursat problems of differential equations described above in $\mathbb{C}$. This fundamental solutions occur in representations of the solutions by boundary integrals, which are a very useful tool to study boundary value problems. See [2], [3], [4]. Also a generalized Pompiu formula in $\mathbb{C}^n$ related to the differential operator (1) can be derived by solving the Goursat problem, see [5]. In studying formal hyperbolic differential equations, the Vekua–Riemann function plays a very important role (see [1]). The author has generalized the Vekua–Riemann function for the differential operator (1) in [6]. For this generalization, existence and uniqueness of the solution of the Goursat problem is very important. By the generalized Vekua–Riemann function a represantation of holomorphic functions by integrals over a Cartesian product of curves can be obtained. As a result one obtains a method of solving Goursat problems, if the Vekua–Riemann function of the considered differential operator is known. See [6]. Also how to solve an inhomogeneous differential equation in $\mathbb{C}^2$ with homogeneous Goursat conditions is shown by the author in [2]. Among others this result will be generalized. We will solve the differential equation (2) with homogeneus Goursat conditions at a point w.

First special differential operators are considered, namely operators of the form $P_1(\partial_{z_1})\dots P_n(\partial_{z_n})$, where the $P_j(\partial_{z_j})$ are polynomials with holomorphic coefficients, depending only on z_j. We will write the differential operator (1) in the form

$$P_1(\partial_{z_1})\dots P_n(\partial_{z_n}) + P(\partial_{z_1},\dots,\partial_{z_n}).$$

The reason is, that this form can be reduced to the first case by an integro–differential operator. By well known methods of functional analysis the homogeneous Goursat problem can be solved. In the last section we will consider the inhomogeneous equation

$$P_1(\partial_{z_1})\dots P_n(\partial_{z_n}) + P(\partial_{z_1},\dots,\partial_{z_n})f = g$$

with arbitrary holomorphic right side. It is shown, that a solution of the inhomogeneous equation with homogeneous Goursat conditions can be obtained by solving a special, recursively defined Goursat problem of the homogeneous equation.

2. A special Goursat problem

We will solve Goursat problems for the differential operators of the form

$$P_1(\partial_{z_1})P_2(\partial_{z_2})\dots P_n(\partial_{z_n}).$$

The differential operator $P_j(\partial_{z_j})$ is defined by

$$P_j(\partial_{z_j}) = \sum_{k=0}^{m_j} A_{k,j}(z_j)\partial_{z_j}^k.$$

That means, we seek for a holomorphic solution F of the equation

$$P_1(\partial_{z_1})P_2(\partial_{z_2})\dots P_n(\partial_{z_n})f(z_1,\dots,z_n)=0$$

with the initial conditions

$$\partial^k_{z_j}F(z_1,\dots,z_n)\,|_{z_j=w_j}=\varphi^k_j(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)$$

for $j=1,\dots,n$ and $k=0,\dots,m_j$, where $m_j=\deg(P_j(\partial_{z_j}))$. The functions φ^k_j are holomorphic functions, defined in a cylindrical domain, which should fulfil some compatibility conditions. Because, if a solution F of the Goursat problem exists, one has

$$\partial^l_{z_i}\partial^k_{z_j}F(z_1,\dots,z_n)\,|_{z_j=w_j,z_i=w_i}=\partial^l_{z_i}\varphi^k_j\,|_{z_i=w_i}=\partial^k_{z_j}\varphi^l_i\,|_{z_j=w_j},$$

the functions φ^k_j should satisfy the relation

$$\partial^l_{z_i}\varphi^k_j\,|_{z_i=w_i}=\partial^k_{z_j}\varphi^l_i\,|_{z_j=w_j}\;.$$

One easily shows, that every solution of the equation

$$P_1(\partial_{z_1})P_2(\partial_{z_2})\dots P_n(\partial_{z_n})f(z_1,\dots,z_n)=0$$

is a sum of solutions of the equations $P_j(\partial_{z_j})f=0$. Every solution of $P_j(\partial_{z_j})f=0$ can be written in the form

$$\sum_{k=0}^{m_j-1}A^k_jf^k_j,$$

where for $k=0,\dots,m_j-1$ the functions f^k_j are a fundamental system of the ordinary differential equation $P_j(\frac{d}{dz_j})f=0$ and the functions A^k_j are holomorphic functions, depending only on $n-1$ variables with the property $\partial_{z_j}A^k_j=0$.

In the following, we will denote by f^k_j a solution of the ordinary differential equation $P_j(\partial_{z_j})f=0$ with the initial conditions

$$f^k_j(w_j)=\dots=\partial^{m_j-2}_{z_j}f^k_j(w_j)=0$$

and

$$\partial^{m_j-1}_{z_j}f^k_j(w_j)=1.$$

The functions f^k_j can be regarded as a fundamental system of the differential operator $P_1(\partial_{z_1})\dots P_n(\partial_{z_n})$ because every solution of the homogeneous equation has the form

$$\sum_{i=1}^{n}\sum_{k=0}^{m_j-1}A^k_jf^k_j.$$

First we will prove the uniqueness of the solution of Goursat's problem. For this sake, we assume, that the Goursat conditions at a point w are homogeneous. That means the functions φ_j^k are all zero and we will prove, that the function, which is identically zero, is the only solution. This can be proved by induction. If the considered differential operator has only one factor, that means it has the form $P(\frac{d}{dz})$, then according the theory of ordinary differential equations only the function $f \equiv 0$ is a solution with homogeneous initial conditions.

Now let $F(z_1, \ldots, z_n)$ be a solution of the differential equation

$$P_1(\partial_{z_1})P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})f(z_1,\ldots,z_n) = 0$$

with homogeneous Goursat conditions at a point w. Since

$$F = \sum_{i=1}^{n} \sum_{k=0}^{m_j-1} A_j^k f_j^k$$

for $l = 0, \ldots, m_1 - 1$ we have

$$0 = \partial_{z_1}^l F \,|_{z_1=w_1} = \sum_{i=1}^{n} \sum_{k=0}^{m_j-1} \partial_{z_1}^l (A_j^k f_j^k) \,|_{z_1=w_1} =$$

$$A_1^l + \sum_{i=2}^{n} \sum_{k=0}^{m_j-1} \partial_{z_1}^l (A_j^k) \,|_{z_1=w_1} = f_j^k,$$

because the functions A_1^l for $l = 0, \ldots, m_1 - 1$ and the functions f_j^k for $j > 1$ don't depend on z_1. So we get

$$A_1^l = -\sum_{i=2}^{n} \sum_{k=0}^{m_j-1} \partial_{z_1}^l (A_j^k) \,|_{z_1=w_1} = f_j^k.$$

Because the right hand side of the equation is a solution of the differential equation

$$P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})f = 0$$

also for $l = 0, \ldots, m_1 - 1$ the functions A_1^l are solutions of the differential equation

$$P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})f = 0.$$

Therefore the solution F of the homogeneous Goursat problem is also a solution of this differential equation, having only $n-1$ factors. By assumption for each z_1 the solution F is identically zero.

Now we will prove the existence of the solution of Goursat problem. This proof gives also a method of computing the solution recursively. The proof will be done by induction with respect to the number n of the factors of the differential operator $P_1(\partial_{z_1})P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})$. If $n = 1$, the Goursat problem is the problem of finding a solution of a ordinary differential equation with initial conditions. So we assume $n > 1$. Let

$$\varphi_j^k(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)$$

be the Goursat conditions at a point w of the solution. First we will look for a solution $\widetilde{F}$ of the equation

$$P_1(\partial_{z_1})P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})f(z_1,\ldots,z_n) = 0$$

with the property

$$\partial_{z_1}^k F(z_1,\ldots,z_n)\mid_{z_1=w_1} = \varphi_1^k(z_2,\ldots,z_n)$$

for $0 \leq k \leq m_1 - 1$. Of course, the solution $\widetilde{F}$ is not uniquely determined, because we have not yet specified the Goursat conditions to

$$\partial_{z_j}^k F(z_1,\ldots,z_n)\mid_{z_j=w_j} \qquad \text{for } j > 1.$$

For example a solution of this uncompleted initial value problem is given by

$$\widetilde{F}(z_1,\ldots,z_n) = \sum_{k=0}^{m_1-1} \varphi_1^k(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)f_1^k(z_1),$$

where for $0 \leq k \leq m_1 - 1$ the functions $f_1^k(z_1)$ are a fundamental system of the ordinary differential operator $P_1(\partial_{z_1})$ with the property

$$f_1^k(w_1) = \ldots = \partial_{z_1}^{m_1-2} f_1^k(w_1) = 0$$

and

$$\partial_{z_1}^{m_1-1} f_1^k(w_1) = 1.$$

Now let be

$$\psi_i^j := \partial_{z_i} j\widetilde{F}\mid_{z_i=w_i}$$

for $i > 1$ and $0 \leq j \leq m_i - 1$. Then we have

$$\partial_{z_1}^j \psi_i^k \mid_{z_1=w_1} = \partial_{z_i}^k \varphi_1^j \mid_{z_i=w_i} .$$

By assumption of induction, we can find the uniquely determined solution G of the differential equation

$$P_2(\partial_{z_2})\ldots P_n(\partial_{z_n})f = 0$$

with the Goursat conditions

$$\partial_{z_j}^k G\mid_{z_j=w_j} = \varphi_j^k - \psi_j^k.$$

The function $F = \widetilde{F} + G$ is the solution of the Goursat problem.

3. Another way of solving a Goursat problem

Here we will learn, how to solve the Goursat problem in another way. Let us remember, that a solution of the differential equation $P_1(\partial_{z_1})\dots P_n(\partial_{z_n})$ has the form

$$\sum_{j=1}^{n}\sum_{i=0}^{m_j-1} A_j^i f_j^i \tag{3}$$

where for each $j\in\{1,\dots,n\}$ and all $i=0,\dots m_j-1$ the functions f_j^i are a fundamental system of the ordinary differential equation $P_j(\partial_{z_j})f=0$. In the following, we assume, that these fundamental systems have the property $\partial_{z_j}^k f_j^i|_{z_j=w_j}=\delta_{k,i}$, where the delta is the Kronecker delta. Of course the functions A_j^i don't depend on z_j. So we have

$$\varphi_i^l = A_j^l + \sum_{i=1,\,i\neq j}^{n}\sum_{k=0}^{m_i-1}\partial_{z_j}^l A_i^k|_{z_i=w_i}$$

or

$$A_j^l = \varphi_i^l - \sum_{i=1,\,i\neq j}^{n}\sum_{k=0}^{m_i-1}\partial_{z_j}^l A_i^k|_{z_i=w_i}. \tag{4}$$

THEOREM 3.1. *Choose functions* $\alpha_{r,j}^{l,s}$, *which do not depend on* z_r *and* z_j *and which fulfil the relation*

$$\partial_{z_r}^s\varphi_j^l = \alpha_{r,j}^{s,l} + \alpha_{j,r}^{l,s} + \sum_{k=1,\,k\neq j,\,k\neq r}^{n}\sum_{i=0}^{m_k-1}\partial_{z_r}^s\alpha_{j,k}^{l,i} f_k^i|_{z_r=w_r,z_j=w_j}.$$

If we set

$$A_j^l = \varphi_j^l - \sum_{k=1,\,k\neq j}^{n}\sum_{i=0}^{m_k-1}\alpha_{j,k}^{l,i} f_k^i,$$

then the function defined by (3) *is a solution of the Goursat problem.*

Proof. By this choice of the functions A_j^l we have $\partial_{z_r}^s A_j^l|_{z_r=w_r}=\alpha_{r,j}^{s,l}$, because

$$\partial_{z_r}^s A_j^l|_{z_r=w_r} = \partial_{z_r}^s\varphi_j^l - \sum_{k=1,\,k\neq j}^{n}\sum_{i=0}^{m_k-1}\partial_{z_r}^s(\alpha_{j,k}^{l,i} f_k^i) =$$

$$= \partial_{z_r}^s\varphi_j^l - \alpha_{j,r}^{l,s} - \sum_{k=1,\,k\neq j,\,k\neq r}^{n}\sum_{i=0}^{m_k-1}\partial_{z_r}^s(\alpha_{j,k}^{l,i} f_k^i) = \alpha_{r,j}^{s,l}.$$

Therefore in (4) the expression $\partial_{z_j}^l A_i^k|_{z_i=w_i}$ can be substituted by $\alpha_{j,i}^{l,k}$. So the Goursat conditions are fulfilled.

Generally it does not seem to be very easy to find functions $\alpha_{j,i}^{l,k}$, but if one solves Goursat problems in $\mathbb{C}^2$, the problem of finding such functions is reduced to find constants satisfying a very easy relation.

4. A special inhomogeneous Goursat problem

THEOREM 4.1. *Let $P(\frac{d}{dz})$ be a linear ordinary differential operator with holomorphic coefficients, which are defined in a simply connected domain G containing the origin. We assume, that the leading coefficient is equal to one. Let $f : G \times G \to \mathbb{C}$ be a holomorphic solution of the differential equation $P(\partial_z)f = 0$ with the initial conditions $\partial_z^k f(w,w) = \delta_{n-1,k}$, where $\delta_{n-1,k}$ is the Kronecker delta. If $u : G \longrightarrow \mathbb{C}$ is a holomorphic function then*

$$U =: \int_0^z f(z,\zeta)u(\zeta)\,d\zeta$$

is a special solution of the differential equation $P(\partial_z)U = u$ with homogeneous initial conditions at $z = 0$.

Proof. In consideration of the initial conditions of f, we have

$$\partial_z U = f(z,z)u(z) + \int_0^z \partial_z f(z,\zeta)u(\zeta)d\zeta = \int_0^z \partial_z f(z,\zeta)u(\zeta)\,d\zeta.$$

By complete induction, one gets

$$\partial_z^k U = \partial_z^{k-1} f(z,z)u(z) + \int_0^z \partial_z^k f(z,\zeta)u(\zeta)d\zeta = \int_0^z \partial_z^k f(z,\zeta)u(\zeta)\,d\zeta,$$

if $k < n$, where n is the degree of the operator $P(\partial_z)$, and

$$\partial_z^n U = \partial_z^{n-1} f(z,z)u(z) + \int_0^z \partial_z^n f(z,\zeta)u(\zeta)\,d\zeta = u(z) + \int_0^z \partial_z^n f(z,\zeta)u(\zeta)\,d\zeta.$$

So, we get the result

$$P(\partial_z)U(z) = u(z) + \int_0^z P(\partial_z)f(z,\zeta)u(\zeta)\,d\zeta = u(z).$$

THEOREM 4.2. *Let $G_i \subset \mathbb{C}$ be simply connected domains with $0 \in G_i$ and $P_j(\partial_{z_j})$ differential operators with holomorphic coefficients, defined on G_i. As above we assume, that the leading coefficient is equal one. The product $P_1(\partial_{z_1}) \ldots P_n(\partial_{z_n})$ is defined on the cylindrical domain $G_1 \times \ldots \times G_n$.*

Let be $f_i : G_i \times G_i \longrightarrow \mathbb{C}$ a holomorphic solution of the differential equation $P_j(\partial_{z_j})f_j = 0$ with the initial conditions $\partial_{z_j}^k f_j(w,w) = \delta_{n-1,k}$. If the function $u : G_1 \times \ldots \times G_n \longrightarrow \mathbb{C}$ is holomorphic, then

$$U(z_1,\ldots,z_n) := \int_0^{z_1} \ldots \int_0^{z_n} f_1(z_1,\zeta_1)\ldots f_n(z_n,\zeta_n)u(\zeta_1,\ldots,\zeta_n)\, d\zeta_1 \ldots d\zeta_n$$

is a special solution of the differential equation $P_1(\partial_{z_1})\ldots P_n(\partial_{z_n})f = u$ with homogeneous Goursat conditions.

Proof. This theorem is an easy conclusion of Theorem 4.1.

5. The general homogeneous Goursat problem

Now we will solve the general Goursat problem for the differential equation

$$P_1(\partial_{z_1})\ldots P_n(\partial_{z_n}) + P(\partial_{z_1},\ldots,\partial_{z_n})f = 0.$$

First we will solve it locally. Afterwards we will think about analytic continuation. Locally the existence and uniqueness of a solution of Goursats problem will be proved by an integro–differential equation and Banach's fixed point theorem. For this purpose a Banach space is needed. For $i = 1,\ldots,n$ let be $K_i \subset \mathbb{C}$ compact circles. On $K_1 \times K_2 \times \ldots \times K_n$ we consider all holomorphic functions f for which all derivatives $\partial_{z_j}^k$ for $1 \le j \le n$ and for $0 \le k \le m_j$ are defined on $K_1 \times K_2 \times \ldots \times K_n$. This means, that the derivatives also exist on the boundary. With the norm

$$\|f\| := \sup\left\{\partial_{z_1}^{k_1}\ldots\partial_{z_n}^{k_n} f(z) \mid z \in K_1\times\ldots\times K_n,\ 0 \le k_i \le m_i,\ \sum_{i=1}^n k_i < \sum_{i=1}^n m_i\right\}$$

this function space becomes a Banach space. We will denote this Banach space by $\mathcal{B}$ and apply Banach's fixed point theorem in it.

LEMMA 5.1. *Let $f_i(z_i,\zeta_i)$ be a holomorphic solution of the differential equation $P_j(\partial_{z_j})f = 0$ with the initial conditions*

$$f_i(\zeta_i,\zeta_i) = f_i'(\zeta_i,\zeta_i) = \ldots = f_i^{m_i-2}(\zeta_i,\zeta_i) = 0 \text{ and } f_i^{m_i-1}(\zeta_i,\zeta_i) = 1.$$

We assume, that the functions $f_i(z_i,\zeta_i)$ are defined in a neighbourhood of circles K_i. On $K_1 \times \ldots \times K_n$ we define the operator

$$H(g)(z_1,\ldots,z_n) :=$$

$$\int_{w_1}^{z_1} \ldots \int_{w_n}^{z_n} f_1(z_1,\zeta_1)\ldots f_n(z_n,\zeta_n)P(\partial_{z_1},\ldots,\partial_{z_n})g\, d\zeta_1 \ldots d\zeta_n.$$

The operator $H : \mathcal{B} \longrightarrow \mathcal{B}$ is a linear and bounded operator. If the circles K_i are small enough, then the norm of the operator H is smaller than one.

Proof. One easily computes, that $H(g) \in \mathcal{B}$. Obviously $H : \mathcal{B} \to \mathcal{B}$ is linear. If one applies differential operators $\partial_{z_1}^{k_1} \dots \partial_{z_n}^{k_n}$ for $0 \le k_i \le m_i$, $\sum_{i=1}^n k_i < \sum_{i=1}^n m_i$ to $H(g)$ one sees, that there exists a constant M with $\|H(g)\| \le M\|g\|$. The constant M depends on the diameters of the circles K_i. So if the diameters of the circles K_i are small enough, the constant M is smaller than one.

THEOREM 5.2. *To the Goursat conditions φ_i^j at the point $w = (w_1, \dots \dots, w_n)$, locally in a neighbourhood $K_1 \times \dots \times K_n$ of the point w there exists in the Banach space $\mathcal{B}$ a uniquely determined solution of the differential equation*

$$P_1(\partial_{z_1}) \dots P_n(\partial_{z_n}) + P(\partial_{z_1}, \dots, \partial_{z_n})f = 0.$$

Proof. The differential equation can be written in the form

$$P_1(\partial_{z_1}) \dots P_n(\partial_{z_n})\{f + H(f)\} = 0.$$

Let G be a solution of the differential equation $P_1(\partial_{z_1}) \dots P_n(\partial_{z_n})f = 0$ with the Goursat conditions φ_i^j, then F, which is a solution of the integro–differential equation

$$f + H(f) = G$$

is a solution of the differential equation

$$P_1(\partial_{z_1}) \dots P_n(\partial_{z_n}) + P(\partial_{z_1}, \dots, \partial_{z_n})f = 0.$$

Because $\partial_{z_j}^k H(f) \mid_{z_j = w_j} = 0$, if $0 \le k \le m_j - 1$, F fulfils the desired Goursat conditions. According the Banach fixed point theorem, the solution F exists and is uniquely determined in the Banach space $\mathcal{B}$.

THEOREM 5.3. *If the domain D, on which the differential operator is defined, is simply connected, then there exists a unique determined solution of the Goursat problem.*

Proof. Because such a solution exists locally, it can be analytically continued along any path. Since the domain is simply connected, the continuation is unique.

6. The general inhomogeneous Goursat problem

Now we will solve a Goursat problem for the inhomogeneous equation

$$P_1(\partial_{z_1}) \dots P_n(\partial_{z_n}) + P(\partial_{z_1}, \dots, \partial_{z_n})f = g.$$

Because we can solve Goursat problems for the homogeneous equation, it is enough to find a solution of the inhomogeneous equation with homogeneous Goursat conditions. In this context the solution of a special Goursat problem of the homogeneous equation is very important. The Goursat conditions of this special Goursat problem are defined recursively at a point

$w = (w_1, \dots w_n)$. It should be remarked, that the solution of the Goursat problem depends holomorphicaly on w. Now let us define the Goursat conditions. If the differential operator is an ordinary differential operator, then we look for a solution F with the initial conditions

$$F(w) = F'(w) = \dots = F^{(n-2)}(w) = 0 \quad \text{and} \quad F^{(n-1)}(w) = 1.$$

Let be $n > 1$ and the differential operator $P_1(\partial_{z_1}) \dots P_n(\partial_{z_n}) + P(\partial_{z_1}, \dots, \partial_{z_n})$ written in the form

$$\sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} a_{i_1,\dots,i_n} \partial_{z_1}^{i_1} \dots \partial_{z_n}^{i_n}.$$

By $F(z, w)$ we denote the uniquely determined solution of the homogeneous equation with following Goursat conditions. For $0 \le j \le m_k - 2$

$$\partial_{z_k}^j F\,|_{z_k=w_k} = 0 \quad \text{and} \quad \partial_{z_k}^{m_k-1} F\,|_{z_k=w_k} = \varphi_k,$$

where φ_k is a solution of the differential equation

$$\sum_{i_1=0}^{m_1} \dots \sum_{i_{k-1}=0}^{m_{k-1}} \sum_{i_{k+1}=0}^{m_{k+1}} \dots \sum_{i_n=0}^{m_n} a_{i_1,\dots i_{k-1}, i_{m_k}, i_{k+1} \dots i_n} \partial_{z_1}^{i_1} \dots \partial_{z_{k-1}}^{i_{k-1}} \partial_{z_{k+1}}^{i_{k+1}} \dots \partial_{z_n}^{i_n} f = 0.$$

Because the differential operator

$$\sum_{i_1=0}^{m_1} \dots \sum_{i_{k-1}=0}^{m_{k-1}} \sum_{i_{k+1}=0}^{m_{k+1}} \dots \sum_{i_n=0}^{m_n} a_{i_1,\dots i_{k-1}, i_{m_k}, i_{k+1} \dots i_n} \partial_{z_1}^{i_1} \dots \partial_{z_{k-1}}^{i_{k-1}} \partial_{z_{k+1}}^{i_{k+1}} \dots \partial_{z_n}^{i_n}$$

is a differential operator of the form 1 in $\mathbb{C}^{n-1}$, the Goursat conditions of φ_k are known by recursion.

Some abbriavations are useful. Let be

$$I_{i_1,\dots,i_n} f = \int_{w_{k_1}}^{z_{k_1}} \dots \int_{w_{k_r}}^{z_{k_r}} f \, dz_{k_1} \dots dz_{k_r},$$

if $i_{k_1} = \dots = i_{k_r} = 1$ and all the other indices are zero. Furthermore if exclusively $i_{l_1} = \dots = i_{l_s} = 0$ let be

$$\varphi_{i_1,\dots,i_n} = \partial_{z_{l_1}}^{m_{l_1}-1} \dots \partial_{z_{l_s}}^{m_{l_s}-1} F(z, \zeta)\,|_{z_{l_1}=\zeta_{l_1},\dots,z_{l_s}=\zeta_{l_s}}\,. \tag{5}$$

Corresponding with the recursively defined Goursat conditions of F, the functions defined by (5) satisfies the differential equation

$$\sum a_{i_1,\dots,i_{k_1-1}, m_{i_1}, i_{k_1+1}, \dots i_{k_2-1}, m_{i_2}, i_{k_2+1}, \dots, i_{k_m-1}, m_{i_m}, i_{k_m+1}, \dots, i_n} *$$
$$\partial_{z_1}^{i_1} \dots \partial_{z_{k_1-1}}^{i_{k_1-1}} \partial_{z_{k_1+1}}^{i_{k_1+1}} \dots \partial_{z_{k_2-1}}^{i_{k_2-1}} \partial_{z_{k_2+1}}^{i_{k_2+1}} \dots \partial_{z_n}^{i_n} f = 0$$

and have Goursat conditions defined above. That means, if $i_k = 1$

$$\partial_{z_k}^{m_k-1} \varphi_{i_1,\dots,i_n}\,|_{z_k=w_k} = \varphi_{j_1,\dots,j_n},$$

where $j_l = i_l$, if $l \neq k$ and $j_k = 0$. We define functions $g_{i_1,\dots,i_n}^{k_1,\dots,k_n}$ by

$$g_{i_1,\dots,i_n}^{k_1,\dots,k_n}(z) := I_{i_1,\dots,i_n}\partial_{z_1}^{k_1}\dots\partial_{z_n}^{k_n}\varphi_{i_1,\dots,i_n}(z,\zeta)g(\zeta)|_{z_{l_1}=\zeta_{l_1},\dots,z_{l_s}=\zeta_{l_s}}$$

for $0 \leq k_i \leq m_i$, where $i_{l_1},\dots i_{l_s}$ are all those indices, which are zero.

LEMMA 6.1. *For $i = 1,\dots,n$ $k_i < m_i$ we have*

$$\partial_{z_1}^{k_1}\dots\partial_{z_n}^{k_n} g_{i_1,\dots,i_n}^{0,\dots,0} = I_{i_1,\dots,i_n}\partial_{z_1}^{k_1}\dots\partial_{z_n}^{k_n}\varphi_{i_1,\dots,i_n} = g_{i_1,\dots,i_n}^{k_1,\dots,k_n}(z)$$

and

$$\partial_{z_{j_1}}^{m_{j_1}}\dots\partial_{z_{j_m}}^{m_{j_m}} g_{i_1,\dots,j_1-1,1,j_1+1,\dots,i_n}^{k_1,\dots,k_{j_1-1},0,k_{j_1+1},\dots,k_n} = \\ + \sum_{i_{j_1},\dots,i_{j_m}=0}^{1} g_{i_1,\dots,j_1-1,j_1,j_1+1,\dots,i_n}^{k_1,\dots,k_{j_1-1},j_1(m_{j_1}-j_1),k_{j_1+1},\dots,k_n},.$$

Moreover

$$g_{0,\dots,0}^{0,\dots,0} = g.$$

Proof. By complete induction.

THEOREM 6.2. *Let $F(z_1,\dots,z_n,\zeta_1,\dots\zeta_n)$ be a solution of the homogeneous equation*

$$\sum_{i_1=0}^{m_1}\dots\sum_{i_n=0}^{m_n} a_{i_1,\dots,i_n}\partial_{z_1}^{i_1}\dots\partial_{z_n}^{i_n} f = 0$$

fulfilling the recursively defined Goursat conditions, described above. Then

$$G(z_1,\dots,z_n) = \int_{w_1}^{z_1}\dots\int_{w_n}^{z_n} F(z_1,\dots,z_n,\zeta_1,\dots\zeta_n)g((\zeta_1,\dots\zeta_n)\,d\zeta_1\dots d\zeta_n)$$

is a solution of the inhomogeneous differential equation

$$\sum_{i_1=0}^{m_1}\dots\sum_{i_n=0}^{m_n} a_{i_1,\dots,i_n}\partial_{z_1}^{i_1}\dots\partial_{z_n}^{i_n} f = g,$$

which fulfils homogeneous Goursat conditions at $z = w$.

Proof. According Lemma 6.1 the function G has homogeneous initial conditions at the point w.

If one applies the differential operator

$$\sum_{i_1,\dots,i_n=0}^{m_1,\dots,m_n} a_{i_1,\dots,i_n}\partial_{z_1}^{i_1}\dots\partial_{z_n}^{i_n}$$

to the function

$$G(z_1,\dots,z_n) = \int_{w_1}^{z_1}\dots\int_{w_n}^{z_n} F(z_1,\dots,z_n,\zeta_1,\dots\zeta_n)g((\zeta_1,\dots\zeta_n)\,d\zeta_1\dots d\zeta_n$$

one receives

$$\sum_{i_1,\ldots,i_n=0}^{m_1,\ldots,m_n} a_{i_1,\ldots,i_n} \partial_{z_1}^{i_1} \ldots \partial_{z_n}^{i_n} g_{1,\ldots,1}^{0,\ldots,0} =$$

$$\sum_{\{k_1,\ldots,k_m\}\subset\{1,\ldots,n\}} \sum_{i_1,\ldots,i_{k_1-1},i_{k_1+1},\ldots,i_n=0}^{m_1,\ldots,m_{k_1-1},m_{k_1+1},\ldots,m_n} a_{i_1,\ldots,i_{k_1-1},m_{k_1},i_{k_1+1},\ldots,i_n} g_{1,\ldots,1,0,\ldots}^{i_1,\ldots,i_{k_1-1},0,\ldots} =$$

$$= g_{0,\ldots,0}^{0,\ldots,0} = g.$$

References

[1] I. N. Vekua, *New Methods For Solving Elliptic Equations*, Amsterdam, 1967.

[2] J. Witte, *Polynomiale Differentialoperatoren in den Ableitungen nach der komplexen Variablen z und der komplex konjugierten Variablen $\bar{z}$*, Dissertation Freie Universität Berlin, Berlin, 1991.

[3] J. Witte, *Linear Partial Differential Operators with Holomorphic Coefficients*, Berlin 1992, to appear in 'Complex Variables', Delaware, USA.

[4] J. Witte, *Systems of Linear Partial Differential Equations with Holomorphic Coefficients*, Berlin 1992, to appear in 'Complex Variables', Delaware, USA.

[5] J. Witte, *A Generalized Pompiu Formula Related to Differential Euqations in $\mathbb{C}^n$ with Higher Order*, Berlin 1993, submitted.

[6] J. Witte, *Integral Represantation in $\mathbb{C}^n$ by a Generalized Riemann Function*, Berlin 1994, submitted.

FREIE UNIVERSITÄT BERLIN
FACHBEREICH MATHEMATIK
1. INSTITUTE
ARNIMALLEE 3
14195 BERLIN, GERMANY

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