DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 4 1995

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GENERALIZED $c(R)$ PROPERTY AND INTERPOLATION PROBLEMS INDUCED BY RIGHT INVERTIBLE OPERATORS

Dedicated to Professor Janina Wolska-Bochenek

Introduction

The general interpolation problem induced by a right invertible operator with initial operators possessing the property (c) was introduced and investigated by Przeworska-Rolewicz [7]. In [5], we give some new general conditions for the general classical interpolation problem to be well-posed and we present its unique solution in a closed form. A necessary and suficient condition for the general interpolation problem to have unique solution was found in [3], [4]. The above listed results are based on the so-called property $c(R)$ of a given system of initial operators [7]. However, for a right invertible operator D with $\dim\ker D \geq 2$, not all the initial operators possess the property (c) (see [2]-[7]).

In this paper, we introduce the generalized $c(R)$-property of a given system of initial operators for a right inverse R of D. Then we give the general interpolation formula for a system of initial operators possessing the generalized $c(R)$-property.

1. Some characterizations of initial operators

Let X be a linear space over a field $\mathcal{F}$ of scalars, where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$. Denote by $R(X)$ the set of all right invertible operators acting in X. For a $D \in R(X)$ we write

$$\mathcal{R}_D = \{R \in L_0(X): \ DR = I\},$$

$$\mathcal{F}_D = \{F \in L_0(X): \ FX = \ker D, \ F^2 = F \text{ and } \exists R \in \mathcal{R}_D, \ FR = 0\}.$$

In the sequel we assume that $\dim\ker D > 0$, i.e. D is not invertible.

DEFINITION 1. (i) Every operator $R \in \mathcal{R}_D$ is said to be a right inverse of D. (ii) Every operator $F \in \mathcal{F}_D$ such that $FR = 0$ for an $R \in \mathcal{R}_D$ is said to be an initial operator of D corresponding to R.

Let $R \in \mathcal{R}_D$. Then

$$\ker D^m = \{z_0 + Rz_1 + \ldots + R^{m-1}z_{m-1},\ z_k \in \ker D,\ k = 0, 1, \ldots, m-1\}.$$

Elements of the form $z_0 + Rz_1 + \ldots + R^n z_n$ are called D-polynomials of degree n if $z_n \neq 0$.

DEFINITION 2 [7]. Let $R \in \mathcal{R}_D$. An operator $F_0 \in \mathcal{F}_D$ possesses the property $c(R)$ if there exists scalars c_k such that

$$F_0 R^k z = \frac{c_k}{k!} z \quad \text{for all } z \in \ker D,\ k \in \mathbb{N}. \tag{1}$$

The set of all initial operators possessing the property $c(R)$ will be denoted by $\mathcal{F}_{D,R}$.

For $N \in \mathbb{N}^+$ we denote by

$$P_N(R) = \operatorname{lin}\{R^k z :\ z \in \ker D,\ k = 0, 1, \ldots, N-1\}. \tag{2}$$

The following result has been proved by Przeworska-Rolewicz in [7].

THEOREM 1. $\mathcal{F}_{D,R} = \mathcal{F}_D$ *if and only if* $\dim\ker D = 1$.

We now charcterize initial operators having the property $c(R)$.

LEMMA 1. *Let* $\dim\ker D < \infty$. *Then an initial operator* F_0 *for* D *has the property* $c(R)$ *for a right inverse* R *if and only if* $F_0 R^k e_j = d_k e_j$, $d_k \in \mathcal{F}$, $k \in \mathbb{N}$, $j = 1, 2, \ldots, s$, *where* $(e_1, \ldots, e_s)$ *is a basis of* $\ker D$.

The proof is an immediate consequence of Definition 2.

DEFINITION 3. Let $R \in \mathcal{R}_D$ and let $F_k \in \mathcal{F}_D$ $(k = 1, 2, \ldots, n)$. Then the system $(F_1, \ldots, F_n)$ is said to have the generalized $c(R)$-property if there are nontrivial subspaces $Z_1, \ldots, Z_s$ of $\ker D$ such that

$$\ker D = \bigoplus_{j=1}^{s} Z_j \tag{3}$$

and $F_k R^j z_\nu = c_{kj\nu} z_\nu$ for all $z_\nu \in Z_\nu$, $c_{kj\nu} \in \mathcal{F}$ $(\nu = 1, 2, \ldots, s;\ k = 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots)$.

Definition 3 and Lemma 1 together imply that every system $(F_1, \ldots, F_n)$ possessing the property $c(R)$ has the generalized $c(R)$-property.

The following example shows that there are systems of initial operators with the generalized $c(R)$-property which do not have the $c(R)$-property.

EXAMPLE. Let $X = C(\mathbb{R})$, $D = d^2/dt^2$, $R = \int_0^t \int_0^s$. A basis of $\ker D^2$ is $\{1, t\}$, i.e. $e_1(t) \equiv 1$, $e_2(t) \equiv t$. Consider operators F_k given by

$$(F_k x)(t) = x(0) + tx'(0) + \frac{1}{2}(1+t)x''(k) + \frac{1}{2}(1-t)x''(-k) \quad (k = 0, 1).$$

It is easy to check that $F_k \in \mathcal{F}_D$, $\ker D = Z_1 \oplus Z_2$, where $Z_1 = \operatorname{lin}\{e_1\}$, $Z_2 = \operatorname{lin}\{e_2\}$ and

$$F_k R^j e_1 = \frac{k^{2j-2}}{(2j-2)!} e_1, \; F_k R^j e_2 = \frac{k^{2j-1}}{(2j-1)!} e_2, \; j = 1, 2, \ldots$$

Hence, the system (F_1, F_2) possesses the generalized $c(R)$-property but does not have the $c(R)$-property.

Let $(F_1, F_2, \ldots, F_n)$ be a system possessing the generalized $c(R)$-property with respect to a system of nontrivial subspaces $(Z_1, \ldots, Z_s)$ of $\ker D$, i.e.

$$F_k R^j z_\nu = c_{kj\nu} z_\nu \quad \text{for all } z_\nu \in Z_\nu, \; c_{kj\nu} \in \mathcal{F} \tag{4}$$

$(\nu = 1, \ldots, s;\; k = 1, \ldots, n;\; j = 0, 1, \ldots)$.

Denote by

$$V_n^{(\nu)} = \det(c_{k_j - 1\nu})_{k,j=1}^n, \tag{5}$$

$$\widehat{F}_i = (F_1, F_1 R, \ldots, F_1 R^{n-1}), \; i = 1, \ldots, n, \tag{6}$$

$$c_i^{(\nu)} = (c_{i0\nu}, c_{i1\nu}, \ldots, c_{in-1\nu}), i = 1, \ldots, n. \tag{7}$$

LEMMA 2. *For every $\nu \in \{1, 2, \ldots, s\}$, the system of vectors $\widehat{F}_1, \ldots, \widehat{F}_n)$ defined by (6) is linearly independent on Z_ν if and only if every system $\{c_1^{(\nu)}, \ldots, c_n^{(\nu)}\}$, $\nu = 1, 2, \ldots, s$, where $c_i^{(\nu)}$ are defined by (7), is linearly independent.*

Proof. Let $\{\widehat{F}_1, \ldots, \widehat{F}_n\}$ be linearly independent on Z_ν, i.e. the equality

$$\sum_{i=1}^n \alpha_i \widehat{F}_i z_\nu = 0 \quad \text{for all } z_\nu \in Z_\nu, \; \alpha_i \in \mathcal{F} \; (i = 1, 2, \ldots, n)$$

implies $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. Let ν be fixed $(1 \le \nu \le s)$ and let $\sum_{i=1}^n \beta_i c_i^{(\nu)} = 0$, i.e. $\sum_{i=1}^n \beta_i c_{ij\nu} = 0$ for $j = 0, 1, \ldots, n-1$. Then

$$\sum_{i=1}^n \beta_i c_{ij\nu} z_\nu = 0 \quad \text{for all } z_\nu \in Z_\nu, \; j = 0, 1, \ldots, n-1,$$

i.e.

$$\sum_{i=1}^n \beta_i \widehat{F}_i z_\nu = 0 \quad \text{for all } z_\nu \in Z_\nu.$$

Hence $\beta_i = 0$ for $i = 1, 2, \ldots, n$.

Conversely, if $\{c_1^{(\nu)}, \dots, c_n^{(\nu)}\}$ is linearly independent for some $\nu \in \{1, 2, \dots, s\}$, then $\{\widehat{F}_1, \dots, \widehat{F}_n\}$ is linearly independent on Z_ν.

COROLLARY 1. *Let $\{F_1, \dots, F_n\}$ be a system of initial operators on D having the generalized $c(R)$-property with respect to subspaces $Z_1, \dots, Z_s$ of* $\ker D$. *Let $\nu \in \{1, 2 \dots, s\}$. Then $V_n^{(\nu)} \neq 0$ if and only if the system $\{F_1 R^k, \dots, F_n R^k\}$ is linearly independent on Z_ν for every $k \in \{0, 1 \dots, \dots n-1\}$.*

In the sequel, we assume that the system of initial operators $\{F_1, \dots, F_n\}$ possesses generalized $c(R)$-property with respect to $Z_1, \dots, Z_s \subset \ker D$.

THEOREM 2. *$V_n^{(\nu)} \neq 0$ if and only if the system $\{F_1, \dots, F_n\}$ is linearly independent on $P_{n\nu}(R)$, where $P_{n\nu}(R) = \mathrm{lin}\{z_\nu, Rz_\nu, \dots, R^{n-1} z_\nu\}$, $z_\nu \in Z_\nu$.*

Proof. By Corollary 1, $V_n^{(\nu)} \neq 0$ if and only if for every $k \in \{0, 1, \dots, \dots n-1\}$ the system $\{F_1 R^k, \dots, F_n R^k\}$ is linearly indepondent on Z_ν, i.e. the equality

$$\sum_{i=1}^{n} \alpha_i F_i R^k z_\nu = 0 \quad \text{for all } z_\nu \in Z_\nu,\ \alpha_i \in \mathcal{F}$$

implies $\alpha_i = 0$ for $i = 1, 2, \dots, n$. It means that

$$\sum_{k=0}^{n-1} \lambda_{k\nu} \sum_{i=1}^{n} \alpha_i F_i R^k z_\nu = 0 \quad \text{for all } \lambda_{k\nu} \in \mathcal{F}$$

if and only if

$$\sum_{i=1}^{n} \alpha_i F_i \Big(\sum_{k=0}^{n-1} \lambda_{k\nu} R^k z_\nu \Big) = 0,$$

i.e.

$$\sum_{i=0}^{n} \alpha_i F_i y_\nu = 0 \quad \text{for all } y_\nu \in P_{n\nu}(R).$$

2. General interpolation problem

We shall consider the following problem:

Given n finite sets I_i of non-negative integers with powers $\# I_i = r_i$, and let $r_1 + r_2 + \dots + r_n = N$.

Find a D-polynomial u of degree $N-1$ satisfying N conditions

$$F_i D^k u = u_{ik} \quad (k \in I_i,\ i = 1, 2, \dots, n), \tag{8}$$

where $u_{ik} \in \ker D$ are given.

The above problem we shall call general interpolation problem.

Since

$$u = z_1 + Rz_2 + \ldots + R^{N-1} z_N, \quad z_j \in \ker D, \ R \in \mathcal{R}_D, \tag{9}$$

we have to determine all elements z_j.

DEFINITION 4. The general interpolation problem is said to be well-posed if it has a unique solution for every $u_{ik} \in \ker D$. If either there exists u_{ik} such that the problem has no solutions or the corresponding problem with $u_{ik} = 0$ ($k \in I_i$, $i = 1, \ldots, n$) has at least one nontrivial solution then this problem is said to be ill-posed. By the assumption, the system of initial operators $F_1, \ldots, F_n$ possesses the generalized $c(R)$-property with respect to subspace Z_ν, i.e.

$$F_i R^k z_{m\nu} = c_{ikm\nu} z_\nu \quad \text{for all } z_{m\nu} \in Z_\nu \tag{10}$$

($i = 1, \ldots, n$; $k \in I_i$, $m = 1, \ldots, N$; $\nu = 1, \ldots, s$). We assume that elements of sets I_i are ordered r_i-tuples $(k_1, \ldots, k_{r_i})$ with $0 \leq k_1 \leq k_2 \ldots \leq k_{r_i}$.

Then (8) is of the form

$$F_i D^{k_j} u = u_{ik_j} \in Z_\nu. \tag{11}$$

Rewrite (9) in the form

$$u = \sum_{\nu=1}^{s} \sum_{m=1}^{N} R^{m-1} z_{m\nu}, \quad z_{m\nu} \in Z_\nu. \tag{12}$$

Then

$$\begin{aligned} F_i D^{k_j} u &= \sum_{\nu=1}^{s} \sum_{m=1}^{N} F_i D^{k_j} R^{m-1} z_{m\nu} \\ &= \sum_{\nu=1}^{s} \sum_{m=k_j-1}^{N} c_{i(m-k-1)m\nu} z_\nu. \end{aligned}$$

These equalities and (11) together imply

$$\sum_{m=k_j-1}^{N} c_{i(m-k_j-1)m\nu} z_{m\nu} = u_{ik_j\nu}. \tag{13}$$

Write: $u_{ik_j\nu} = u_{l\nu}$ for $l = r_0 + r_1 + \ldots + r_i + k_j - 1$, $r_0 = 0$. Then (13) is of the form

$$\sum_{m=1}^{N} d_{lm}^{(\nu)} z_{m\nu} = u_{l\nu},$$

where $d_{im}^{(\nu)}$ are determined by coefficients $c_{i(m-k-1)m\nu}$ from (13).

LEMMA 3. *Let* $G^{(\nu)} = (d_{lm}^{(\nu)})_{l,m=1}^{N}$, *where* $\nu \in \{1,2,\dots,s\}$. *Then the system of vectors*

$$\widehat{F}_i^{(k_j)} = (F_iD^{k_j}, F_iD^{k_j-1},\dots,F_i,F_iR,\dots,F_iR^{N-1-k_j}),$$
$$i=1,\dots,n;\ j=1,\dots r_i$$

is linearly independent on Z_ν *if and only if* $\operatorname{Rank} G^{(\nu)} = N$.

Proof. Suppose that $\operatorname{Rank} G^{(\nu)} = N$, i.e. the system $d_{lm}^{(\nu)}$ is linearly independent and

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\widehat{F}_i^{(k_j)} z_{m\nu} \quad \text{for all } z_{m\nu} \in Z_\nu,\ a_{ij} \in \mathcal{F}.$$

Hence, we have for a fixed m $(1 \le m \le N)$

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}R^{m-1}z_{m\nu} = 0 \quad \text{for all } z_{m\nu} \in Z_\nu.$$

By (10), this implies

$$\begin{aligned}\sum_{m=1}^{N} b_m \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}R^{m-1}z_{m\nu} &= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i \sum_{m=1}^{N} b_mD^{k_j}R^{m-1}z_{m\nu}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_i \sum_{m=k_j-1}^{N} b_mR^{m-k_j-1}z_{m\nu}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij} \sum_{m=k_j-1}^{N} b_mF_iR^{m-k_j-1}z_{m\nu}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij} \sum_{m=k_j-1}^{N} b_mc_{i(m-k_j-1)m\nu}z_{m\nu}.\end{aligned}$$

The arbitrary choice both $z_{m\nu} \in Z_\nu$ and $b_m \in \mathcal{F}$ imply that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}d_{lm}^{(\nu)} = 0,$$

i.e. $a_{ij} = 0$ for $i = 1,2,\dots,n$; $j = 1,2,\dots,r_i$.

Conversely, suppose that the system $\{\widehat{F}_i^{(k_j)}\}$ $(i = 1,2,\dots,n;\ j = 1,2,\dots \dots,r_i)$ is linearly independent on Z_ν and that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}d_{lm}^{(\nu)} = 0, \quad a_{ij} \in \mathcal{F}.$$

This means that

$$(14)\qquad \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}c_{i(m-k-1)m\nu} = 0, \quad m = 1,2,\ldots,N.$$

Since $\dim Z_\nu \neq 0$, (14) is equivalent to

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}c_{i(m-k-1)m\nu}z_{m\nu} = 0, \quad \text{for all } z_{m\nu} \in Z_\nu.$$

According to (10), these equalities can be written as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\widehat{F}_i^{(k_j)}z_{m\nu} = 0, \quad \text{for all } z_{m\nu} \in Z_\nu.$$

Now our assumption implies that $a_{ij} = 0$ ($i = 1,2,\ldots,n$; $j = 1,2,\ldots,r_i$), which was to be proved.

THEOREM 3. *Let ν be fixed ($\nu \in \{1,2,\ldots,n\}$). Then $V_n^{(\nu)} := \det G^{(\nu)} \neq 0$ if and only if the system $\{F_{ik_j}; i = 1,\ldots,n; j = 1,\ldots,r_i\}$, where*

$$\widehat{F}_i^{(k_j)} = (F_{ik_j}, F_{ik_j}R, \ldots, F_{ik_j}R^{N-1})$$

is linearly independent on $P_{N\nu}(R)$.

Proof. Suppose that $\det G^{(\nu)} \neq 0$. By Lemma 3, the vector operators $\{\widehat{F}_i^{(k_j)}\}$ ($i = 1,\ldots,n$; $j = 1,\ldots,r_i$) are linearly independent on Z_ν. Hence, for every fixed index m ($1 \leq m \leq N$) the operators $F_{ik_j}R^{m-1}$ ($i = 1,\ldots,n$; $j = 1,\ldots,r_i$) are linearly independent on Z_ν. This means that the system of operators $F_{ik_j} = F_iD^{k_j}$ ($i = 1,\ldots,n$; $j = 1,\ldots,r_i$) is linearly independent on $P_{N\nu}(R)$.

Conversely, suppose that F_{ik_j} ($i = 1,\ldots,n$; $j = 1,\ldots,r_i$) are linearly independent on $P_{N\nu}(R)$. By Lemma 3, in order to prove that $\det G^{(\nu)} \neq 0$ it is enough to show that the system $\{\widehat{F}_i^{(k_j)}\}$ is linearly independent on Z_ν. Let

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}\widehat{F}_i^{(k_j)}z_{m\nu} = 0, \quad \text{for all } z_{m\nu} \in Z_\nu;\ a_{ij} \in \mathcal{F},$$

i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}R^{m-1}z_{m\nu} = 0$$

for every fixed m ($1 \leq m \leq N$) and for all $z_{m\nu} \in Z_\nu$. Hence

$$\sum_{m=1}^{N} b_m \sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}R^{m-1}z_{m\nu} = 0,$$

so

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}\sum_{m=1}^{N} b_mR^{m-1}z_{m\nu}=0 \quad \text{for all } z_{m\nu}\in Z_\nu,\ b_m\in\mathcal{F}.$$

This means that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} a_{ij}F_iD^{k_j}R^{m-1}x=0 \quad \text{for all } x\in P_{N\nu}(R).$$

Thus, by the assumption on the system $\{F_iD^{k_j}\}$ we get $a_{ij}=0$ $(i=1,\ldots,n;$ $j=1,\ldots,r_i)$.

Lemma 3 and Theorem 3 together imply the following

THEOREM 4. *The general interpolation problem is well-posed if and only if the system of vectors* $\{F_iD^{k_j};\ i=1,\ldots,n; j=1,\ldots,r_i\}$ *is linearly independent on every set* $P_{N\nu}(R)$ $(\nu=1,\ldots,s)$.

THEOREM 5. *If* $V_N^{(\nu)}=\det G^{(\nu)}\neq 0$ *for all* $\nu\in\{1,\ldots,s\}$, *then the unique solution of the general interpolation problem is of the form*

$$u=u^{(1)}+\ldots+u^{(s)}, \tag{15}$$

where

$$u^{(\nu)}=\sum_{j=1}^{N} V_{Nj}^{(\nu)}(R)u_{j\nu},$$

$$V_{Nj}^{(\nu)}(R)=\frac{1}{V_N^{(\nu)}}\sum_{k=1}^{N}(-1)^{k+j}V_{Njk}^{(\nu)}R^{k-1}$$

and $V_{Njk}^{(\nu)}$ *is the minor determinant obtained by cancelling in* $V_N^{(\nu)}$ *the (j-1)-st row and* $(k-1)$*-st column.*

P r o o f. Every solution of the general interpolation problem is of the form

$$u=\sum_{\nu=1}^{s}\sum_{m=1}^{N} R^{m-1}z_{m\nu},$$

where $z_{1\nu},\ldots,z_{N\nu}$ $(\nu=1,2,\ldots,s)$ are to be determined by the system (13). By the assumption, the determinant $V_N^{(\nu)}$ of the system (13) is different from zero. Thus, by the Cramer formulae, the unique solution of (13) is of the form (15).

As applications of Theorems 3-5, we shall give classical interpolation problems for right invertible operators

(a) *Hermite interpolation problem.* If $I_i = \{0, 1, \ldots, r_i - 1\}$, then we have the following interpolation problem: Find a D-polynomial u of degree $N-1$ which for given n ($n \leq N$) different initial operators $F_1, \ldots, F_n$ admits given values together with $D^k u$ up to order $r_j - 1$, i.e. find a solution of the Hermite interpolation problem:

$$F_i D^j u = u_{ij}, \quad i = 1, \ldots, n; \; j = 1, \ldots, r_j - 1,$$

where $r_1 + r_2 + \ldots + r_n = N$; $u_{ij} \in \ker D$ are given, $u = z_1 + Rz_2 + \ldots + R^{N-1} z_N$ for $R \in \mathcal{R}_D$ and $z_1, \ldots, z_N$ are to be determined.

THEOREM 6. *Suppose that $D \in R(X)$ and the system of initial operators $\{F_1, \ldots, F_n\}$ possesses the generalized $c(R)$-property with respect to subspaces $Z_1, \ldots, Z_s$ of* $\ker D$. *Then the Hermite interpolation problem has a unique solution if and only if the system of operators $\{F_i D^j\}$ ($i = 1, \ldots, n$; $j = 1, \ldots, r_i - 1$) is linearly independent on every $P_{N\nu}(R)$. If this condition is satisfied, then the unique solution is determined by*

$$u = \sum_{\nu=1}^{s} \sum_{j=1}^{N-1} V_{Nj}^{(\nu)}(R) u_{j\nu}.$$

Here

$$V_{Nj}^{(\nu)}(R) = \frac{1}{V_N^{(\nu)}} \sum_{k=0}^{N-1} (-1)^{k+j} V_{Njk}^{(\nu)} R^k$$

and $V_{Njk}^{(\nu)}$ is the minor determinant obtained by cancelling in $V_N^{(\nu)}$ the j-th row and the k-th column and elements $u_{0\nu}, \ldots, u_{N-1\nu}$ are defined by the equalities

$$\begin{aligned} u_{q\nu} &= u_{1q\nu} \quad \text{for } q = 0, 1, \ldots, r_1 - 1, \\ u_{(r_1+q)\nu} &= u_{2q\nu} \quad \text{for } q = 0, 1, \ldots, r_2 - 1, \\ &\ldots \\ u_{(r_1+\ldots+r_{n-1}+q)\nu} &= u_{nq\nu} \quad \text{for } q = 0, 1, \ldots, r_n - 1. \end{aligned}$$

(b) *Lagrange interpolation problem.* If $I_i = \{0\}$ for $i = 1, \ldots, n$, then we obtain the Lagrange interpolation problem: Find a D-polynomial of degree $n-1$ which for given different initial operators $F_1, \ldots, F_n$ admits given values

$$F_i u = u_i, \quad u_i \in \ker D \text{ are given, } u = z_1 + Rz_2 + \ldots + R^{n-1} z_n$$

for $R \in \mathcal{R}_D$ and $z_1, z_2 \ldots, z_n$ are to be determined.

THEOREM 7. *Suppose that $D \in R(X)$ and the system $\{F_1, \ldots F_n\}$ possesses the generalized $c(R)$-property with respect to subspaces $Z_1, Z_2, \ldots, Z_s$ of* $\ker D$. *Then the Lagrange interpolation problem has a unique solution if* *and only if the system $\{F_1, \ldots, F_n\}$ is linearly independent on $P_n(R)$. If this condition is satisfied, then the unique solution is of the form*

$$u = \sum_{\nu=1}^{s} \sum_{j=1}^{n-1} V_{nj}^{(\nu)}(R) u_{j\nu},$$

where

$$V_{nj}^{(\nu)}(R) = \frac{1}{V_N^{(\nu)}} \sum_{k=0}^{n-1} (-1)^{k+j} V_{njk}^{(\nu)} R^k$$

and $V_{njk}^{(\nu)}$ is the minor determinant obtained by cancelling in $V_n^{(\nu)}$ the j-th row and k-th column.

Acknowledgement. The authors wish to thank Professor Przeworska-Rolewicz for her helpful comments.

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Received June 6, 1994.