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ON DIFFERENTIAL AND INTEGRAL EQUATIONS IN LOCALLY CONVEX SPACES

Dedicated to Professor Janina Wolska-Bochenek

In this paper we investigate continuous solutions of some integral equations in sequentially complete locally convex spaces containing a compact barrel. Moreover, we prove a Kneser-type theorem for the Darboux problem in this class of spaces.

1. Introduction

Consider the following initial value problem

$$x' = f(t,x), \quad x(0) = x_0, \tag{1}$$

where f is a bounded continuous function with values in a quasi-complete locally convex space E. Millionščikov [9] and Hukuhara [4] proved that the problem (1) has a solution if the function f is compact or it satisfies the Kamke condition. Later many authors have studied the existence of solutions of (1) under diffrent assumptions on E or f (e.g. see [1], [8], [10]). Moreover, in recent years there have appeared papers concerning integral equations in locally convex spaces (e.g. see [5], [11]).

In this paper we shall assume that the considered locally convex Hausdorff space over $\mathbb{R}$ is sequentially complete and contains a compact barrel.

We shall prove a Kneser-type theorem for continuous solution set of the nonlinear Volterra integral equation

$$x(t) = g(t) + \int_{A(t)} f(t,s,x(s))\,ds, \quad t \in A \tag{2}$$

and an existence theorem for continuous solutions of the Urysohn integral equation

$$(3) \qquad x(t) = g(t) + \lambda \int_A f(t,s,x(s))\,ds, \quad t \in A,\ \lambda \in \mathbb{R},$$

considered in the space E, where $A = [0,a_1]\times[0,a_2]\times\ldots\times[0,a_n]$, ($a_i > 0$, $i = 1,\ldots,n$) and $A(t) = \{s \in \mathbb{R}^n : 0 \le s_i \le t_i,\ i = 1,\ldots,n\}$. In the above equations the sign "$\int$" stands for the Riemann integral.

Moreover, in Section 3 we obtain a Kneser-type theorem for the Darboux problem as a corollary from the corresponding result for the equation (2).

2. Volterra integral equation

A useful tool in our considerations will be the following

LEMMA 1 (Astala [1]). *E is sequentially complete locally convex space containing a compact barrel iff*

$$E = (X', \tau),$$

where X' is the dual of a barrelled normed space X and τ is a locally convex topology of X' that is stronger than the w^*-topology but weaker than the topology of precompact convergence; briefly

$$\sigma(X',X) \le \tau \le \lambda(X',X).$$

By the above lemma we can use in the space E the notion of the norm.

Consider the equation (2). We assume that the functions $g : A \to E$ and $f : A^2 \times E \to E$ are continuous. Now we shall prove the following Kneser-type theorem which is the main result of our paper.

THEOREM 1. *Under the above assumptions there exists a set*

$$J = [0,d_1] \times [0,d_2] \times \ldots \times [0,d_n] \subset A$$

such that the set S of all continuous solutions of (2), defined on J, is nonempty, compact and connected in the space $C(J,E)$ of all continuous functions $J \to E$ with the topology of uniform convergence.

The above result extends Th. 3.1 from [2].

Proof. Let r be any positive number. Since the ball $B_r = \{x \in E : \|x\| \le r\}$ is convex, ballanced, closed, bounded and sequentially complete, so in view of the Banach–Mackey theorem ([6], p.91) it is absorbing by the barrel and therefore it is compact. Hence for every number $r > 0$ there exists a number $m_r > 0$ such that

$$\|f(t,s,x)\| \le m_r \quad \text{for } t,s \in A \text{ and } x \in B_r.$$

Let $c = \sup_{t\in A} \|g(t)\|$ and $\rho = \sup_{r>0}(r-c)/(m_r)$. Choose a number $e < \rho$. Then there exists $b > 0$ such that $c + m_b e < b$. Choose numbers d_i, $1 \le i \le n$

in such a way that $0 < d_i \le a_i$ for $i = 1, \ldots, n$ and $d_1 d_2 \ldots d_n < e$. Let $J = [0, d_1] \times [0, d_2] \times \ldots \times [0, d_n]$. In the space $\mathbb{R}^n$ we introduce the norm defined by the formula

$$\|t\| = \max\left(|t_1|, \frac{d_1}{d_2}|t_2|, \ldots, \frac{d_1}{d_n}|t_n|\right) \quad \text{for } t = (t_1, t_2, \ldots, t_n).$$

Then $J = \{t \in \mathbb{R}^n : t \ge 0,\ \|t\| \le d_1\}$. Denote by $\widetilde{B}$ the set of all continuous functions $J \to B_b$. We shall consider $\widetilde{B}$ as a subspace of $C(J, E)$. Put

$$G(x)(t) = g(t) + \int_{A(t)} f(t, s, x(s))\, ds \quad t \in J,\ x \in \widetilde{B}.$$

Since

$$G(x)(t) - G(x)(\tau) = g(t) - g(\tau) + \int_{A(t)} f(t, s, x(s))\, ds - \int_{A(\tau)} f(\tau, s, x(s))\, ds =$$

$$= g(t) - g(\tau) + \int_{A(t)} (f(t, s, x(s)) - f(\tau, s, x(s)))\, ds +$$

$$+ \int_{A(t)\setminus A(\tau)} f(\tau, s, x(s))\, ds - \int_{A(\tau)\setminus A(t)} f(\tau, s, x(s))\, ds,$$

$$\int_{A(t)\dot{-}A(\tau)} \|f(\tau, s, x(s))\|\, ds \le \mu(A(t)\dot{-}A(\tau)) m_b \le m_b d_2 d_3 \ldots d_n \|t - \tau\|$$

and

$$\|G(x)(t)\| \le \|g(t)\| + \int_{A(t)} \|f(t, s, x(s))\|\, ds \le c + \mu(A(t)) m_b \le$$

$$\le c + m_b d_2 d_3 \ldots d_n \|t\|$$

for $x \in \widetilde{B}$, $t, \tau \in J$ (μ denotes here the Lebesgue measure in $\mathbb{R}^n$), so $G(\widetilde{B}) \subset \widetilde{B}$ and the family $G(\widetilde{B})$ is equiuniformly continuous. Moreover, in view of the Krasnoselskii–Krein-type lemma (cf. [7]) we deduce that G is continuous.

For any $\varepsilon > 0$ denote by S_ε the set of all $x \in \widetilde{B}$ such that $\|x(t) - G(x)(t)\| < \varepsilon$ for every $t \in J$. Before passing to further considerations, we shall quote the following

LEMMA 2 ([3]). *For every $\varepsilon > 0$ such that $\varepsilon < b - c - m_b d_1 d_2 \ldots d_n$, the set S_ε is nonempty and connected.*

Now we return to the proof of Th. 1. First, we shall show that the set S is nonempty. By Lemma 2, there exists a sequence (u_k) such that $u_k \in \widetilde{B}$ for $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \sup_{t\in J} \|u_k(t) - G(u_k)(t)\| = 0. \tag{4}$$

Let $V = \{u_k : k\in\mathbb{N}\}$. Since $u_k = (u_k - G(u_k)) + G(u_k)$ for $k \in \mathbb{N}$, so V is equiuniformly continuous. As E contains a compact barrel, so by the Banach–Mackey theorem every bounded subset of E is relatively compact. Hence $V(t) = \{u_k(t) : k \in \mathbb{N}\}$ is relatively compact for every $t \in J$. In view of Ascoli's theorem ([6], pp. 80–81) we conclude that the sequence (u_k) has a limit point u. From (4) and the continuity of G it is clear that $u = G(u)$, so $u \in S$.

Further, since G is continuous, so S is closed in $C(J, E)$. As $S = G(S)$, therefore using similar arguments as above, we deduce that S is compact in $C(J, E)$.

To prove that S is connected it is enough to apply standard arguments as for example in [3]. The proof is completed.

3. Darboux problem

Let $B = \{z \in E : \|z\| \le b\}$, $A = [0, a_1] \times [0, a_2]$ $(a_1, a_2 > 0)$ and let $f : A \times B \to E$ be a continuous function. By the Banach–Mackey theorem there exists a number $M > 0$ such that $\|f(x, y, z)\| \le M$ for $(x, y, z) \in A\times B$. Choose positive numbers d_1, d_2 such that $d_1 \le a_1$, $d_2 \le a_2$ and $Md_1d_2 < b$. Put $J = [0, d_1] \times [0, d_2]$. Let us consider the following problem

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z), \quad (x, y) \in J, \tag{5}$$
$$z(x, 0) = 0, \quad 0 \le x \le d_1, \quad z(0, y) = 0, \quad 0 \le y \le d_2,$$

where $\frac{\partial^2 z}{\partial x \partial y}$ denotes the mixed second derivative of z.

We shall prove the following

THEOREM 2. *Under the above assumptions the set S of all solutions of* (5), *defined on J, is nonempty, compact and connected in $C(J, E)$.*

Proof. As E is sequentially complete and f is continuous, so the problem (5) is equivalent to the following integral equation

$$z(x, y) = \int_0^x\int_0^y f(\xi, \eta, z(\xi, \eta))\, d\xi\, d\eta, \quad (x, y) \in J, \tag{6}$$

where the sign "$\int\int$" stands for the Riemann integral.

In the space $\mathbb{R}^2$ we introduce the norm defined by the formula

$$\|t\| = \max\left(|t_1|, \frac{d_1}{d_2}|t_2|\right) \quad \text{for } t = (t_1, t_2).$$

Then $J = \{t \in \mathbb{R}^2 : t \geq 0,\ \|t\| \leq d_1\}$. Putting $A(t) = \{s \in \mathbb{R}^2 : 0 \leq s \leq t\}$ for $t \in J$, we can write the equation (6) in the form

$$z(t) = \int_{D(t)} f(s, z(s))\, ds, \tag{7}$$

Applying now Th. 1 we see that the set S is nonempty, compact and connected in $C(J, E)$, because it coincides with the set of all continuous solutions of (7) on J.

4. Urysohn integral equation

In this section we shall consider the equation (3). As in the Section 2 assume that the functions $g : A \to E$ and $f : A^2 \to E$ are continuous. Now we shall prove the following

THEOREM 3. *Under the above assumptions there exists $\eta > 0$ such that for $\lambda \in \mathbb{R}$ with $|\lambda| < \eta$, the equation* (3) *has a continuous solution defined on A.*

Proof. Similarly as in Section 2 we deduce that for any number $r > 0$ there exists a number $m_r > 0$ such that

$$\|f(t, s, x)\| \leq m_r \quad \text{for } t, s \in A \text{ and } \|x\| \leq r.$$

Let $c = \sup_{t \in A} \|g(t)\|$ and $\eta = \sup_{r > 0}(r - c)/(am_r)$. Choose $\lambda \in \mathbb{R}$ with $|\lambda| < \eta$. Hence there exists a number $b > 0$ such that $c + |\lambda| am_b < b$. Denote by $\widetilde{B}$ the set of all continuous functions $A \to B_b$. We shall consider $\widetilde{B}$ as a topological subspace $C(A, E)$. Define

$$G(x)(t) = g(t) + \lambda \int_A f(t, s, x(s))\, ds, \quad t \in A,\ x \in \widetilde{B}.$$

It can be easily verified that G is a continuous mapping of $\widetilde{B}$ into itself. Again, by the Krasnoselskii–Krein-type lemma the family $G(\widetilde{B})$ is equiuniformly continuous. Let $V = \overline{\text{conv}}\, G(\widetilde{B})$. Then V is also equiuniformly continuous and $V(t) = \{x(t) : x \in V\}$ is relatively compact for every $t \in A$. By Ascoli's theorem we deduce that V is compact. From the Schauder–Tychonoff theorem it follows that the mapping $G|_V$ has a fixed point. Obviously this completes the proof of Th. 3.

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