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*Dedicated to Professor Janina Wolska-Bochenek*

## 1. Introduction

In this note we obtain a sufficient condition for univalence of an analytic function in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and so we generalize the results obtained in papers [2] and [7]. This condition involves two arbitrary functions $g(z) = 1+b_1z+\dots$ and $h(z) = 1+c_1z+\dots$ analytic in $U$. Replacing $g$ and $h$ by some particular functions, we find the well-known conditions for univalency established by Nehari [5], by Becker [1], by Lewandowski [3] and by Lewandowski and Stankiewicz [4]. Likewise we find other new sufficient conditions.

THEOREM A (cf. [5]). *Let $f(z) = z + a_2z^2 + \dots$ be an analytic function in $U$. If for all $z \in U$*

$$(1)\qquad |\{f;z\}| \le \frac{2}{(1-|z|^2)^2}, \quad \textit{where } \{f;z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

*then the function $f$ is univalent in $U$.*

THEOREM B (cf. [1]). *Let $f(z) = z + a_2z^2 + \dots$ be an analytic function in $U$. If for all $z \in U$*

$$(2)\qquad (1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1,$$

*then the function $f$ is univalent in $U$.*

THEOREM C (cf. [3]). *Let $f(z) = z + a_2z^2 + \dots$ be an analytic function in $U$. If there exists an analytic function $p$ with positive real part in $U$ with*

$p(0)=1$, *such that the inequality*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^2-(1-|z|^2)\left[\frac{zp'(z)}{p(z)+1}+\frac{zf''(z)}{f'(z)}\right]\right|\le 1 \tag{3}$$

*holds true for all* $z\in U$, *then the function* $f$ *is univalent in* $U$.

THEOREM D (cf. [4]). *If the function* $f$ *is analytic and locally univalent in* $U$ *and there exists an analytic function* $p$, $\operatorname{Re} p(z)>0$, *in* $U$ *such that*

$$\left|\frac{z\{f;z\}}{4\bar{z}}(1-|z|^2)^2(1+p(z))+\frac{zp'(z)}{p(z)+1}(1-|z|^2)+|z|^2\frac{1-p(z)}{1+p(z)}\right|\le 1 \tag{3'}$$

*for all* $z\in U$, *then* $f$ *is univalent in* $U$.

THEOREM E (cf. [7]). *Let* $f(z)=z+a_2z^2+\ldots$. *Let* $g(z)=1+b_1z+\ldots$ *and* $h(z)=1+c_1z+\ldots$ *be analytic functions in* $U$ *and* $f'(z)g(z)h(z)\neq 0$ *for all* $z\in U$. *If*

$$\left|\frac{g(z)}{h(z)}-1\right|<1 \quad \text{for all } z\in U, \tag{4}$$

$$\begin{aligned}\Bigg|\left(\frac{g(z)}{h(z)}-1\right)|z|^4&+z(1-|z|^2)|z|^2\left[\frac{f''(z)}{f'(z)}+3\frac{h'(z)}{h(z)}-\frac{g'(z)}{g(z)}\right]\\ &+z^2(1-|z|^2)^2\left[\frac{f''(z)h'(z)}{f'(z)g(z)}+2\frac{(h'(z))^2}{g(z)h(z)}-\frac{h''(z)}{g(z)}\right]\Bigg|\le |z|^2\end{aligned} \tag{5}$$

*for all* $z\in U\setminus\{0\}$, *then the function* $f$ *is univalent in* $U$.

## 2. Preliminaries

Let $A$ denote the class of functions $f$ that are analytic in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ with $f(0)=0$ and $f'(0)=1$.

We denote by $U_r$ the disk of the $z$-plane $U_r=\{z\in\mathbb{C}:|z|<r\}$, $U_1=U$ and $U^*=U\setminus\{0\}$.

DEFINITION. A function $L(z,t)$, $z\in U$, $t\in I=[0,\infty)$ is called a Loewner chain if

$$L(z,t)=e^tz+a_2(t)z^2+\ldots,\quad |z|<1$$

is analytic and univalent in $U$ for each $t\in I$ and if for all $s,t\in I$, $0\le s\le t$, one has $L(z,s)\prec L(z,t)$, where $\prec$ denotes the relation of subordination.

THEOREM F (cf. [11]). *Let* $r$ *be a real number,* $r\in(0,1]$. *Let* $L(z,t)=a_1(t)z+a_2(t)z^2+\ldots$, $a_1(t)\neq 0$ *be analytic in* $U_r$ *for all* $t\in I$, *locally absolutely continuous in* $I$ *and locally uniform with respect to* $U_r$. *For almost all* $t\in I$ *suppose*

$$z\frac{\partial L(z,t)}{\partial z}=p(z,t)\frac{\partial L(z,t)}{\partial t}\quad \text{for all } z\in U_r,$$

*where* $p(z,t)$ *is analytic in* $U$ *such that* $\operatorname{Re} p(z,t) > 0$ *for* $z \in U$, $t \in I$.

*If* $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{L(z,t)/a_1(t)\}$ *forms a normal family in* $U_r$, *then* $L(z,t)$ *has, for each* $t \in I$, *an analytic and univalent extension to the whole disk* $U$.

## 3. Main results

THEOREM 1. *Let* $a$ *and* $c$ *be complex numbers such that* $\operatorname{Re} a > 1$, $|c| < 1$, *let* $f \in A$ *and* $g(z) = 1 + b_1 z + \ldots$, $h(z) = 1 + c_1 z + \ldots$ *be analytic functions in* $U$ *with* $f'(z)(g(z)h(z) \neq 0$ *for all* $z \in U$. *If*

$$|a-2| + 2|c| < |a|, \tag{6}$$

$$\left|\frac{2(1+c)}{a}\frac{g(z)}{h(z)} - 1\right| < 1, \tag{7}$$

$$\begin{aligned}&\Bigg|(2-a)|z|^a + 2\left[(1+c)\frac{g(z)}{h(z)} - 1\right]|z|^{2a} + 2(1-|z|^a)z \\ &\times\left|\left(\frac{f''(z)}{f'(z)} + 3\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)}\right)|z|^a\right. \\ &\left.+\frac{1-|z|^a}{1+c}z\left(\frac{f''(z)}{f'(z)}\frac{h'(z)}{g(z)} + 2\frac{(h'(z))^2}{g(z)h(z)} - \frac{h''(z)}{g(z)}\right)\right]\Bigg| \le |a||z|^{\operatorname{Re} a}\end{aligned} \tag{8}$$

*for all* $z \in U^*$, *then the function* $f$ *is univalent in* $U$.

Proof. Let us consider the function

$$h_1(z,t) = g(e^{-t}z) + \frac{1}{1+c}(e^{at}-1)e^{-t}z.h'(e^{-t}z). \tag{9}$$

It is clear that if $z \in U$, then $e^{-t}z \in U$ for all $t \in I$ and, from the analyticity of $g$ and $h'$ in $U$ it follows that $h_1(z,t)$ is also analytic in $U$ for all $t \geq 0$. Since $h_1(0,t) = g(0) = 1$ and $h_1$ is analytic in $U$, there is a disk $U_{r_1}$, $0 < r_1 \leq 1$, in which $h_1(z,t) \neq 0$ for all $t \geq 0$. If we denote

$$\begin{aligned} h_2(z,t) &= f(e^{-t}z)g(e^{-t}z)[h_1(z,t)]^{-1}, \\ h_3(z,t) &= e^{-t}z[f'(e^{-t}z)h(e^{-t}z) + f(e^{-t}z)h'(e^{-t}z)][h_1(z,t)]^{-1}, \end{aligned} \tag{10}$$

we can observe that $h_2(z,t) = e^{-t}z + \ldots$ and $h_3(z,t) = e^{-t}z + \ldots$. It follows that the function

$$L(z,t) = h_2(z,t) + \frac{1}{1+c}(e^{at}-1)h_3(z,t) \tag{11}$$

is analytic in $U_{r_1}$ and $L(0,t) = 0$ for all $t \geq 0$. We have

$$L(z,t) = e^{-t}zh_4(z,t), \tag{12}$$

where $h_4(z,t) = \dfrac{c+e^{at}}{1+c} + \sum_{n=1}^{\infty} \alpha_n (e^{-t}z)^n$.

Using (9), (10) and (11), it follows that the function $L(z,t)$, defined for $z \in U_{r_1}$ and $t \geq 0$, can be written as

$$(13)\quad L(z,t) = \left[g(e^{-t}z) + \frac{1}{1+c}(e^{at}-1)e^{-t}z.h'(e^{-t}z)\right]^{-1} \times\left\{f(e^{-t}z)g(e^{-t}z) + \frac{e^{at}-1}{1+c}e^{-t}z[f'(e^{-t}z)h(e^{-t}z) + f(e^{-t}z)h'(e^{-t}z)]\right\}$$

and has a power series expansion

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, \quad \text{where } a_1(t) = \frac{e^{(a-1)t} + ce^{-t}}{1+c}.$$

Let us prove that $a_1(t) \neq 0$ for all $t \geq 0$. Observe that if $a_1(t_0) = 0$, then $c = -e^{at_0}$ and it follows that $|c| = e^{t_0 \operatorname{Re} a}$ which is impossible, because $|c| < 1$ and $\operatorname{Re} a > 1$. For $\operatorname{Re} a > 1$ we have $\lim_{t\to\infty} |e^{(a-1)t}| = \lim_{t\to\infty} e^{(\operatorname{Re} a-1)t} = \infty$ and then $\lim_{t\to\infty} |a_1(t)| = \infty$.

The function $L(z,t)$ is analytic in $U_{r_1}$ for all $t \geq 0$ and then it follows that there exist a number $r_2$, $0 < r_2 < r_1$, and a constant $K = K(r_2)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K \quad \text{for all } z \in U_{r_2},\ t \geq 0.$$

Then, by Montel's theorem, it follows that $\{L(z,t)/a_1(t)\}$ is a normal family in $U_{r_2}$. From (12) we have

$$(14)\qquad \frac{\partial L(z,t)}{\partial t} = e^{-t}z\left[\frac{\partial h_4(z,t)}{\partial t} - h_4(z,t)\right].$$

It is clear that $\frac{\partial h_4(z,t)}{\partial t}$ is analytic function in $U_{r_2}$ and $\frac{\partial L(z,t)}{\partial t}$ too. Then, for all fixed numbers $T > 0$ and $r_3$, $0 < r_3 < r_2$, there exists a constant $K_1 > 0$ such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1 \quad \text{for all } z \in U_{r_3} \text{ and } t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to $U_{r_3}$. Since $\frac{\partial L(z,t)}{\partial t}$ is analytic in $U_{r_3}$, from (14) it follows that there is a number $r$, $0 < r < r_3$, such that $\frac{1}{z}\frac{\partial L(z,t)}{\partial t} \neq 0$ for all $z \in U_r$, and then the function

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

is analytic in $U_r$, for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in $U$, for all $t \geq 0$, it is sufficient to show that the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}, \quad z \in U_r,$$

can be continued analytically in $U$ and that

$$(15) \qquad |w(z,t)| < 1 \quad \text{for all } z \in U \text{ and } t \geq 0.$$

After computation we obtain

$$(16) \quad w(z,t) = \frac{2}{a} - 1 + \frac{2}{a}\left[(1+c)\frac{g(e^{-t}z)}{h(e^{-t}z)} - 1\right]e^{-at} + \frac{2}{a}(1-e^{-at})e^{-t}z\left[\left(\frac{f''(e^{-t}z)}{f'(e^{-t}z)} + 3\frac{h'(e^{-t}z)}{h(e^{-t}z)} - \frac{g'(e^{-t}z)}{g(e^{-t}z)}\right) + \frac{e^{at}-1}{1+c}e^{-t}z\left(\frac{f''(e^{-t}z)}{f'(e^{-t}z)}\frac{h'(e^{-t}z)}{g(e^{-t}z)} + 2\frac{(h'(e^{-t}z))^2}{g(e^{-t}z)h(e^{-t}z)} - \frac{h''(e^{-t}z)}{g(e^{-t}z)}\right)\right]$$

Since $f'(z)g(z)h(z) \neq 0$ for all $z \in U$, the function $w(z,t)$ is analytic in $U$. For $z = 0$ we have

$$(17) \qquad |w(0,t)| = \left|\frac{2c}{a}e^{-at} + \frac{2}{a} - 1\right| \leq \frac{|a-2|}{|a|} + \frac{2|c|}{|a|}$$

and, by (6), it results that $|w(0,t)| < 1$ for all $t \geq 0$. For $z \neq 0$ and $t = 0$ in view of (7) we obtain

$$(18) \qquad |w(z,0)| = \left|\frac{2(1+c)}{a} \cdot \frac{g(z)}{h(z)} - 1\right| < 1.$$

Let now be a fixed number $t > 0$, $z \in U$, $z \neq 0$. In this case the function $w(z,t)$ is analytic in $\overline{U}$, because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \overline{U}$. Using the maximum principle, for all $z \in U$ and $t > 0$, we have

$$(19) \qquad |w(z,t)| < \max_{|\zeta|=1}|w(\zeta,t)| = |w(e^{i\theta},t)|,$$

where $\theta = \theta(t)$ is a real number.

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (16) we obtain

$$(20) \ |w(e^{i\theta},t)| = \frac{1}{|a||u|^{\mathrm{Re}\,a}}\Bigg|(2-a)|u|^a + 2\left[(1+c)\frac{g(u)}{h(u)} - 1\right]|u|^{2a} + 2(1-|u|^a)u\left[\left(\frac{f''(u)}{f'(u)} + 3\frac{h'(u)}{h(u)} - \frac{g'(u)}{g(u)}\right)|u|^a + \frac{1-|u|^a}{1+c}u\left(\frac{f''(u)}{f'(u)}\frac{h'(u)}{g(u)} + 2\frac{(h'(u))^2}{g(u).h(u)} - \frac{h''(u)}{g(u)}\right)\right]\Bigg|.$$

Since $u \in U$, the relation (8) implies $|w(e^{i\theta}, t)| \leq 1$.

From (17), (18), (19) and (20) we conclude that (15) holds true for all $z \in U$ and $t \geq 0$. It follows that $L(z,t)$ is a Loewner chain, and hence, the function $L(z,0) = f(z)$ is univalent in $U$.

THEOREM 2. *Let $a$ and $c$ be complex numbers satisfying* $\operatorname{Re} a > 1$, $|c| < 1$ *and let $f \in A$. If*

$$|a-2| + 2|c| < |a| \tag{6'}$$

*and if there exists an analytic function $p$ with positive real part in $U$ such that $p(0) = (a-1-c)/(1+c)$, and*

$$\left|1 - \frac{a}{2} + \frac{a-1-p(z)}{1+p(z)}|z|^a + (1-|z|^a)\left[\frac{zp'(z)}{p(z)+1} + \frac{zf''(z)}{f'(z)}\right]\right| \leq \frac{|a|}{2} \tag{21}$$

*for all $z \in U^*$, then the function $f$ is univalent in $U$.*

Proof. Let us consider the function $h(z) \equiv 1$ in Theorem 1. Denoting

$$p(z) = \frac{a-(1+c)g(z)}{(1+c)g(z)}, \tag{22}$$

we get $p(0) = (a-1-c)/(1+c)$, and from (22) we obtain

$$g(z) = \frac{a}{(1+c)(1+p(z))} \quad \text{and} \quad \frac{g'(z)}{g(z)} = -\frac{p'(z)}{1+p(z)}.$$

The condition (7) will be replaced by

$$\left|\frac{2(1+c)}{a}g(z) - 1\right| = \left|\frac{2}{1+p(z)} - 1\right| = \left|\frac{p(z)-1}{p(z)+1}\right| < 1$$

equivalent to $\operatorname{Re} p(z) > 0$ for all $z \in U$, and (8) becomes

$$\left|(2-a) + 2[(1+c)g(z)-1]|z|^a + 2(1-|z|^a)z\left(\frac{f''(z)}{f'(z)} - \frac{g'(z)}{g(z)}\right)\right| \leq |a|.$$

In view of (22), we obtain the relation (21).

## 4. Remarks

1. For $a = 2$ and $c = 0$ Theorem 1 becomes Theorem E.
2. For $a = 2$ we find the results from the paper [9].
3. For $c = 0$ we find the results from the paper [8].
4. For $g(z) \equiv h(z)$ we find the results from the paper [10]. If $a = 2$ and $c = 0$, then we obtain the results from the paper [2].
5. For $g(z) \equiv 1$ and $h(z) \equiv 1$, $a = 2$ and $c = 0$ we have Theorem B.

6. For $g(z) = (f'(z))^{-1/2}$ and $h(z) = (f'(z))^{-1/2}$ we find the generalization of Nehari's univalence criterion obtained in [6]. If $a = 2$ and $c = 0$, then we have Theorem A.

7. For $h(z) \equiv 1$ Theorem 1 becomes Theorem 2 generalizing Lewandowski's univalence criterion. If $a = 2$ and $c = 0$, then we have Theorem C.

8. For $a = 2$, $c = 0$, $h(z) = (f'(z))^{-1/2}$ and $g(z) = \frac{2(f'(z))^{-1/2}}{1+p(z)}$, where $p$ is an analytic function in $U$, $\operatorname{Re} p(z) > 0$ for all $z \in U$, Theorem 1 becomes Theorem D. In this case, the inequalities (6) and (7) are true and we have

$$\frac{g(z)}{h(z)} - 1 = \frac{1 - p(z)}{1 + p(z)},$$

$$\frac{f''(z)}{f'(z)} + 3\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)} = \frac{p'(z)}{1 + p(z)},$$

$$\frac{f''(z)}{f'(z)}\frac{h'(z)}{g(z)} + 2\frac{(h'(z))^2}{g(z)h(z)} - \frac{h''(z)}{g(z)} = \frac{1 + p(z)}{4}\{f; z\}.$$

The condition (8) becomes

$$\left|\frac{1 - p(z)}{1 + p(z)}|z|^4 + (1 - |z|^2)\frac{zp'(z)}{p(z) + 1}|z|^2 + z^2(1 - |z|^2)^2\frac{1 + p(z)}{4}\{f; z\}\right| \le |z|^2$$

which is equivalent to the inequality (3') in Theorem D.

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