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ON THE BOUNDS OF SOLUTIONS OF A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

In this paper the conditions for the bounds of solution for a system of linear Volterra integral equations are presented. We use certain new variants of the Volterra inequalites.

Consider the following system of linear integral equations of the Volterra type

$$(1)\qquad u_i(x) = f_i(x) + \sum_{j=1}^{m} \int_0^x k_{ij}(x,t)u_j(t)\,dt, \qquad i = 1,2,\ldots,m,$$

where f_i, $i = 1,2,\ldots,m$, k_{ij}, $i,j = 1,2,\ldots,m$, are continuous functions in $I =: \{x : 0 \le x < \infty\}$ and $D =: \{(x,t) : 0 \le t \le x < \infty\}$, respectively.

It is clear that from (1) we get

$$(2)\qquad u(x) \le f(x) + \int_0^x k(x,t)u(t)dt,$$

where

$$f(x) = \sum_{i=1}^{m} |f_i(x)| \ge 0, \qquad u(x) = \sum_{i-1}^{m} |u_i(x)| \ge 0,$$

$$k(x,y) = \sum_{i=1}^{m} \max_{1\le j\le m} |k_{ij}(x,y)| \ge 0.$$

From the theory of linear Volterra integral equations it is well known that

$$(3)\qquad u(x) \le f(x) + \int_0^x R(x,t)f(t)\,dt,$$

where $R(x,t) = \sum_{n=0}^{\infty} k_n(x,t)$ is the resolvent kernel,

$$k_n(x,t) = \int_t^x k(x,s)k_{n-1}(s,t)ds, \qquad n = 1,2,\ldots, k_0(x,t) \equiv k(x,t).$$

2. Various cases

I. If $k(x,t) = b(t) \geq 0$, then $R(x,t) = b(t)\exp[\int_t^x b(s)ds]$ (see [2]) and (3) can be written in the form

$$u(x) \leq f(x) + \int_0^x b(t)f(t)\exp\Big[\int_t^x b(s)ds\Big]\,dt. \tag{4}$$

Introducing the notation $F(x) = \sup\{f(t) :\ 0 \leq t \leq x\}$, we obtain from (4)

$$\begin{aligned} u(x) &\leq f(x) + F(x)\int_0^x b(t)\exp\Big[\int_t^x b(s)\,ds\Big]\,dt = \\ &= f(x) + F(x)\Big\{\exp\Big[\int_0^x b(t)\,dt\Big] - 1\Big\}. \end{aligned} \tag{4'}$$

THEOREM 1. *If the functions* f_i, $i = 1,2,\ldots,m$ *and* k_{ij}, $i,j = 1,2,\ldots,m$, *are continuous functions in* I *and* D, *respectively, and if*

$$\sum_{i=1}^{m} \max_{1\leq j\leq m} |k_{ij}(x,t)| \leq b(t) \quad \text{in } D, \tag{i}$$

$$\sum_{i=1}^{m} |f_i(x)| = f(x) \quad \text{in } I, \tag{ii}$$

then the following inequality

$$\sum_{i=1}^{m} |u_i(x)| \leq f(x) + F(x)\Big\{\exp\Big[\int_0^x b(t)\,dt\Big] - 1\Big\} \tag{5}$$

holds, where $\{u_i(x)\}$, $i = 1,2,\ldots,m$, *is a solution of the system* (1).

Moreover, if f *is bounded in* I *and* b *is bounded and integrable in* I, *then the solution* $\{u_i(x)\}$, $i = 1,2,\ldots,m$ *of the system (1) is bounded as* $x \to \infty$.

Remark 1. If f is nondecreasing in I, then $F(x) = f(x)$ and we obtain

$$\sum_{i=1}^{m} |u_i(x)| \leq f(x)\exp\Big[\int_0^x b(t)\,dt\Big].$$

COROLLARY 1. *The estimate* (5) *in Theorem* 1 *holds, if*

$$k(x,t) = \sum_{i=1}^{m} \max_{1\le j\le m} |k_{ij}(x,t)|$$

is nonincreasing with respect to variable x *or* $k(x,t) \le k(t,t)$ *for* $x \ge t$. *Then* $b(t) = k(t,t)$.

II. In the case of $k(x,t) = a(x)b(t)$, $a(x) > 0$ and $b(t) \ge 0$, the inequality (2) leads to

$$u(x) \le f(x) + a(x) \int_0^x b(t)u(t)\, dt \tag{6}$$

or, equivalently, to

$$v(x) \le g(x) + a(x) \int_0^x a(t)b(t)v(t)\, dt, \tag{6'}$$

where

$$v(x) = \frac{u(x)}{a(x)}, \qquad g(x) = \frac{f(x)}{a(x)}.$$

Using the inequality (4'), we obtain the following estimate

$$v(x) \le g(x) + G(x)\Big\{ \exp\Big[\int_0^x a(t)b(t)\, dt\Big] - 1\Big\},$$

where

$$G(x) = \sup_{0\le t\le x} g(t).$$

Hence

$$u(x) \le f(x) + a(x)G(x)\Big\{ \exp\Big[\int_0^x a(t)b(t)\, dt\Big] - 1\Big\} =: p(x). \tag{7}$$

If g is nonincreasing, then $G(x) = g(x)$, and we get an inequality of the Gronwall type

$$u(x) \le f(x) \exp\Big[\int_0^x a(t)b(t)\, dt\Big]. \tag{7'}$$

Introduce now the notation

$$h(x) = \max_{x\in I}\{f(x), a(x)\}. \tag{8}$$

Then we can rewrite inequality (6) in the form

$$u(x) \le h(x) + h(x) \int_0^x b(t)u(t)\, dt, \tag{9}$$

or

$$w(x) \le 1 + \int_0^x b(t)h(t)w(t)\,dt. \tag{10}$$

Using the classical Gronwall inequality, we get

$$u(x) \le h(x)\exp\Big[\int_0^x b(t)h(t)\,dt\Big] =: q(x) \tag{11}$$

or

$$w(x) \le \exp\Big[\int_0^x b(t)h(t)\,dt\Big], \tag{11'}$$

where $w(x) = \frac{u(x)}{h(x)}$, $h(x) > 0$. In this way the following theorem is proved.

THEOREM 2. *Let k_{ij}, $i,j = 1,2,\dots,m$ and f_i, $i = 1,2,\dots,m$ be continous functions in D and I, respectively, and*

$$\sum_{i=1}^{m} \max_{1\le j\le m} |k_{ij}(x,t)| = a(x)b(t), \qquad a(x) > 0,\ b(t) \ge 0, \tag{i}$$

$$\sum_{i=1}^{m} |f_i(x)| = f(x). \tag{ii}$$

Then a solution $\{u_i(x)\}$ $i = 1,2,\dots,m$, of the system (1) *is estimated by*

$$\sum_{i=1}^{m} |u_i(x)| \le \min_{x\in I}\{p(x), q(x)\},$$

where $p(x)$ and $q(x)$ are right hand sides of inequalities (7) *and* (11), *respectively.*

Moreover, if the conditions

$$\begin{cases} \lim\limits_{x\to\infty} f(x) < \infty, \quad \lim\limits_{x\to\infty} a(x) \sup\limits_{0\le t\le x} \frac{f(t)}{a(t)} < \infty, \\ \lim\limits_{x\to\infty} \int_0^x a(t)b(t)dt < \infty, \end{cases} \tag{A}$$

or

$$\lim_{x\to\infty} h(x) < \infty, \quad \lim_{x\to\infty} \int_0^x b(t)h(t)dt < \infty \tag{B}$$

are satisfied, then the solution $\{u_i(x)\}$, $i = 1,2,\dots,m$, of the system (1) *is bounded as $x \to \infty$.*

Remark 2. If
$(*)$ $a(x) = f(x)$, then $p(x) = q(x)$ and the conditions (A) and (B) reduce to

$$\lim_{x\to\infty} a(x) < \infty \quad \text{and} \quad \lim_{x\to\infty} \int_0^x a(t)b(t)dt < \infty,$$

respectively,
$(**)$ $a(x) < f(x)$, then for $\lim_{x\to\infty} a(x) \neq 0$ the conditions (A) and (B) lead to

$$(\mathrm{A}^*) \qquad \lim_{x\to\infty} f(x) < \infty, \quad \lim_{x\to\infty} \int_0^x a(t)b(t)dt < \infty$$

and

$$(\mathrm{B}^*) \qquad \lim_{x\to\infty} f(x) < \infty, \quad \lim_{x\to\infty} \int_0^x f(t)b(t)dt < \infty,$$

respectively, and $(B^*) \Rightarrow (A^*)$;
$(***)$ $\quad a(x) > f(x)$, then $(B) \Rightarrow (A)$.

The case $(***)$ was considered by Butlewski in [1]. Our results are better, because the condition (A) is sufficient for the bounds of solutions of the system (1) at infinity.

Remark 3. If $\frac{f}{a}$ is nondecreasing in I, then the condition (A) leads to (A^*).

COROLLARY 2. *If k is nonincreasing with respect to t or $k(x,t) \leq k(x,x)$ for $t \leq x$ and assumptions of Theorem 2 are satisfied, then we get the result with $a(x) = k(x,x)$, $b(t) = 1$.*

III. Let be $k(x,t) = \sum_{k=1}^n a_k(x)b_k(t)$, where a_k, b_k, $k = 1,2,\ldots,n$, are nonnegative continuous functions in D and $\sum_{k=1}^n a_k(x) > 0$.

Consider two following cases:

(a) If $A_1(x) = \sup_{1\leq k\leq n} a_k(x)$, then $k(x,t) \leq A_1(x)B_1(t)$,

where

$$B_1(t) = \sum_{k=1}^n b_k(t),$$

(b) If $B_2(x) = \sup_{1\leq k\leq n} b_k(x)$, then $k(x,t) \leq A_2(x)B_2(t)$,

where

$$A_2(x) = \sum_{k=1}^n a_k(x).$$

In both cases the inequality (2) reduces to

$$u(x) \le f(x) + A_r(x) \int_0^x B_r(t)u(t)\,dt, \qquad r = 1,2. \tag{12}$$

Using Theorem 2 ($A_r(x) > 0$, $B_r(t) \ge 0$ for $r = 1,2$), we obtain the following result.

THEOREM 3. *If f_i, $i = 1,2,\dots,m$ and k_{ij}, $i,j = 1,2,\dots,m$, are continuous in I and D, respectively, and*

$$k(x,t) = \sum_{k=1}^{m} a_k(x)b_k(t), \quad \sum_{k=1}^{n} a_k(x) \ne 0,$$

$$a_k(x), b_k(t) \ge 0, \quad \sum_{i=1}^{m} |f_i(x)| = f(x),$$

then

$$\sum_{i=1}^{m} |u_i(x)| \le \min_{x\in I}\{p_1(x), p_2(x), q_1(x), q_2(x)\}$$

where $\{u_i(x)\}$, $i = 1,2,\dots,m$, is a solution of the system (1) *and*

$$p_r(x) = f(x) + A_r(x) \sup_{0\le s\le x} \frac{f(s)}{A_r(s)}\Big\{\exp\Big[\int_0^x A_r(t)B_r(t)\,dt\Big] - 1\Big\},$$

$$q_r(x) = \max_{x\in I}[f(x), A_r(x)] \exp\Big\{\int_0^x B_r(t) \max_{t\in I}[f(t), A_r(t)]\,dt\Big\}.$$

Moreover, if one of conditions

$$(\mathrm{A}')\qquad \begin{cases} \lim\limits_{x\to\infty} f(x) < \infty, \quad \lim\limits_{x\to\infty} A_r(x) \sup\limits_{0\le t\le x} \frac{f(t)}{A_r(t)} < \infty, \\ \lim\limits_{x\to\infty} \int_0^x A_r(t)B_r(t)\,dt < \infty, \end{cases}$$

$$(\mathrm{B}')\qquad \begin{cases} \lim\limits_{x\to\infty} \max\limits_{x\in I}[f(x), A_r(x)] < \infty, \\ \lim\limits_{x\to\infty} \int_0^x B_r(t) \max\limits_{t\in I}[f(t), A_r(t)]\,dt < \infty, \end{cases}$$

for $r = 1$ or $r = 2$ is fulfilled, then a solution of the system (1) *is bounded at infinity.*

Remark 4. If $\frac{f}{A_r}$ is nondecreasing in I, then the condition (A') is reduced to

$$(\mathrm{A}'')\qquad \lim_{x\to\infty} f(x) < \infty, \quad \lim_{x\to\infty} \int_0^x A_r(t)B_r(t)dt < \infty, \quad r = 1 \text{ or } r = 2.$$

IV. Introduce the notation

$$K(x) = \sup_{0 \le t \le x} k(x,t)$$

for every $x \in I$. It is clear that $k(x,t) \le K(x)$ for $x \ge t$. Then the inequality (2) can be replaced by

$$u(x) \le f(x) + K(x) \int_0^x u(t)\, dt. \tag{13}$$

Let us notice that (13) is a particular case of inequality (6), where $a(x) = K(x)$ and $b(t) = 1$. By the similar way as in Theorem 2 the following results were obtained.

THEOREM 4. *If f_i, $i = 1, 2, \ldots, m$ and k_{ij}, $i, j = 1, 2, \ldots, m$, are continuous functions in I and D, respectively, and*

$$\sup_{0 \le t \le x} k(x,t) + K(x) > 0, \quad \sum_{i=1}^{m} |f_i(x)| = f(x),$$

then a solution $\{u_i(x)\}$, $i = 1, 2, \ldots, m$, of the system (1) satisfies the following inequality

$$\sum_{i=1}^{m} |u_i(x)| \le \min_{x \in I}[P(x), Q(x)],$$

where

$$P(x) = f(x) + K(x) \sup_{0 \le s \le x} \frac{f(s)}{K(s)} \Big\{ \exp\Big[\int_0^x K(t)\, dt \Big] - 1 \Big\},$$

$$Q(x) = \max_{x \in I}[f(x), K(x)] \exp \int_0^x K(t)\, dt.$$

Moreover, if one of conditions

$$\text{(A''')} \qquad \begin{cases} \lim\limits_{x \to \infty} f(x) < \infty, \quad \lim\limits_{x \to \infty} K(x) \sup\limits_{0 \le t \le x} \frac{f(t)}{K(t)} < \infty, \\ \lim\limits_{x \to \infty} \int_0^x K(t) dt < \infty, \end{cases}$$

$$\text{(B''')} \qquad \lim_{x \to \infty} \max_{x \in I}[f(x), K(x)] < \infty, \quad \lim_{x \to \infty} \int_0^x K(t) dt < \infty,$$

is satisfied, then a solution of the system (1) is bounded at infinity.

Remark 5. If $f(x) < K(x)$ and $\frac{f}{K}$ is nondecreasing in I, then conditions (A‴) and (B‴) are reduce to

$$\lim_{x\to\infty} f(x) < \infty, \quad \lim_{x\to\infty} \int_0^x K(t)dt < \infty.$$

The above results can be extendend on the classes L and L^2.

References

[1] Z. Butlewski, *Sur la limitation des solutions d'un système d'équations intégrales de Volterra*, Ann. Polon. Math. 6 (1959), 253–257.

[2] A. Corduneanu, *A note on Gronwall inequality in two independent variables*, J. Int. Equ. 4 (1982), 271–276.

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