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LOCAL EXISTENCE OF SOLUTIONS OF $2m$-th ORDER SEMILINEAR PARABOLIC EQUATIONS

Dedicated to Professor Janina Wolska-Bochenek

1. Motivation

Semigroup "dynamical" approach is frequently used to deal with the wide class of nonlinear problems (e.g. [2], [3], [16]). However, it seems to be worth pointing out that many of these problems possess their satisfactory treatment as well in the frame of classical for the theory of differential equations ideas as in the technique. Hence, the aim of this paper is to prove in a classical way local solvability of the $2m$-th order semilinear parabolic equations and furthermore, to derive suitable estimate of the $W^{k,\infty}$ norm of the solution, which enables to find a minimal time of its existence. Particularly, we also want to show that the classical Peano method (used in the existence theorems for the ordinary differential equations initial value problems) can be successfully applied in the proof of solvability of higher order semilinear parabolic equations.

2. Introduction and notation

We shall study the initial boundary value problem

$$(1)\qquad \begin{cases} u_t = -Pu + f(t,x,d^{2m-1}u) =: \Gamma(t,x,u) & \text{in } D^T \\ B_0 u = \ldots = B_{m-1}u = 0 \quad \text{on } \partial G \\ u(0,x) = u_0(x) \quad \text{in } G \end{cases}$$

where P is a $2m$-th order strongly elliptic differential operator given by

$$Pu = \sum_{|\alpha|,|\beta|\le m} (-1)^{|\beta|} D^\beta(a_{\alpha,\beta}(x)D^\alpha u),$$

$B_0, \ldots B_{m-1}$ are linear and time independent boundary operators, D^T stands for a product $(0,T)\times G$ and $d^{2m-1}u$ (which appears as an argument of the nonlinear function f) denotes the vector of the length $d = (2m+n-1)!((2m-1)!n!)^{-1}$ consisted of all partial derivatives of u up to the order $2m-1$ with respect to the space variable x, i.e.

$$d^{2m-1}u = \left(u, \frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \dots \frac{\partial^{2m-1} u}{\partial x_n^{2m-1}}\right).$$

We shall estimate (from below) the life time T_0 of the solutions of the problem (1) and show that the problem possesses, under suitable conditions on f, a unique solution in the space of Hölder functions $C^{2m+\mu_0, 1+\mu_0/2m}(D^{T_0})$ (with certain $\mu_0 \in (0,1)$).

Generally, the proof of existence is based on the *a priori* estimates technique and on the method of continuity. But its inspiration comes from the Peano concept well known in the theory of ordinary differential equations, since the range of the arguments of the nonlinear function f is limited to a (multidimensional) rectangle and then a positive time T_0 is determined in such a way that all these arguments stay inside the fixed rectangle until the time T_0. Equivalently, we limit the $W^{2m-1,\infty}$ norm of the solution and next use this fact to obtain better estimate of the same norm, finding simultaneously the life time of the solution.

The technique we present here was used previously by T. Dlotko (see [5], [6]) in case of 2-nd and 4-th order equations. It has also been used by myself [4] to prove the existence of solutions for parabolic problems of the general type (1) but with the function f depending only on the derivatives of u of the order not exceeding m.

Notation of Sobolev and Hölder spaces which we use throughout the paper comes from the monographs [1], [10], [11]; but we denote the spaces of continuous functions by C instead of H (as in [7], [8]). Space variable x belongs to the fixed bounded domain $G \subset R^n$ having sufficiently regular boundary ∂G (which is at least of the class $C^{2m+\mu}$ where $\mu \in (0,1)$ is fixed from now on). We also write $|G|$ for the Lebesgue measure of G. Different positive constants are denoted by C_l with various $l \in R$. Whereas the unspecified integrals are always understood to be taken over G. We also use (for simplicity of the notation) common letters L and M for (resp.) various Lipschitz constants and numerous upper bounds constants appearing in our considerations.

3. Assumptions

The following assumptions are valid throughout the paper:

(I) We consider for the simplicity of calculations only space dimensions $n \leq 3$.

(II) Coefficients $a_{\alpha,\beta}$ (of the operator P) belong to the class (resp.)

$$C^{4m+|\beta|+\mu}(\mathrm{cl}\, G).$$

(III) Initial function u_0 is of the class $C^{6m+\mu}(\mathrm{cl}\, G)$ and satisfies the following compatibility conditions

$$B_0 u_0 = \ldots = B_{m-1} u_0 = 0,$$
$$B_0 \Gamma(0, x, u_0) = \ldots = B_{m-1} \Gamma(0, x, u_0) = 0.$$

(IV) In $R^+ \times \mathrm{cl}\, G \times R^d$ the function $f = f(t, x, p_1, \ldots p_d)$ has locally bounded time derivatives up to the third order. Moreover f is $6m$-times f_t is $4m$-times and f_{tt} is $2m$-times differentiable with respect to the space and functional arguments, (hence also both f and its all first order partial derivatives are locally Lipschitz continuous in $R^+ \times clG \times R^d$ with respect to $t, x, p_1, \ldots p_d$).

(V) The triple $(P, \{B_j\}, G)$ forms a "regular elliptic boundary value problem" in the sense of the definition stated in [14] p. 125 or in [7] p. 76 (i.e. according to [7], it satisfies the *root condition*, the *smoothness condition*, certain *complementary condition* and the system $\{B_j\}$ is normal).

(VI) For all $w \in D(P) = \{\varphi \in W^{2m,2}(G) : B_0\varphi = \ldots = B_{m-1}\varphi = 0$ on $\partial G\}$ connected with the operator P bilinear form p, given by

$$p(u, v) = \sum_{|\alpha|,|\beta| \leq m} a_{\alpha,\beta}(x) D^\alpha u D^\beta v,$$

satisfies the following conditions

— coerciveness inequality

$$\text{(i)} \qquad \int p(w, w)\, dx + C_2 \|w\|_{0,2}^2 \geq C_1 \|w\|_{m,2}^2$$

— Green's Identity

$$\text{(ii)} \qquad \int (Pw) w dx = \int p(w, w)\, dx.$$

The operator P and boundary operators $\{B_j\}$ are given in general form but in our considerations any particular form of them would be superfluous. We only note that in spite of certain complicity of the above assumptions there are number of examples fulfilling all of them (comp. [4], [6], [7], [13], [14]).

4. Preliminaries

Instead of (1) we consider

$$\text{(2)} \qquad \begin{cases} v_t = -Pv + g(t, x, d^{2m-1} v) & \text{in } D^T \\ B_0 v = \ldots = B_{m-1} v = 0 & \text{on } \partial G \\ v(0, x) = 0 & \text{in } G \end{cases}$$

with $g(t,x,d^{2m-1}v) = f(t,x,d^{2m-1}v + d^{2m-1}u_0) - Pu_0$. It is clear that if f is locally Lipschitz continuous in $R^+ \times \operatorname{cl} G \times R^d$, then g is bounded and globally Lipschitz continuous in a compact set Y which we define as

$$Y := \Big\{(t,x,p_1,\ldots,p_d) : t \in [0,T],\ x \in \operatorname{cl} G,\ \sum_{i=1}^{d} |p_i| \le R\Big\},$$

where R, T are fixed positive numbers (R will be taken sufficiently large, see (29)). Moreover the same occurs for the derivatives $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial(D^\alpha v)}$, $\frac{\partial^2 g}{\partial t^2}$, $\frac{\partial^2 g}{\partial(D^\alpha v)\partial(D^\beta v)}$, $\frac{\partial^2 g}{\partial t \partial(D^\alpha v)}$, which as long as v stays in Y (i.e. as long as for each $x \in clG$ the full vector $(t,x,d^{2m-1}v(t,x))$, belongs to Y) can be regarded as bounded by the common constant M which is taken relatively to Y. Thus in particular, in the set Y (for all α, β with $|\alpha|,|\beta| \le 2m-1$) the following inequalities

$$\left|\frac{\partial g}{\partial t}\right|, \left|\frac{\partial g}{\partial(D^\alpha v)}\right|, \left|\frac{\partial^2 g}{\partial t^2}\right|, \left|\frac{\partial^2 g}{\partial(D^\alpha v)\partial(D^\beta v)}\right|, \left|\frac{\partial^2 g}{\partial t\partial(D^\alpha v)}\right| \le M \tag{3}$$

hold.

It is well known that linear theory is necessary in order to deal successfully with the nonlinear problems. We will make use of it to derive the following two estimates:

$$\|D^\alpha v_t(t,\cdot)\|_\infty \le C_3, \tag{4}$$

$$\|D^\alpha v_{tt}(t,\cdot)\|_\infty \le C_4, \tag{5}$$

which are valid for all α with $|\alpha| \le 2m-1$ as long as v remains inside Y (constants C_3, C_4 depends in particular on the choice of R in the definition of Y).

To obtain (4) let us note that since boundary operators $\{B_j\}$ are time independent then time derivative v_t solve

$$\begin{cases} \omega_t = -P\omega + g_t(t,x,v) + \sum_{|\alpha|\le 2m-1} g_\alpha(t,x,v)D^\alpha\omega \\ \qquad =: L(t,x,v,\omega) \quad \text{in } D^T \\ B_0\omega = \ldots = B_{m-1}\omega = 0 \quad \text{on } \partial G \\ \omega(0,x) = \Gamma(0,x,u_0) \quad \text{in } G, \end{cases} \tag{6}$$

with

$$g_t(t,x,v) = \frac{\partial g}{\partial t}(t,x,d^{2m-1}v(t,x))$$

and

$$g_\alpha(t,x,v) = \frac{\partial g}{\partial(D^\alpha v)}(t,x,d^{2m-1}v(t,x)).$$

Applying to (6) the estimate for the linear $2m$-th order parabolic equations given in [11] (see Th. 10.4, Chapt. VII, §10 cited here with $l = 0$, $s = 0$, $t = 2m$, $\rho = -2m$, $q = 2n + 2 + \Theta$, and any $\Theta \in (0,1)$) we get (as long as v stays inside Y)

$$(7) \quad \|\omega_t\|_{L^{2n+2+\Theta}(D^t)} + \sum_{|\alpha|\le 2m} \|D^\alpha \omega\|_{L^{2n+2+\Theta}(D^t)} \le c\{M(|G|T)^{\frac{1}{2n+2+\Theta}} + \|\Gamma(0,\cdot,u_0)\|_{W^{2m-\frac{2m}{2n+2+\Theta}}_{2n+2+\Theta}(G)}\}.$$

Next by differentiation we obtain from (6) the initial boundary value problem for ω_t (ω_t corresponds to v_{tt})

$$(8) \quad \begin{cases} (\omega_t)_t = -P\omega_t + \sum_{|\alpha|\le 2m-1} g_\alpha(t,x,v) D^\alpha \omega_t + \Lambda(t,x,v,\omega) & \text{in } D^T \\ B_0(\omega_t) = \ldots = B_{m-1}(\omega_t) = 0 \text{ on } \partial G \\ \omega_t(0,x) = L(0,x,0,\Gamma(0,x,u_0)) \text{ in } G, \end{cases}$$

where

$$\Lambda(t,x,v,w) = \sum_{|\alpha|\le 2m-1} \frac{\partial^2 g}{\partial(D^\alpha v)\partial t}(t,x,d^{2m-1}v(t,x)) D^\alpha v_t(t,x)$$
$$+ \frac{\partial^2 g}{\partial t^2}(t,x,d^{2m-1}v(t,x)) + \sum_{|\alpha|\le 2m-1} \frac{\partial^2 g}{\partial t \partial(D^\alpha v)}(t,x,d^{2m-1}v(t,x)) D^\alpha \omega(t,x)$$
$$+ \sum_{|\alpha|\le 2m-1} \sum_{|\beta|\le 2m-1} \frac{\partial^2 g}{\partial(D^\beta v)\partial(D^\alpha v)}(t,x,d^{2m-1}v(t,x)) D^\alpha \omega(t,x) D^\beta v_t(t,x).$$

According to (7) (ω corresponds to v_t) as long as v stays in Y, the above function Λ is estimated independently of t in the space $L^{n+1+\Theta/2}(D^t)$. Thus using again Th. 10.4, Chapt. VII, §10 from [11] we find $L^{n+1+\Theta/2}(D^t)$-bound for $D^\alpha v_{tt}$ (with $|\alpha| \le 2m-1$) and consequently, since for each α with $|\alpha| \le 2m-1$,

$$D^\alpha v_t,\ D^\alpha v_{tt},\ \frac{\partial}{\partial x_1} D^\alpha v_t, \ldots, \frac{\partial}{\partial x_n} D^\alpha v_t \in L^{n+1+\frac{\Theta}{2}}(D^t),$$

then, thanks to Sobolev Embedding Theorem in $n+1$ dimensional space, we come immediately to (4).

The proof of the inequality (5) is entirely analogous to given above evidence of (4), and we will omit it. We end this section by formulating a lemma making possible to estimate both the solution v and its time derivative v_t in a certain flexible manner which will be useful in our further considerations.

PRELIMINARY LEMMA. *As long as the solution v of the problem* (2) *stays inside Y, there exist positive constants ν_0 and θ_0 such that for all $\nu \in (0,\nu_0]$*

and each $\theta \in (0, \theta_0]$ (*respectively*) *the following estimates*

$$\|v(t,\cdot)\|^2_{2m-1,n+2} \le \nu(\|v_t(t,\cdot)\|^2_{0,2} + C_5) + C_\nu \|v(t,\cdot)\|^2_{0,2}, \tag{9}$$

and

$$\|v_t(t,\cdot) - \Gamma(0,\cdot,u_0)\|^2_{2m-1,n+2} \le \theta(\|v_{tt}(t,\cdot)\|^2_{0,2} + C_6) + C_\theta \|v_t(t,\cdot) - \Gamma(t,\cdot,u_0)\|^2_{0,2}. \tag{10}$$

hold.

Since the proof of the above lemma is rather technical, it will be left until the Appendix.

5. Local existence

MAIN THEOREM. *Under conditions* (I)–(VI) *there exists a positive time* T_0 *and a constant* $\mu_0 \in (0,1)$ *such that the solution* v *of the problem* (2) *is estimated a priori in the Hölder space* $C^{2m+\mu_0, 1+\mu_0/(2m)}(\mathrm{cl}\, D^{T_0})$.

Before we deal with the Hölder norms stated in the above theorem, we will first derive the estimates of the "weaker" norms of the solution.

LEMMA 1 (First a priori estimate). *As long as* v *remains inside* Y

$$\|v(t,\cdot)\|^2_{0,2} \le C_7 t \exp(C_8 t), \tag{17}$$

where $C_7 = M^2|G|$ *and* $C_8 = 2C_2 + 1$.

Proof. Multiplying (2) by v and integrating over G we have

$$\frac{1}{2}\frac{d}{dt}\int v^2\,dx = -\int (Pv)v\,dx + \int g(t,x,d^{2m-1}v)v\,dx. \tag{12}$$

Because of the assumptions (i), (ii) equality (12) gives

$$\frac{1}{2}\frac{d}{dt}\int v^2\,dx \le -C_1\|v\|^2_{m,2} + C_2\|v\|^2_{0,2} + \int |g(t,x,d^{2m-1}v)||v|\,dx. \tag{13}$$

Since g is bounded on compact sets, then increasing the right-hand side of (13) we come out to the inequality

$$\frac{d}{dt}\int v^2\,dx \le M^2|G| + (2C_2+1)\|v\|^2_{0,2}, \tag{13}$$

which leads directly to (11). The proof of Lemma 1 is finished.

Now we give the L^2 estimate of the time derivative v_t.

LEMMA 2 (Second a priori estimate). *As long as* v *remains inside* Y

$$\|v_t(t,\cdot) - \Gamma(0,\cdot,u_0)\|^2_{0,2} \le C_9 t \exp(C_{10} t), \tag{15}$$

with $C_9 = |G|(M^2 + M_H^2 + M^2 C_3^2 d^2)$ *and* $C_{10} = 2C_2 + 3$ (Γ *as in* (1)).

Proof. Substitute in (6), $v_t = \omega = z + \Gamma(0,x,u_0)$. We get

$$(16)\qquad \begin{cases} z_t = -Pz + g_t(t,x,v) + \sum_{|\alpha|\le 2m-1} g_\alpha(t,x,v)D^\alpha z + H(t,x,v) \\ B_0 z = \ldots = B_{m-1}z = 0 \\ z(0,x) = 0, \end{cases}$$

where $H(t,x,v) = -P\Gamma(0,x,u_0) + \sum_{|\alpha|\le 2m-1} g_\alpha(t,x,v)D^\alpha(\Gamma(0,x,u_0))$. Multiplying the first equation in (16) by z and next integrating it over G we get

$$(17)\qquad \frac{1}{2}\frac{d}{dt}\int z^2\,dx = -\int (Pz)z\,dx + \int g_t(t,x,v)z\,dx + \int \sum_{|\alpha|\le 2m-1} g_\alpha(t,x,v)(D^\alpha z)z\,dx + \int H(t,x,v)z\,dx.$$

Next from (17), (4), (3), making use of the assumptions (i), (ii) and of the boundedness of Γ (M_H denotes upper bound for H) we find that

$$(18)\qquad \frac{1}{2}\frac{d}{dt}\int z^2\,dx \le -C_1\|z\|^2_{m,2} + C_2\|z\|^2_{0,2} + M\int |z|\,dx + MC_3 \sum_{|\alpha|\le 2m-1} \int |z|\,dx + M_H\int |z|\,dx.$$

Applying both Hölder and Cauchy inequalities, we obtain from (18) the estimate

$$(19)\qquad \frac{d}{dt}\int z^2\,dx \le M^2|G| + M_H|G| + M^2C_3^2d^2|G| + (2C_2+3)\|z\|^2_{0,2},$$

and from that we get immediately condition (15). Lemma 2 is thus proved.

So far we have done two important steps in the proof of the Main Theorem having obtained a priori bounds for $\|v(t,\cdot)\|_{0,2}$ and $\|v_t(t,\cdot)\|_{0,2}$. Next we shall derive analogous estimate for v_{tt}.

LEMMA 3 (Third a priori estimate). *As long as v stays inside Y*

$$(20)\qquad \|v_{tt}(t,\cdot)\|^2_{0,2} \le (\|L(0,\cdot,0,\Gamma(0,\cdot,u_0))\|^2_{0,2} + C_{11}t)\exp(C_{12}t),$$

where $C_{11} = ((1+C_3d)^4 + C_4^2d^2)M^2|G|$ *and* $C_{12} = 2C_2 + 2$.

Proof. Multiplying the first equation in (8) by ω_t and integrating it over G we get

$$(21)\qquad \frac{1}{2}\frac{d}{dt}\int (\omega_t)^2\,dx = -\int (P\omega_t)\omega_t\,dx + \sum_{|\alpha|\le 2m-1} \int g_\alpha(t,x,v)D^\alpha(\omega_t)\omega_t\,dx + \int \Lambda(t,x,v,w)\omega_t\,dx.$$

As long as v stays inside Y then applying the estimates, (3), (5), (21) and the assumptions (i), (ii), we find from (4) that

$$(22)\qquad \frac{1}{2}\frac{d}{dt}\int(\omega_t)^2\,dx \le -C_1\|\omega_t\|_{m,2}^2 + C_2\|\omega_t\|_{0,2}^2 + dC_4M\int|\omega_t|\,dx + (2C_3dM + M + C_3^2d^2M)\int|\omega_t|\,dx.$$

Similarly as in the proof of Lemmas 1 and 2 we further obtain

$$(23)\qquad \frac{d}{dt}\int(\omega_t)^2\,dx \le ((2C_3d + 1 + C_3^2d^2)^2 + C_4^2d^2)M^2|G| + (2C_2+2)\int(\omega_t)^2\,dx,$$

and also (since w_t corresponds to v_{tt}) we come out to (20). Lemma 3 is proved.

Now we are fully prepared to determine the time T_0 and Hölder constant μ_0 introduced in the formulation of the Main Theorem and complete its proof.

Proof of the Main Theorem. First we shall estimate (from below) a time T_0 until which each classical solution of the problem (2) remains inside Y. Let us note that as long as v is in Y, then Calderon–Zygmund estimate gives

$$(24)\qquad \begin{aligned}\|v(t,\cdot)\|_{2m,n+2} &\le C_{13}(\|v_t(t,\cdot) - g(t,\cdot,d^{2m-1}v)\|_{0,n+2} + \|v(t,\cdot)\|_{0,n+2})\\ &\le C_{13}\|v_t(t,\cdot)\|_{0,n+2} + C_{13}\|g(t,\cdot,d^{2m-1}0)\|_{0,n+2}\\ &\quad + C_{13}L\sum_{|\alpha|\le 2m-1}\|D^\alpha v\|_{0,n+2} + C_{13}\|v(t,\cdot)\|_{0,n+2}).\end{aligned}$$

where C_{13} is the constant in Calderon–Zygmund estimate.

Denoting by

$$C_{14} = C_{13}|G|^{\frac{1}{n+2}}\sup_{\operatorname{cl}D_T}\{|g(t,x,d^{2m-1}0)|\},\qquad C_{15} = C_{13}(L+1)$$

we can increase the right side of (24), and then by using Sobolev Inequality we obtain ($C_{16} = d\times$ (*embedding constant*))

$$(25)\qquad \begin{aligned}\|v(t,\cdot)\|_{2m-1,\infty} &\le C_{16}\|v(t,\cdot)\|_{2m,n+2}\\ &\le C_{13}C_{16}\|v_t(t,\cdot)\|_{0,n+2}\\ &\quad + C_{14}C_{16} + C_{15}C_{16}\|v(t,\cdot)\|_{2m-1,n+2}.\end{aligned}$$

Taking square of the both sides of (25) we get

$$(26)\qquad \begin{aligned}\|v(t,\cdot)\|_{2m-1,\infty}^2 &\le 3C_{13}^2C_{16}^2\|v_t(t,\cdot)\|_{0,n+2}^2\\ &\quad + 3C_{14}^2C_{16}^2 + 3C_{15}^2C_{16}^2\|v(t,\cdot)\|_{2m-1,n+2}^2.\end{aligned}$$

From estimates (9), (10), (26) we conclude that

$$(27)\quad \begin{aligned}&\|v(t,\cdot)\|^2_{2m-1,\infty}\\ &\le 6C_{13}^2C_{16}^2\theta(\|v_{tt}(t,\cdot)\|^2_{0,2}+C_6)\\ &\quad+6C_{13}^2C_{16}^2C_\theta\|v_t(t,\cdot)-\Gamma(0,\cdot,u_0)\|^2_{0,2}+6C_{13}^2C_{16}^2\|\Gamma(0,\cdot,u_0)\|^2_{0,n+2}\\ &\quad+3C_{14}^2C_{16}^2+3C_{15}^2C_{16}^2\nu(\|v_t(t,\cdot)\|^2_{0,2}+C_5)+3C_{15}^2C_{16}^2C_\nu\|v(t,\cdot)\|^2_{0,2}.\end{aligned}$$

Because of the inequalities (11), (15) condition (27) gives

$$(28)\quad \begin{aligned}&\|v(t,\cdot)\|^2_{2m-1,\infty}\\ &\le 6C_{13}^2C_{16}^2\theta\|L(0,\cdot,0,\Gamma(0,\cdot,u_0))\|^2_{0,2}\exp(C_{12}t)\\ &\quad+6C_{13}^2C_{16}^2\theta(C_{11}t\exp(C_{12}t)+C_6)+6C_9C_{13}^2C_{16}^2C_\theta t\exp(C_{10}t)\\ &\quad+6C_{13}^2C_{16}^2\sup_{x\in clG}{}^2\{|\Gamma(0,x,u_0)|\}|G|^{\frac{2}{n+2}}+3C_{14}^2C_{16}^2\\ &\quad+3C_{15}^2C_{16}^2\nu(2\|\Gamma(0,\cdot,u_0)\|^2_{0,2}+2C_9t\exp(C_{10}t)+C_5)\\ &\quad+3C_7C_{15}^2C_{16}^2C_\nu t\exp(C_8t).\end{aligned}$$

Let us note, that the right side of (28) increases exponentially with respect to t and its value in the initial moment $t=0$ is equal to

$$6C_{13}^2C_{16}^2\sup_{x\in clG}{}^2\{|\Gamma(0,x,u_0)|\}|G|^{\frac{2}{n+2}}+3C_{14}^2C_{16}^2$$
$$+6C_{13}^2C_{16}^2(\|L(0,\cdot,0,\Gamma(0,\cdot,u_0))\|^2_{0,2}+C_6)\theta+3C_{15}^2C_{16}^2(2\|\Gamma(0,\cdot,u_0)\|^2_{0,2}+C_5)\nu.$$

Moreover, the constants C_{13}, C_{14}, C_{16} are independent of the choice of R in Y. Hence, if we choose in the definition of Y sufficiently large constant R such that

$$(29)\quad R^2>36C_{13}^2C_{16}^2\sup_{x\in clG}{}^2\{|\Gamma(0,x,u_0)|\}|G|^{\frac{2}{n+2}}+18C_{14}^2C_{16}^2,$$

then the right-hand side of (28) for stated below values of parameters θ and v with

$$\theta=\min\left\{(3C_{13}^2C_{16}^2(\|L(0,\cdot,0,\Gamma(0,\cdot,u_0))\|^2_{0,2}+C_6))^{-1}\frac{R^2}{6},\theta_0\right\}$$

and

$$\nu=\min\left\{(3C_{15}^2C_{16}^2(\|\Gamma(0,\cdot,u_0)\|^2_{0,2}+C_5))^{-1}\frac{R^2}{6},\nu_0\right\},$$

will increase from the value not exceeding $R^2/2$ for $t=0$ to the value R^2 which will be reached in a positive time T_1. Hence taking

$$(30)\quad T_0=\min\{T,T_1\}$$

we are sure that the solution v of the problem (2) will remains inside Y at least until the time T_0 (thus all the estimates which have been derived so far hold almost until the time T_0). In particular, obtained estimates ensure that

$$(31)\quad D^\alpha v, \frac{\partial}{\partial t} D^\alpha v, \frac{\partial}{\partial x_1} D^\alpha v, \ldots, \frac{\partial}{\partial x_n} D^\alpha v \in L^{n+2}(D^{T_0}) \quad |\alpha| \leq 2m-1,$$

thus in consequence of (31) and Sobolev Inequality (in $n+1$ dimensional space)

$$(32)\qquad D^\alpha v \in C^{\frac{1}{n+2}, \frac{1}{n+2}}(\mathrm{cl}\, D^{T_0}) \quad |\alpha| \leq 2m-1.$$

Since g is Hölder continuous with respect to x (Hölder exponent is denoted by μ) and Lipschitz continuous with respect to both t and all functional arguments, then applying condition (32) we find that

$$(33)\qquad g^* \in C^{\mu_0, \mu_0}(\mathrm{cl}\, D^{T_0}),$$

where

$$\mu_0 = \min\left\{\mu, \frac{1}{n+2}\right\}, \qquad g^*(t,x) = g(t,x,d^{2m-1}v(t,x)).$$

Finally, making use of the linear theory stated in [11] (see Th. 10.1, Chapt. VII; with $l = \mu_0$, $s = 0$, $t = 2m$) we come out to the required property

$$(34)\qquad v \in C^{2m+\mu_0, 1+\frac{\mu_0}{2m}}(\mathrm{cl}\, D^{T_0}).$$

The proof of the Main Theorem is completed.

Estimates of the solution derived in the Main Theorem are sufficient to justify local solvability of (2). Since the proof of existence (based on the method of continuity) is very standard will be given here only an outline of it.

EXISTENCE THEOREM. *Under conditions* (I)–(VI) *there exists a unique classical solution of the problem* (2) *which belongs to the Hölder space*

$$C^{2m+\mu_0, 1+\mu_0/(2m)}(\mathrm{cl}\, D^{T_0}).$$

Proof. For the proof of uniqueness let us come back to the problem (1) and assume that u_1, u_2 are two different solutions of (1). It is clear that $U = u_1 - u_2$ solves

$$(35)\qquad \begin{cases} U_t = -PU + f(t,x,d^{2m-1}u_1) - f(t,x,d^{2m-1}u_2) & \text{in } D^{T_0} \\ B_0 U = \ldots = B_{m-1} U = 0 & \text{on } \partial G \\ U(0,x) = 0 & \text{in } G. \end{cases}$$

We can write the first equation in (35) as

$$U_t = -PU + f(t,x,d^{2m-1}u_1) - f\left(t,x,u_2,\frac{\partial u_1}{\partial x_1},\dots,\frac{\partial^{2m-1}u_1}{\partial x_n^{2m-1}}\right)$$
$$+ f\left(t,x,u_2,\frac{\partial u_1}{\partial x_1},\dots,\frac{\partial^{2m-1}u_1}{\partial x_n^{2m-1}}\right) + \dots - f(t,x,d^{2m-1}u_2).$$

Further, making use of the differentiability of the function f and the mean value theorem we can transform the last equation into "linear" form

$$U_t = -PU + \sum_{|\alpha|\le 2m-1} b_\alpha(t,x)D^\alpha U$$

where the coefficients b_α are given by

$$b_\alpha(t,x) = \frac{\partial f}{\partial(D^\alpha u)}\left(t,x,u_1,\dots,\zeta_\alpha(t,x),\dots\frac{\partial^{2m-1}u_2}{\partial x^{2m-1}}\right)$$

(ζ_α in the line above is on the place which has a number α). Since (what follows from obtained estimates) until T_0 each $D^\alpha u$ with $|\alpha| \le 2m-1$ is bounded, we are sure that all functions b_α ($|\alpha| \le 2m-1$) are bounded in D^{T_0}. Thus using again Th. 10.4 Chapt. VII of [11] we obtain required uniqueness property.

For the proof of existence let us define the Banach space Ξ

$$\Xi := \left\{\varphi : \mathrm{cl}\, D^{T_0} \to R;\ \sum_{|\alpha|\le 2m-1} \|D^\alpha\varphi\|_{\mu_0,\frac{\mu_0}{2}} < \infty\right\}$$

and the nonlinear operator $\Phi : \Xi \times [0,1] \to \Xi$, such that $v = \Phi(w,\lambda)$ iff v is a unique solution of

$$\begin{cases} v_t = -Pv + \lambda g(t,x,d^{2m-1}w) & \text{in } D^{T_0} \\ B_0 v = \dots = B_{m-1}v = 0 & \text{on } \partial G \\ v(0,x) = 0 & \text{in } G. \end{cases}$$

Solvability of the equation $\Phi(v,\lambda) = v$ is equivalent to solvability of the problem (2). Also, since the parameter λ belongs to the interval $[0,1]$, the Main Theorem which guarantees necessary *a priori* estimate of Hölder norms of the solution remains true for the whole family of parabolic problems

$$\begin{cases} v_t = -Pv + \lambda g(t,x,d^{2m-1}v) & \text{in } D^{T_0} \\ B_0 v = \dots = B_{m-1}v = 0 & \text{on } \partial G \\ v(0,x) = 0 & \text{in } G. \end{cases}$$

(let us note that because λ is limited to $[0,1]$ the constants in the estimates will not increase). Thus existence of a fixed point of the operator $\Phi(\cdot,1)$

can be justified by the standard use of the Leray–Schauder Principle. The theorem is thus proved.

6. The final example

We use the above theory to justify solvability of the problem

$$\begin{cases} u_t + \varepsilon u_{xxxx} + \gamma u_{xxx} + u u_x = 0, \ 0 < t \le T, \ 0 < x < 1, \ \varepsilon, \gamma > 0, \\ \frac{\partial^j u}{\partial x^j}(t,0) = \frac{\partial^j u}{\partial x^j}(t,1) \qquad j = 0,1,2,3, \\ u(0,x) = u_0(x) \qquad 0 < x < 1 \end{cases}$$

which is known as the parabolic regularization of the celebrated KdV's problem (cf [13], p. 363). Using our notation we shall specify the operator P appearing in (36) as $Pw = \varepsilon D^2(D^2 w)$ and the bilinear form p as $p(w,v) = \varepsilon D^2 w D^2 v$. Note that if the data of the problem (36) are sufficiently regular the assumptions **(II)–(IV)** are satisfied. Hence we need only to check Calderon-Zygmund estimate and stated in **VI** properties (i), (ii).

Because of the boundary conditions we can easily verify that

$$\int_0^1 D^j u(t,x)\,dx = D^{j-1}u(t,1) - D^{j-1}u(t,0) = 0 \qquad j = 1,2,3,4. \tag{37}$$

Moreover, Poincaré inequality ensures us that the expression

$$\|w\|_* = \sqrt{\int_0^1 (D^j w)^2\,dx + \int_0^1 w^2\,dx} \tag{38}$$

defines on $\{w \in W^{j,2} : D^k w(0) = D^k w(1), \ k = 0,\dots,j-1\}$ the norm which is equivalent to the standard $W^{j,2}$-norm. Thanks to (38) both coerciveness property and Calderon–Zygmund estimate are clearly satisfied. Furthermore, integrating by parts we find out that

$$\int_0^1 (Pu)u\,dx = \int_0^1 \varepsilon(D^4 u)u\,dx = \int_0^1 \varepsilon(D^2 u)^2\,dx = p(u,u),$$

which ensures required in **(VI)** Green's Identity. Since all needed assumptions are verified, thus our general result guarantees that the problem (36) has a unique local Hölder solution (in fact, as it was shown in [4], it is possible to justify the existence of the global solution for this problem).

7. Appendix

We shall give here the proof of the Preliminary lemma that has been formulated at the end of the paragraph 4.

P r o o f o f t h e P r e l i m i n a r y L e m m a. Let us choose α, with $|\alpha| \le 2m - 1$. From Nirenberg–Gagliardo, Young, and Cauchy inequalities we obtain

$$(39)\qquad \|D^\alpha v\|^2_{0,n+2} \le (C_{17}\|D^\alpha v\|^{\frac{9}{10}}_{1,2}\|D^\alpha v\|^{\frac{1}{10}}_{0,2})^2$$
$$\le C^2_{17}\left(\frac{9}{10}\varepsilon^{\frac{10}{9}}\|D^\alpha v\|_{1,2} + \frac{1}{10}\varepsilon^{-10}\|D^\alpha v\|_{0,2}\right)^2$$
$$\le \frac{\delta}{2}\|v\|^2_{2m,2} + C'_\delta\|D^\alpha v\|^2_{0,2}.$$

Next we find out that

$$(40)\qquad \|D^\alpha v\|^2_{0,2} \le |v|^2_{|\alpha|,2} \le \frac{\delta}{2C'_\delta}|v|^2_{2m,2} + C_{\delta,\alpha}|v|^2_{0,2},$$

where (according to [1], p. 75) $|v|_{l,2} = (\sum_{|\alpha|=l}\|D^\alpha v\|^2_{0,2})^{1/2}$. Inserting (40) to (39) we have further

$$(41)\qquad \|D^\alpha v\|^2_{0,n+2} \le \delta\|v\|^2_{2m,2} + C_{\delta,\alpha}C'_\delta\|v\|^2_{0,2},$$

and consequently

$$(42)\qquad \|D^\alpha v\|^2_{0,n+2} \le \delta\|v\|^2_{2m,2} + C_\delta\|v\|^2_{0,2},$$

with $C_\delta = \sum_{|\alpha|\le 2m-1} C'_\delta C_{\delta,\alpha}$. Using now Calderon-Zygmund estimate we get from (42) that

$$(43)\qquad \|D^\alpha v\|^2_{0,n+2} \le \delta C^2_{13}(\|Pv\|_{0,2} + \|v\|_{0,2})^2 + C_\delta\|v\|^2_{0,2}$$
$$\le 2\delta C^2_{13}\|Pv\|^2_{0,2} + (2\delta C^2_{13} + C_\delta)\|v\|^2_{0,2}.$$

Next, we shall estimate L^2-norm of Pv. Because of (2) and global Lipschitz continuity of g inside Y we have (C_5 stands for $|G|\sup_{(t,x)\in \mathrm{cl}\, D^T}$ $\{|g(t,x,d^{2m-1}0)|\}$)

$$(44)\qquad \|Pv\|^2_{0,2} = \|g(t,\cdot,d^{2m-1}v) - v_t\|^2_{0,2}$$
$$\le 3\|v_t\|^2_{0,2} + 3\|g(t,\cdot,d^{2m-1}0)\|^2_{0,2}$$
$$+ 3\|g(t,\cdot,d^{2m-1}v) - g(t,\cdot,d^{2m-1}0)\|^2_{0,2}$$
$$\le 3\|v_t\|^2_{0,2} + 3C_5 + 3dL^2\sum_{|\alpha|\le 2m-1}\|D^\alpha v\|^2_{0,2}.$$

From (43), (44) we find that

$$(45)\qquad \|D^\alpha v\|^2_{0,n+2} \le 6\delta C^2_{13}dL^2\sum_{|\alpha|\le 2m-1}\|D^\alpha v\|^2_{0,2} + 6\delta C^2_{13}\|v_t\|^2_{0,2}$$
$$+ 6\delta C_5C^2_{13} + (C_\delta + 2\delta C^2_{13})\|v\|^2_{0,2}.$$

Moreover, summing both sides of (45) with respect to α, with $|\alpha| \le 2m-1$

we get

$$(46)\quad \sum_{|\alpha|\le 2m-1} \|D^\alpha v\|^2_{0,n+2} \le 6\delta C_{13}^2 d^2 L^2 \sum_{|\alpha|\le 2m-1} \|D^\alpha v\|^2_{0,2} + 6\delta C_{13}^2 d \|v_t\|^2_{0,2} + 6\delta C_5 C_{13}^2 d + (C_\delta + 2\delta C_{13}^2) d \|v\|^2_{0,2}.$$

Now using inequality

$$\|D^\alpha v\|^2_{0,2} \le |G|^{\frac{n}{n+2}} \|D^\alpha v\|^2_{0,n+2}$$

and choosing δ_0 so small that

$$6\delta_0 C_{13}^2 d^2 L^2 |G|^{\frac{n}{n+2}} \le \frac{1}{2}$$

we obtain, for all $\delta \in (0, \delta_0]$

$$(47)\quad \|v\|^2_{2m-1,n+2} \le d \sum_{|\alpha|\le 2m-1} \|D^\alpha v\|^2_{0,n+2} \le 12\delta C_{13}^2 d^2 \|v_t\|^2_{0,2} + 12 C_5 C_{13}^2 d^2 + 2(C_\delta + 2\delta C_{13}^2) d^2 \|v\|^2_{0,2}.$$

Hence, substituting in (47) $\nu = 12\delta C_{13}^2 d^2$ and defining

$$(48)\qquad \nu_0 = (L^2 |G|^{\frac{n}{n+2}})^{-1}$$

we come finally to inequality (9).

To justify (10) we start from (16). Choosing arbitrary α, with $|\alpha| \le 2m-1$ and following the proof of inequality (9) (between formulas (39)–(43)) we obtain the estimate

$$(49)\qquad \|D^\alpha z\|^2_{0,n+2} \le 2\delta C_{13}^2 \|Pz\|^2_{0,2} + (2\delta C_{13}^2 + C_\delta) \|z\|^2_{0,2}.$$

From the first equation in (16) we have

$$(50)\quad \|Pz\|^2_{0,2} = \|g_t(t,\cdot,v) + \sum_{|\alpha|\le 2m-1} g_\alpha(t,\cdot,v) D^\alpha z + H(t,\cdot,v) - z_t\|^2_{0,2} \le (d+3)(\|z_t\|^2_{0,2} + \|g_t(t,\cdot,v)\|^2_{0,2} + \|H(t,\cdot,v)\|^2_{0,2} + \sum_{|\alpha|\le 2m-1} \|g_\alpha(t,\cdot,v) D^\alpha z\|^2_{0,2}).$$

The estimates (50), (3) give (let us recall that H, g_t, g_α are bounded on compact sets and upper bounds are denoted by the same constant M, except the bound for H which is denoted by M_H)

$$(51)\quad \|Pz\|^2_{0,2} \le (d+3)(\|z_t\|^2_{0,2} + M^2|G| + M_H^2 |G| + M^2 \sum_{|\alpha|\le 2m-1} \|D^\alpha z\|^2_{0,2}).$$

Putting (51) into (49) we obtain that

$$
\begin{aligned}
(52)\ \|D^\alpha z\|_{0,2}^2 \le\ & 2\delta C_{13}^2(d+3)M^2 \sum_{|\alpha|\le 2m-1} \|D^\alpha z\|_{0,2}^2 \\
& + 2\delta C_{13}^2(d+3)\|z_t\|_{0,2}^2 \\
& + 2\delta C_{13}^2(d+3)(M^2+M_H^2)|G| + (2\delta C_{13}^2 + C_\delta)\|z\|_{0,2}^2.
\end{aligned}
$$

Since inequality (52) is fully analogous to (45), then the rest of the proof follows exactly in the same way as in the case of the estimate (9) (between formulas (46)–(48)). Our considerations are completed.

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