DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 4 1995

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ON A NONEXISTENCE OF POSITIVE SOLUTION OF LAPLACE EQUATION IN UPPER HALF-SPACE WITH CAUCHY DATA

Dedicated to Professor Janina Wolska-Bochenek

We study the following problem for Laplace equation in cylinder coordinates

(1) $u_{rr} + \frac{1}{r}u_r + u_{zz} = 0,\ r > 0,\ z > 0,$

(2) $u_z(r,0) + g(r, u(r,0)) = 0,\ r \geq 0,$

where the given function g is continuous, nondecreasing and bounded below by the power function of order α with respect to the second variable.

By establishing a suitable recurrent sequence, we prove that, if $0 < \alpha \leq 2$, the problem (1), (2) has no positive solution.

1. Introduction

We consider the following Laplace equation in cylinder coordinates

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0,\ r > 0,\ z > 0, \tag{1.1}$$

with nonlinear boundary condition

$$u_z(r,0) + g(r, u(r,0)) = 0,\ r \geq 0. \tag{1.2}$$

In [1], the authors have studied the problem (1.1)-(1.2) in case of

$$g(r,u) = I_0 \cdot \exp(-r^2/r_0^2) + u^\alpha, \tag{1.3}$$

where α, I_0, r_0 are given positive constants. The problem (1.1)-(1.3) is associated with ignition by radiation. In the case of $0 < \alpha \leq 2$, the authors in [1] have proved that the problem (1.1)-(1.3) has no positive solution.

We consider Laplace equation (1.1) with the Cauchy conditions

$$u(r,0) = u_0(r), \; u_z(r,0) = u_1(r). \tag{1.4}$$

Problem (1.1), (1.4) is an ill-posed problem in the sense of Hadamard [2].

In this article we want to indicate a relation between the Cauchy data u_0 and u_1 in the form

$$u_1(r) + g(r, u_0(r)) = 0, \; \forall r \geq 0, \tag{1.5}$$

in order to show that the problem (1.1), (1.2) has no positive solution.

2. Hypotheses and statement of theorem

First, we use the notation $R_+ = \{x/ \; x \geq 0\}$ and admit following hypotheses for the function $g : R_+ \times R_+ \to R$:

(H_1) g is continuous,

(H_2) g is nondecreasing with respect to the second variable, i.e.,

$$(g(r,u) - g(r,v))(u - v) \geq 0, \; \forall r, u, v \in R_+,$$

(H_3) $\exists \alpha > 0$, $\exists C_1 > 0$: $g(r,u) \geq C_1 u^\alpha$, $\forall r, u \in R_+$,

(H_4) $\exists r_0 > 0$ such that the integral

$$\int_0^\infty \frac{s.g(r_0 s, 0)}{1+s} ds$$

exists and is positive.

Remark 1. The function g in (1.3) satisfies the hypotheses (H_1)-(H_4).

The main result of this paper is as follows.

THEOREM. *Suppose that* $0 < \alpha \leq 2$ *and the function* g *satisfies the conditions* (H_1)-(H_4). *Then the problem* (1.1)-(1.2) *has no positive solution.*

3. Proof of theorem

Idea of proof is to establish a functional sequence $\{u_n(r)\}$ such that: if the positive solution $u(r,z)$ of the problem (1.1), (1.2) exists, then the sequence $\{u_n(r)\}$ is monotonic, increasing and bounded above by $u(r,0)$. Hence, $\lim_{n\to\infty} u_n(r)$ exists (pointwise) for every $r \geq 0$ and we have

$$\lim_{n\to\infty} u_n(r) \leq u(r,0), \quad \forall r \geq 0. \tag{3.1}$$

Afterwards, we prove that there is such n that $u_n(r) = +\infty$ for every $r \geq 0$ or there is $r > 0$ (enough large) such that $\lim_{n\to\infty} u_n(r) = +\infty$. Hence, the theorem is proved.

First, the problem (1.1), (1.2) is equivalent to

$$u(r,z) = A[g(s, u(s,0))](r,z), \tag{3.2}$$

where linear operator A is defined by formula

$$(3.3) \qquad A[v(s)](r,z) = \int_0^{+\infty} sv(s)ds \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\sqrt{r^2+s^2+z^2-2rs\cos t}}.$$

We substitute $z = 0$ in (3.2), (3.3) and use notation $u(r,0) \equiv u(r)$. Then we have

$$(3.4) \qquad u(r) = A[g(s,u(s))](r)$$

with

$$(3.5) \qquad A[g(s,u(s))](r) = \int_0^{+\infty} sg(s,u(s))ds \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\sqrt{r^2+s^2-2rs\cos t}}.$$

We need some lemmas.

LEMMA 1. (*cf.*[1]):

(3.6) *If* $0 < \alpha \leq 1$, *we have:* $A[(s+r_0)^{-\alpha}](r) = +\infty, \ \forall r \geq 0,$

(3.7) *If* $\alpha \geq 1$, *we have:* $A[(s+r_0)^{-\alpha}](r) \geq \dfrac{(r+r_0)^{1-\alpha}}{2(\alpha-1)}, \ \forall r \geq 0,$

(3.8) *If* $\alpha = 2$, *then* $A[(s+r_0)^{-2}](r) \geq \dfrac{1}{4r} \ln\left(1+\dfrac{r}{r_0}\right), \ \forall r > 0.$

Without loss of generality, we can suppose that $r_0 = 1$.

The following lemma is necessary to construct functional sequence $\{u_n(r)\}$.

LEMMA 2. *Putting*

$$(3.9) \qquad m_1 = \int_0^{\infty} \frac{s.g(s,0)}{1+s} ds,$$

we have

$$(3.10) \qquad A[g(s,0)](r) \geq m_1 \cdot (r+1)^{-1}, \ \forall r \geq 0.$$

Proof. Remark that

$$(3.11) \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{\sqrt{r^2+s^2-2rs.\cos t}} \geq \frac{1}{r+s}, \ \forall r,s > 0.$$

We obtain

$$(3.12)\qquad A[g(s,0)](r) \geq \int_0^{+\infty} sg(s,0)\frac{ds}{r+s}$$

$$= \frac{1}{r+1}\int_0^{+\infty} \frac{(r+1)(1+s)}{r+s}\cdot\frac{s.g(s,0)}{1+s}ds,$$

where $(r+1)(1+s) \geq r+s$, for all $r,s \geq 0$. From (3.12) we obtain (3.10), which ends the proof of Lemma 2.

Afterwards, we construct recurrent sequence as follows:

$$(3.13)\qquad u_1(r) = m_1\cdot(r+1)^{-1},\ r\geq 0,$$

$$(3.14)\qquad u_{n+1}(r) = A[g(s,u_n(s))](r),\ r\geq 0,\ n\geq 1.$$

We suppose that the problem (1.1), (1.2) has positive solution $u(r,z)$. Then, by proving recurrently and using the monotony of operator A, we have the following results:

$\{u_n\}$ is an increasing sequence, i.e.,

$$(3.15)\qquad u_n(r) \leq u_{n+1}(r),\ \forall n\geq 1,\ \forall r\geq 0;$$

$\{u_n\}$ is bouned above, i.e.,

$$(3.16)\qquad u_n(r)\leq u(r),\ \forall n\geq 1,\ \forall r\geq 0.$$

Hence, $\{u_n\}$ converges pointwise satisfying the inequality

$$(3.17)\qquad \lim_{n\to\infty} u_n(r)\leq u(r),\ \forall r\geq 0.$$

LEMMA 3. *If g satisfies* (H_3) *with* $0<\alpha\leq 1$, *then*

$$(3.18)\qquad u_2(r) = +\infty,\ \forall r\geq 0,$$

and the problem (1.1), (1.2) *has no positive solution.*

Proof. By hypothesis (H_3) and (3.13), (3.14), (3.17), we obtain

$$u(r)\geq u_2(r) = A[g(s,u_1(s))](r)\geq A[C_1u_1^\alpha(s)](r)$$
$$= C_1m_1^\alpha A[(s+1)^{-\alpha}](r) = +\infty$$

for $0<\alpha\leq 1$.

LEMMA 4. *If g satisfies* (H_3) *with* $1<\alpha<2$, *then there exists* $N = N(\alpha)>1$ *such that*

$$(3.19)\qquad u_{N+1}(r) = +\infty,\quad \forall r\geq 0.$$

Hence we deduce that the problem (1.1), (1.2) *has no positive solution.*

Proof. By Lemma 3 and (3.7), we obtain

$$u_2(r) \geq C_1 m_1^\alpha A[(s+1)^{-\alpha}](r) \geq m_2(r+1)^{-\lambda_2}, \ \forall r \geq 0, \tag{3.20}$$

with

$$\lambda_2 = \alpha - 1, \quad m_2 = C_1 \frac{m_1^\alpha}{2\lambda_2}. \tag{3.21}$$

By recurrence, we can prove that, if

$$u_{n-1}(r) \geq m_{n-1}(r+1)^{-\lambda_{n-1}}, \quad \forall r \geq 0, \tag{3.22}$$

and

$$\alpha\lambda_{n-1} > 1, \tag{3.23}$$

then

$$u_n(r) \geq m_n(r+1)^{-\lambda_n}, \quad \forall r \geq 0, \tag{3.24}$$

with

$$\lambda_n = \alpha\lambda_{n-1} - 1, \quad m_n = C_1 \frac{m_{n-1}^\alpha}{2\lambda_n}. \tag{3.25}$$

From (3.25) we deduce that

$$\lambda_n = \frac{1-(2-\alpha)\alpha^{n-1}}{\alpha - 1}, \quad \forall n \geq 1. \tag{3.26}$$

Since $1 < \alpha < 2$, we can choose a natural number N, depending on α, such that

$$\frac{-\ln(2-\alpha)}{\ln\alpha} \leq N < 1 - \frac{\ln(2-\alpha)}{\ln\alpha},$$

i.e., such N that

$$0 < \alpha\lambda_N \leq 1. \tag{3.27}$$

On the other hand, using the hypotheses on function g and (3.6), we obtain

$$u_{N+1}(r) \geq A[C_1 u_N^\alpha(s)](r) \geq C_1 m_N^\alpha A[(s+1)^{-\alpha\lambda_N}](r) = +\infty.$$

LEMMA 5. *Putting $\alpha = 2$ and*

$$v_k(r) = \begin{cases} 0, & 0 \leq r \leq 1, \\ D_k \frac{1}{r}(\ln r)^{2^{k-2}}, & r \geq 1, \end{cases} \tag{3.28}$$

where the constant D_k is defined by

$$D_k = C_1^{2^{k-1}-1}\left(\frac{m_1}{2}\sqrt{\ln 2}\right)^{2^{k-1}} \cdot \frac{1}{\ln 2}, \tag{3.29}$$

we have

$$u_k(r) \geq v_k(r), \quad \forall r \geq 0, \ \forall k \geq 2. \tag{3.30}$$

Proof. We prove the inequality (3.30) with $k = 2$. In effect, by (3.8), we have

$$u_2(r) \geq C_1 m_1^2 A[(s+1)^{-2}](r) \geq C_1 m_1^2 \frac{1}{4r} \ln(1+r), \quad \forall r > 0. \tag{3.31}$$

Substituting $k = 2$ in (3.29), we have $D_2 = (C_1 m_1^2)/4$. From (3.28), (3.31) we deduce

$$u_2(r) \geq v_2(r), \quad \forall r \geq 0. \tag{3.32}$$

Suppose that (3.30) is true with $k \geq 2$, then it is clear that

$$u_{k+1}(r) \geq 0, \quad \forall r \geq 0, \tag{3.33}$$

and for $r \geq 1$

$$\begin{aligned} u_{k+1} &\geq C_1 A[u_k^2(s)](r) \geq C_1 A[v_k^2(s)](r) \\ &\geq \int_r^\infty C_1 s v_k^2(s) \frac{ds}{s+r} \geq C_1 D_k^2 \int_r^\infty \frac{1}{s} (\ln s)^{2^{k-1}} \frac{ds}{s+r} \\ &\geq C_1 D_k^2 (\ln r)^{2^{k-1}} \int_r^\infty \frac{ds}{s(r+s)} = C_1 D_k^2 (\ln r)^{2^{k-1}} \cdot \frac{\ln 2}{r}. \end{aligned} \tag{3.34}$$

Since $C_1 D_k^2 \cdot \ln 2 = D_{k+1}$, we have

$$u_{k+1}(r) \geq D_{k+1} \frac{1}{r} (\ln r)^{2^{k-1}}, \quad \forall r \geq 1. \tag{3.35}$$

From (3.33) and (3.35) we have

$$u_{k+1}(r) \geq v_{k+1}(r), \quad \forall r \geq 0,$$

which proves that Lemma 5 is true.

Remark 2. In the case of $\alpha = 2$, this Lemma gives an estimate (3.30) simpler than in [1], where $v_k(r)$ is given in a form of functional series.

From the result of Lemma 5, we can rewrite (3.30) for $r \geq 1$ in the form

$$u_k(r) \geq v_k(r) = \frac{1}{C_1 r \cdot \ln r} \left(\frac{C_1 m_1}{2} \sqrt{\ln 2 \cdot \ln r} \right)^{2^{k-1}}.$$

Choosing r such that $\frac{C_1 m_1}{2} \sqrt{\ln 2 \cdot \ln r} > 1$, i.e.,

$$r > \exp[4/(C_1^2 m_1^2 \cdot \ln 2)] = r_1, \tag{3.36}$$

we get

$$\lim_{k \to +\infty} u_k(r) = +\infty, \quad \forall r > r_1,$$

hence the theorem is proved completely.

Acknowledgement. The authors would like to thank the referee for his corrections and suggestions.

References

[1] F.V. Bunkin, V.A. Galaktionov, N.A. Kirichenko, S.P. Kurdyumov, A.A. Samarsky, *On a nonlinear boundary value problem of ignition by radiation*, J. Comp. Math. Phys. 28 (1988), 549–559. (in Russian).

[2] A. Tikhonov, V. Arsénine, *Méthodes de résolution de problèmes mal posés*, Editions Mir. 1976, (traduction française).

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Received July 21st, 1993; revised version February 20, 1995.