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A NOTE ON AN INEQUALITY SIMILAR TO LYAPUNOV'S INEQUALITY

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

Lyapunov [10] (see also [8] p. 345) proved that, if $y(t)$ is a nontrivial solution of

$$y''(t) + q(t)y(t) = 0 \tag{1}$$

on an interval containing the points a, b $(a < b)$ and such that $y(a) = y(b) = 0$, the function q being real and continuous, then

$$(b-a) \int_a^b |q(s)ds| > 4. \tag{2}$$

Many generalizations and applications of this result of Lypunov can be found in [1], [2], [6]-[12] and in the papers cited therein. The main purpose of this note is to provide an inequality similar to (2) for the differential equation

$$(|y'(t)|^{p-2}y'(t))' + q(t)|y(t)|^{p-2}y(t) = 0, \quad p > 1, \tag{3}$$

where $t \in I = [0, \infty)$ and I contains the points a, b $(a < b)$, the function q is real-valued and continuous on I. The problems of existence, uniqueness and other properties of solutions to equations of the form (3) are recently studied in [3]-[5]. In what follows it is assumed that solutions of (3), and also of some generalizations of (3), exist on I.

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2. Main results

THEOREM 1. *Let $y(t)$ be a solution of (3) with $y(a) = y(b) = 0$, and $y(t) \neq 0$ for $t \in (a,b)$. Let $|y(t)|$ be maximized in a point $c \in (a,b)$. Then*

$$1 \le 2^{-p}(b-a)^{p-1} \int_a^b |q(s)|ds, \tag{4}$$

$$1 \le (c-a)^{p-1} \int_a^c |q(s)|ds, \tag{5}$$

$$1 \le (b-c)^{p-1} \int_c^b |q(s)|ds. \tag{6}$$

Proof. Let $M = \max|y(t)| = |y(c)|$, $c \in (a,b)$. By assumptions, M is a positive constant. Since $y(a) = y(b) = 0$, we have the inequalities

$$M = |y(c)| = \Big|\int_a^c y'(s)ds\Big| \le \int_a^c |y'(s)|ds, \tag{7}$$

$$M = |y(c)| = \Big|-\int_c^b y'(s)ds\Big| \le \int_c^b |y'(s)|ds, \tag{8}$$

implying

$$M \le \frac{1}{2}\int_a^b |y'(s)|ds. \tag{9}$$

By taking p-th power on both sides of (9), applying Hölder's inequality with indices $p, \frac{p}{p-1}$, integrating by parts and using the fact that $y(t)$ is a solution of (3) such that $y(a) = y(b) = 0$, we have

$$\begin{aligned} M^p &\le 2^{-p}(b-a)^{p-1}\int_a^b |y'(s)|^p ds \\ &= 2^{-p}(b-a)^{p-1}\int_a^b (|y'(s)|^{p-2}y'(s))y'(s)ds \\ &= 2^{-p}(b-a)^{p-1}\Big\{-\int_a^b (|y'(s)|^{p-2}y'(s))'y(s)ds\Big\} \\ &= 2^{-p}(b-a)^{p-1}\Big\{\int_a^b (q(s)|y(s)|^{p-2}y(s))y(s)ds\Big\} \end{aligned} \tag{10}$$

$$\leq 2^{-p}(b-a)^{p-1} \int_a^b |q(s)||y(s)|^p ds$$

$$\leq 2^{-p}M^p(b-a)^{p-1} \int_a^b |q(s)|ds.$$

Now, dividing both sides of (10) by M^p, we get (4). Inequalities (5), (6) follow in similar fashion.

COROLLARY 1. *Assume that the hypotheses of Theorem 1 hold. Then*

(i) *the inequality* (4) *yields the lower bound on the distance between the consecutive zeros of the nontrivial solution* $y(t)$ *of* (3) *by means of an integral measurement of* $|q(t)|$,

(ii) *the inequalities* (5), (6) *yield the lower bounds that relate to the integral measurement of* $|q(t)|$ *not only the points* a, b *at which the nontrivial solution* $y(t)$ *of* (3) *vanishes but also the point* $c \in (a,b)$ *at which* $|y(t)|$ *is maximized.*

Remark 1. We note that in [12] are established the inequalities of the forms (4)-(6) when $p = 2$ by using different analysis.

We next establish an inequality similar to Lyapunov's inequality for higher order differential equation of the form

$$(11) \qquad (|y'(t)|^{p-2}y'(t))^{(n-1)} + q(t)|y(t)|^{p-2}y(t) = 0, \quad p > 1,\ n \geq 3,$$

with the same function q as in equation (3). We shall use the following notation

$$(*) \qquad E(t, h(s)) = \int_t^{\alpha_1} \int_{s_2}^{\alpha_2} \cdots \int_{s_{n-2}}^{\alpha_{n-2}} h(s)ds\, ds_{n-2} \ldots ds_3\, ds_2,$$

with a real-valued nonnegative continuous function h on I and $\alpha_1, \alpha_2, \ldots$ $\ldots, \alpha_{n-2}$ suitable points in I. We denote by $\overline{E}(t, h(s))$ the integral on the right-hand side of $(*)$, when the upper limits $\alpha_1, \alpha_2, \ldots, \alpha_{n-2}$ of integrals are all replaced by the greatest number from α_i, $i = 1, 2, \ldots, n-2$.

THEOREM 2. *Let* $\alpha_1 > \alpha_2 > \ldots > \alpha_{n-2}$ *be, respectively, zeros of*

$$(|y'(t)|^{p-2}y'(t))', (|y'(t)|^{p-2}y'(t))'', \ldots, (|y'(t)|^{p-2}y'(t))^{(n-2)},$$

where $y(t)$ *is a solution of* (11), *let* $a < \alpha_{n-2}$ *and* $b > \alpha_1$ *be zeros of* $y(t)$, *and* $|y(t)|$ *be maximized in* $c \in (a,b)$. *Then*

$$(12) \qquad 1 \leq 2^{-p}(b-a)^{p-1} \int_a^b \overline{E}(s_1, |q(s)|)ds_1,$$

(13) $$1 \le (c-a)^{p-1} \int_a^c \overline{E}(s_1, |q(s)|)ds_1,$$

(14) $$1 \le (b-c)^{p-1} \int_c^b \overline{E}(s_1, |q(s)|)ds_1.$$

Proof. Integrating $n-2$ times the equation (11), by the hypotheses, we get

(15) $$(-1)^{n-2}(|y'(t)|^{p-2}y'(t))' + E(t, q(s)|y(s)|^{p-2}y(s)) = 0.$$

Then following the proof of Theorem 1 with suitable modifications and using the fact that, by the hypotheses, the solution of (11) satisfies the equivalent integral equation (15), we get (12)-(14).

Remark 2. We note that the inequalities (4)-(6), (12)-(14) can very easily be extended to the following more general equations

(16) $$(r(t)|y'(t)|^{p-2}y'(t))' + q(t)|y(t)|^{p-2}y(t) = 0, \ p > 1,$$

(17) $$(r(t)|y'(t)|^{p-2}y'(t))^{(n-1)} + q(t)|y(t)|^{p-2}y(t) = 0, \ p > 1,$$

where $n \ge 3$, the function r is real-valued positive and continuous on I, and q is the same as in (3). For similar results related to various types of equations we refer to [1], [2], [11].

3. An Application

THEOREM 3. *Suppose that* $|q(t)| \in L^{\alpha}[0,\infty)$, $1 \le \alpha < \infty$. *If* $y(t)$ *is any oscillatory solution of* (3), *then the distance between consecutive zeros of* $y(t)$ *tends to infinity as* $t \to \infty$.

Proof. Assume that the conclusion is not true. Then there exists a solution $y(t)$ with its sequence of zeros $\{t_n\}$ having a subsequence $\{t_{n_k}\}$ such that $|t_{n_{k+1}} - t_{n_k}| \le N < \infty$ for all k. Let s_{n_k} be a point in $(t_{n_k}, t_{n_{k+1}})$ at which $|y(t)|$ is maximized. Then $|s_{n_k} - t_{n_k}| < N$ for all k. Let β be the index conjugate with α, namely $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Since $|q(t)| \in L^{\alpha}[0,\infty)$, $1 \le \alpha < \infty$, for k large enough we can write

(18) $$\left(\int_{t_{n_k}}^{\infty} |q(s)|^{\alpha} ds\right)^{\frac{1}{\alpha}} \le N^{1-p-\frac{1}{\beta}}.$$

By (5), we have

(19) $$1 \le (s_{n_k} - t_{n_k})^{p-1} \int_{t_{n_k}}^{s_{n_k}} |q(s)|ds.$$

Using (18), (19) and the Hölder inequality with indices α, β, we get the inequality

$$
(20) \qquad 1 \le (s_{n_k} - t_{n_k})^{p-1} \Big(\int_{t_{n_k}}^{s_{n_k}} |q(s)|^\alpha ds \Big)^{\frac{1}{\alpha}} (s_{n_k} - t_{n_k})^{\frac{1}{\beta}}
$$

$$
\le (s_{n_k} - t_{n_k})^{p-1+\frac{1}{\beta}} \Big(\int_{t_{n_k}}^{\infty} |q(s)|^\alpha ds \Big)^{\frac{1}{\alpha}}
$$

$$
< N^{p-1+\frac{1}{\beta}} \cdot N^{1-p-\frac{1}{\beta}} = 1,
$$

being a contradiction and the proof is complete.

Remark 3. We note that in [1] and [2] were established the Lyapunov type inequalities for the equations

$$
(21) \quad (r(t)h(y'(t)))^{(n-1)} + a(t)y(t)f(y(t-\tau(t))) = b(t),
$$

$$
(22) \quad L_n x(t) + \sum_{i=1}^{m} p_i(t)x(t)f_i(x(t)) = q(t),
$$

and used to ensure that the oscillatory solution tends to zero as $t \to \infty$ (see, [1], Theorem 2) and that the distance between consecutive zeros of the solutions becomes unbounded (see, [2], Theorem 2). Here it is to be noted that the results similar to those given in [1], [2] can be very easily extended for the equation (11) by using the Lyapunov type inequalities (12)-(14). The precise formulations of these results and their proofs are entirely similar to those in [1], [2] with suitable modifications.

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