DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 4 1995

Grzegorz Bartuzel, Andrzej Fryszkowski

A TOPOLOGICAL PROPERTY OF THE SOLUTION SET TO THE STURM–LIOUVILLE DIFFERENTIAL INCLUSIONS

Dedicated to Professor Janina Wolska-Bochenek

Introduction

Differential inclusions of the form $\mathcal{P}u(t) \in \mathcal{F}(t, u(t))$, where $\mathcal{P}$ is a differential operator, are immediate generalization of the differential equations. The theory of properties of ordinary differential inclusions of the first order has been thriving since the early seventies and a lot is known on the existence of solutions and on their properties both in the framework of the Euclidean space $\mathbf{R}^n$ as well as in the framework of the Banach space X. In general differential inclusions with ordinary differential operator s of the higher order are much less examined although a remarkable amount of interest in this field has been observed lately.

The present paper deals with some topological properties of the solution set to the inclusion

$$-\frac{d^2x(t)}{dt^2} \in \mathcal{F}(t, x(t)) \tag{1}$$

with boundary conditions

$$x(0) = 0 = x(\pi), \tag{2}$$

where $t \in (0; \pi)$, and multifunctions $\mathcal{F}(t, \cdot)$ are Lipschitz with compact but not necessarily convex values in the real line $\mathbf{R}$. We prove that the set of solutions is a retract of the Sobolev space of the type $W^{2,1}$ with a weight ρ. The role of the weight ρ is similar to the role of the Bielecki norm and reduces the problem to the result by Bressan–Cellina–Fryszkowski [4] on the fixed point set for a contraction with decomposable values in $L^1(\rho)$. In section 2 we construct the appropiate weight ρ taking into account the theory of the Sturm–Liouville equation

$$-\frac{d^2x}{dt^2} - m(t)x = \lambda x, \qquad t \in T. \tag{3}$$

The main result is formulated and proved in the section 3 while the section 1 contains preliminary facts and definitions which are needed in this paper.

Let us observe that the Sturm–Liouville equation (3) is a particular case of the problem (1) with $\mathcal{F}(t,z) = \{\lambda z + m(t)z\}$. In this sense the Sturm–Liouville theory of the equation (3) is carried on the diferential inclusions case.

Preliminaries

Let $T = (0,\pi)$, $\mathcal{L}$ be the σ–algebra of Lebesgue measurable subsets of T. The spaces of functions integrable with p-th power on T for $1 \le p \le \infty$, equipped with the usual norms $|x|_p$, we shall denote by L^p and let $W^{m,p}$ and $W_0^{m,p}$ be Sobolev spaces endowed with the usual norms $|x|_{m,p}$.

Any non-negative function $0 \not\equiv \rho \in L^\infty$ will be called a weight.

For a given weight ρ by the weighted Lebesgue space $L^p(\rho)$ we mean the space of all functions x such that

$$x\rho^{1/p} \in L^p \qquad \text{with the norm } \|x\|_\rho = |x\rho^{1/p}|_p.$$

For $p = 1$ we shall also consider the space

$$W^{s,1}(\rho) := \left\{ x \in L^1(\rho) : \frac{d^\ell x}{dt^\ell} \in L^1(\rho),\ \ell \le s \right\}$$

with the norm

$$\|x\|_{\rho,s,1} := \|x\|_\rho + \sum_{1 \le \ell \le s} \left\| \frac{d^\ell x}{dt^\ell} \right\|_\rho$$

and the space $W_0^{s,1}(\rho)$ which is the completion of the $C_0^\infty(T)$ in the norm $\|\cdot\|_{\rho,s,1}$. Now the embedding $i : L^1 \to L^1(\rho)$ is continuous. It is clear that in case when ρ and ρ^{-1} are the weights then just introduced weighted spaces are isomorphic to L^p, $W^{s,1}$, $W_0^{s,1}$ respectively.

Let us fix a function $w \in L^1$ and consider the Sturm–Liouville problem

$$L_w x \equiv -\frac{d^2x}{dt^2} - wx = 0, \qquad \text{with } x(0) = 0 = x(\pi). \tag{4}$$

The operator L_w, called the Sturm–Liouville operator, satisfies

$$\langle L_w x, z\rangle = \int_0^\pi (x'(t)z'(t) - w(t)x(t)z(t))\,dt \tag{5}$$

for all $x \in W^{2,1}$ and $z \in W_0^{1,\infty}$. Denote the bilinear form on the right hand side of (5) by $a_w(x,z)$ and observe that it can be uniquely extended to a bounded bilinear form on $W_0^{1,p} \times W_0^{1,q}$ with $\frac{1}{p} + \frac{1}{q} = 1$. The spectral theory of the operator L_w as well as of the problem (3) with the Dirichlet boundary conditions is well developed. In particular, by using the Prüffer transform there exists a sequence $\lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots$, $\lim_{i\to\infty} \lambda_i = \infty$ of eigenvalues i.e. of such reals that the problems

$$(6) \qquad -\frac{d^2u}{dt^2} - w(t)u = \lambda_i u, \qquad u(0) = 0 = u(\pi),\ i = 0,1,\ldots$$

have not vanishing solutions. Moreover, we may assume that the eigenfunction ψ_0 corresponding to the smallest eigenvalue λ_0, called also the principal eigenvalue, is positive in the interval T and the corresponding eigenspace is one-dimensional (see [2], [5], [6], [9], [11], [12]).

The simplest Sturm–Liouville operator is the operator $L_0 x = -\dfrac{d^2x}{dt^2}$ (for $w = 0$). The equation

$$(7) \qquad L_0 x = u$$

with the boundary conditions (2)

$$(8) \qquad x(0) = 0 = x(\pi).$$

has a solution $x = Au \in W^{2,1} \cap W_0^{1,2}$ for any $u \in L^1$. This solution is expressed by the formula

$$(9) \qquad Au(t) = \int_0^\pi G_0(t,s)u(s)\,ds$$

where $G_0(t,s) \geq 0$ is the corresponding Green function.

The operator $A : L^1 \to W^{2,1}$ is linear, and bounded, and positive, i.e. for any function $u \leq 0$ we have $Au \leq 0$. In particular, it means that for any $u \in L^1$ the following estimate

$$(10) \qquad |Au(t)| \leq A(|u|)(t) \quad \text{a.e. in } \overline{T}$$

holds.

The problem (4) is a special case of the differential inclusion

$$(11) \qquad -\frac{d^2u}{dt^2} - wu \in \mathcal{F}(t,u) \qquad u(0) = 0 = u(\pi)$$

for a given multifunction $\mathcal{F}$.

By a $W^{1,p}$-weak solution u (for all $p \in [1;+\infty]$) of the problem (11) (and also (4)) we mean a function $u \in W^{1,p}(T)$ such that there exists

$z \in L^1(T)$ such that $z(t) \in \mathcal{F}(t,u(t))$ a.e. and such that $a_w(u,\varphi) = \langle z,\varphi\rangle$ for all $\varphi \in W_0^{1,q}(T)$, where $\frac{1}{p}+\frac{1}{q} = 1$. In other words $L_0 : W_0^{1,p}(T) \to W^{-1,q}(T)$ is in the coincidence with the multifunction $\mathcal{K}_{\mathcal{F}}$. [Here $\mathcal{K}_{\mathcal{F}}(u) = \{z \in L^1 : z(t) \in \mathcal{F}(t,u(t))\}$].

The above definition is equivalent to the following one of the Caratheodory solution of the problem

$$-\frac{d^2u}{dt^2} - wu \in \mathcal{F}(t,u), \qquad u(0) = 0 = u(\pi)$$

i.e. the continuous function u such that there exist a $v \in L^1(T)/\mathbf{R}$ (it means that $\int_0^\pi v(t)\,dt = 0$) and L^1-selection $z(t) \in w(t)u(t) + \mathcal{F}(t,u(t))$ a.e. with the representations

$$u(t) = -\int_0^t v(s)\,ds \quad \text{and} \quad v(t) - \int_0^t z(s)\,ds \in \mathbf{R} \text{ a.e.}$$

Let us consider the multifunction $\mathcal{F}(t,x)$ satisfying the following hypotheses:

(H1) the sets $\mathcal{F}(t,x)$ are compact subsets of $\mathbf{R}$ for any $t \in \overline{T}$ and $x \in \mathbf{R}$, and the multifunction s $t \mapsto \mathcal{F}(t,x)$ are measurable for any $x \in \mathbf{R}$;

(H2) there exists $m \in L^1$ such that for any $x, y \in \mathbf{R}$ we have

$$d_H(\mathcal{F}(t,x),\mathcal{F}(t,y)) \le m(t)|x-y|,$$

where $d_H(K,L)$ stands for Hausdorff distance between sets K and $L \subset \mathbf{R}$;

(H3) $\sup\{|x| : x \in \mathcal{F}(t,0)\} \le a(t)$ a.e. and $a \in L^\infty$.

Consider the boundary value problem to the inclusion (1) with boundary conditions (2) In the present paper we deal with properties of the solution set $\mathcal{R}_{\mathcal{F}} \subset W^{2,1}$ of the problem (1), (2). We prove that $\mathcal{R}_{\mathcal{F}}$ is a retract of the whole space $W^{2,1}$.

A bypass is the existence of solutions of the problem (1), (2), since any retract $\mathcal{R}_{\mathcal{F}} \neq \emptyset$. Our work was motivated by a result due to De Blasi and Pianigiani [3], where the authors assumed that the Lipschitz constant $m(t) = \text{const} < 1$, $t \in [0;1]$ (in the more general setting $\mathbf{R}^n$). In the situation considered in our paper, the above hypothesis has been weakened substantially, and we intend to discuss the problem (1) in the $\mathbf{R}^n$ in our subsequent paper.

The method of proving the existence of a retraction is based on the result Bressan–Cellina–Fryszkowski [4] on the fixed point set for a contraction in $L^1(\rho)$. In order to formulate this theorem let us recall the notion of decomposability. The set $\mathcal{K} \subset L^1(\rho)$ is called decomposable, if

$$\lambda_A u + (1 - \lambda_A)v \in \mathcal{K} \tag{12}$$

for any u, $v \in \mathcal{K}$ and $A \in \mathcal{L}$. The family of all non-empty, closed, and decomposable subsets of $L^1(\rho)$ let us denote by $\operatorname{dec} L^1(\rho)$.

Consider the map $\mathcal{K} : L^1(\rho) \to \operatorname{dec} L^1(\rho)$ and assume that the map $\mathcal{K}$ is a contraction i.e. there exists $\alpha < 1$ such that

$$d_H(\mathcal{K}(u), \mathcal{K}(v)) \leq \alpha \|u - v\|_\rho. \tag{13}$$

THEOREM 1 [4]. *For any contraction $\mathcal{K} : L^1(\rho) \to \operatorname{dec} L^1(\rho)$ the set* $\operatorname{Fix}(\mathcal{K})$ *of fixed points of $\mathcal{K}$ is a retract of $L^1(\rho)$.*

Now, we shall establish a kind of the stability result with respect to L^1 perturbations.

PROPOSITION 1. *For any $m \in L^1$ let $\lambda_0 = \lambda_0(m)$ be the principal eigenvalue of the Dirichlet problem for the Sturm–Liouville operator*

$$-\frac{d^2u}{dt^2} - mu = \lambda_0 u, \qquad u(0) = 0 = u(\pi).$$

Then the mapping $m \mapsto \lambda_0(m)$ is continuous in the strong topology of L^1.

Consider $\{m_n\}_{n=0}^{\infty} \subset L^1$. By the (6) for any $n = 0, 1, \ldots$, there exists a sequence λ_i^n , $i = 0, 1, \ldots$ of eigenvalues i.e. of reals such that

$$-\frac{d^2u}{dt^2} - m_n(t)u = \lambda_i^n u, \quad u(0) = 0 = u(\pi), \quad n = 0, 1, \ldots$$

have not vanishing solutions. Moreover the corresponding eigenspaces are one-dimensional. Pick real λ^* diffrent from all of $\{\lambda_i^n\}_{n,i=0}^{\infty}$. Take any $f \in L^1$ and consider the sequence of the non-homogeneous problems

$$-\frac{d^2u_n}{dt^2} - (m_n + \lambda^*)u_n = f, \quad u_n(0) = 0 = u_n(\pi), \quad n = 1, 2, \ldots \tag{15}$$

By the choice of λ^* the only solution to each homogeneous problem (with $f = 0$) is equal to zero. Therefore each non-homogeneous problem (15) has unique solution which we denote by $T_n f$. Observe that $T_n f \in C(T)$ and if ψ_n is an eigenfunction corresponding to the principal eigenvalue λ_0^n of (14) then ψ_n is also an eigenfunction of T_n since

$$T_n \psi_n = (\lambda_0^n - \lambda^*)^{-1} \psi_n. \tag{16}$$

We shall prove that each T_n is a linear compact operator in $C(T)$.

The proof we shall split into few parts.

LEMMA 1. *Assume* $m_n \to m_0$ *in* L^1 *and* $f_n \xrightarrow{L^1} f_0$ *and let* $u_n = T_n f_n$ *be solutions of* (15) *for* $n = 1, 2, \ldots$.

(a) *If* $\{u_n : n = 1, 2, \ldots\}$ *is bounded in* $C(T)$ *then* $\{u_n : n = 1, 2, \ldots\}$ *is relatively compact in* $C(T)$.

(b) *If* $u_n \rightrightarrows u_0$ *and* u_n *are solutions to the* (15), $n = 1, 2, \ldots$ *then* u_0 *is a solution to the* (15) *for* $n = 0$.

Proof. Since $\{u_n\}$ is bounded in L^∞ then by the Banach–Alaoglu theorem we may assume that it is weakly* convergent to a $u_0 \in L^\infty$. It is clear that

$$m_n u_n \xrightarrow{L^1} m_0 u_0$$

because of the strong convergence of m_n in L^1. By (15) this implies that $\left\{\frac{d^2 u_n}{dt^2}\right\}$ is weakly convergent in L^1. We shall prove that $\left\{\frac{du_n}{dt}\right\}$ is weakly convergent in L^1 too. Fix $h \in L^\infty$ and define a function

$$\varphi(t) = \int_0^t h(s)\,ds - \frac{t}{\pi}\int_0^\pi h(s)\,ds.$$

Observe that

$$\varphi(0) = 0 = \varphi(\pi) \tag{17}$$

and

$$\int_0^\pi \frac{du_n}{dt}(t)h(t)\,dt = -\int_0^\pi \frac{d^2 u_n}{dt^2}(t)\varphi(t)\,dt. \tag{18}$$

Indeed, the right hand side of (18) equals

$$-\frac{d}{dt}\varphi\Big|_0^\pi + \int_0^\pi \frac{du_n}{dt}(t)\frac{d\varphi}{dt}(t)\,dt$$
$$= \int_0^\pi \frac{du_n}{dt}(t)h(t)\,dt - \frac{1}{\pi}\int_0^\pi h(s)\,ds \int_0^\pi \frac{du_n}{dt}(t)\,dt,$$

what together with Dirichlet boundary data from (15) gives the required identity (18). Now the weak convergence of $\left\{\frac{du_n}{dt}\right\}$ in L^1 follows from (18) and weak convergence of $\left\{\frac{d^2 u_n}{dt^2}\right\}$.

(a) The weak convergence $\left\{\frac{du_n}{dt}\right\}$ together with the Dirichlet boundary data gives by the Arzela – Ascoli theorem the uniform convergence of the sequence $\{u_n\}$.

(b) Assume that $\left\{\frac{d^2u_n}{dt^2}\right\}$ is weakly convergent say to a $w_0 \in L^1$ and that $\left\{\frac{du_n}{dt}\right\}$ is weakly convergent to v_0. We shall show that $\frac{d^2u_0}{dt^2}$ exists and is equal to w_0. Since u_n are absolutely continuous then

$$\int_0^\pi u_n(t)\frac{d^2\varphi}{dt^2}(t)\,dt = \frac{du_n}{dt}\frac{d\varphi}{dt}\Big|_0^\pi - \int_0^\pi \frac{du_n}{dt}(t)\frac{d\varphi}{dt}(t)\,dt$$

for every $\varphi \in C^2(T)$. Taking $\varphi(t) = \frac{1}{2\pi}(t-\pi)^2$ we obtain the convergence of $\frac{du_n}{dt}(0)$ and taking $\varphi(t) = \frac{1}{2\pi}t^2$ we get the convergence of $\frac{du_n}{dt}(\pi)$.

This together with $\frac{d}{dt}\frac{du_n}{dt} \rightharpoonup w_0$ gives $\frac{du_n}{dt} \rightrightarrows v_0$. Hence v_0 is absolutely continuous and $\frac{dv_0}{dt} = w_0$. Since $\frac{du_n}{dt} \rightrightarrows v_0$ and $u_n(0) = 0 = u_n(\pi)$ then u_0 is absolutely continuous and $\frac{d}{dt}u_0 = v_0$. Thus $\frac{d^2}{dt^2}u_0 = w_0$.

Now passing to the limit in (15) we obtain

$$-\frac{d^2u_0}{dt^2} - (m_0 + \lambda^\star)u_0 = f_0, \qquad u_0(0) = 0 = u_0(\pi)$$

and this completes the proof. ∎

LEMMA 2. *Let $m_n \to m_0$ and let z_n be a solution of* (15). *If $\{f_n : n = 1,2,\ldots\} \subset L^\infty$ is bounded then $\{z_n : n = 1,2,\ldots\}$ is relatively compact in $C(T)$.*

Proof. By the Banach–Alaoglu theorem we may assume that $\{f_n\}$ is weakly convergent in L^1. By Lemma 1 it suffices to show that $\{z_n : n = 1,2,\ldots\}$ is bounded in $C(T)$.

Assuming opposite, by passing to a subsequence, we may assume that $\|z_n\|_\infty \overset{n\to\infty}{\longrightarrow} +\infty$. Denote by $u_n = z_n/\|z_n\|_\infty$; thus

$$\|u_n\|_\infty = 1$$

and

$$-\frac{du_n}{dt} - (m_n + \lambda^\star)u_n = g_n, \qquad u_n(0) = 0 = u_n(\pi),$$

where $g_n = f_n/\|u_n\|_\infty \overset{L^1}{\to} 0$ since $\{f_n\} \subset L^1$ is bounded.

According to Lemma 1 $\{u_n\}$ contains a uniformly convergent subsequence to u_0 and

$$-\frac{du_0}{dt} - (m_0 + \lambda^\star)u_0 = 0, \qquad u_0(0) = 0 = u_0(\pi).$$

But $\lambda^\star$ in not any eigenvalue, hence $u_0 = 0$ and we come to the contradiction with $\|u_0\|_\infty = \lim_{n\to\infty} \|u_n\|_\infty = 1$. ∎

As a consequence of previous lemmas we have the following

COROLLARY 1. (a) *T_n is a compact operator for $n = 0, 1, \ldots$.*
(b) *If $m_n \to m_0$ in L^1 then T_n tends to T_0 in the operator topology.*

Proof of the Proposition 1. Let $m_n \to m_0$ in L^1 as $n \to \infty$ and λ_0^n satisfy (14) ($i = 0$). We need to show that λ_0^n tends to λ_0^0.

Employing the Collorary 1 and Lemma VII.6.3 [7] applied to the sequence of operators T_n we conclude from (16) that

$$(\lambda_0^n - \lambda^\star)^{-1} \overset{n\to\infty}{\longrightarrow} (\lambda_0^0 - \lambda^\star)^{-1},$$

what completes the proof. ∎

Acknowledgement. The authors wish to deeply thank to Professor Giovanni Vidossich for his kind and fruitful cooperation in proving Proposition 1.

Differential inequalities for Sturm–Liouville operator

In this section we explain the question of existence of the non-ve solutions to the inequality:

$$L_w x \equiv -\frac{d^2x}{dt^2} - w(t)x \geq 0 \tag{19}$$

with boundary conditions

$$x(0) = 0 = x(\pi). \tag{20}$$

On the interval T and measurable function w we impose the following hypothesis

(H_+) the operator $L_m = -\frac{d^2}{dt^2} - m$ has only positive eigenvalues.

In particular, the assumption (H_+) assures that the operator L_m is positively defined in $\mathcal{H} = L^2$ and is invertible. Observe the assumption (H_+) is satisfied in three important cases:

(a) $\int_0^\pi m(t)\,dt < 4/\pi$ (cf. [9] p. 405);

(b) $m \in L^\infty$ and $|m|_\infty < 1$ as a consquence of Wirtinger inequality.

(c) $\gamma < 1$, where $\gamma = \pi^{-1} \int_0^\pi t(\pi - t)m(t)\,dt$.

The fundamental lemma in this paper is as follows

LEMMA 3. *Let us assume the operator* L_m *is positively defined. Then there exist* $\varepsilon_0 > 0$ *and* $x_0 \in W^{2,1} \cap W_0^{1,2}$ *such that*

$$-\frac{d^2 x_0}{dt^2} - (m(t) + \lambda_0/2)(1+\varepsilon_0)x_0(t) \geq 0 \tag{21}$$

and

$$x_0(t) > 0 \quad \text{for } t \in T.$$

Proof. Consider the operators C_ε defined on $\operatorname{dom} C_\varepsilon = \operatorname{dom}(C_0 - \lambda_0/2) = W_0^{1,2}$ generated by the form $\langle L_{(w+\lambda_0/2)(1+\varepsilon)}x, z\rangle$ where λ_0 is the principal eigenvalue of the operator L_w. Let $\lambda(\varepsilon)$ stand for the first eigenvalue of the operator C_ε. Thus by the Proposition 1 $\lambda(\varepsilon)$ is continuous at $\varepsilon = 0$.

Since $\lambda(0) = \lambda_0/2 > 0$ therefore there is $\varepsilon_0 > 0$ such that $\lambda(\varepsilon_0) > 0$.

Let $\psi_0 \in W_0^{1,2}$ be the eigenfunction of the operator C_{ε_0} so we have

$$-\frac{d^2\psi_0}{dt^2} - (m + \lambda_0/2)(1+\varepsilon_0)\psi_0(t) = \lambda(\varepsilon_0)\psi_0(t) \tag{22}$$

a.e. in T. Since $\lambda(\varepsilon_0)$ is the principal eigenvalue, the function ψ_0 does not vanish in the interval T, so we can assume $\psi_0(t) > 0$ for $t \in T$. The inequality (21) is now a straightforward conclusion. ∎

LEMMA 4. *Under the same assumptions as in Lemma 3 there exist* $\alpha \in (0;1)$ *and a weight* ρ *such that*

(i) A *is defined in* $L^1(\rho)$;

(ii) $A(w\rho)(t) \leq \alpha\rho(t)$ *a.e. in* $T = [0;\pi]$.

Proof. It suffices to take $\alpha = (1+\varepsilon_0)^{-1}$, and ρ as the function:

$$\rho(t) := -\frac{d^2\psi_0(t)}{dt^2}(m + \lambda_0/2)^{-1}$$

where ε_0 and ψ_0 satisfy (22). Then

$$A((m + \lambda_0/2)\rho) = \psi_0,$$

and (22) means that

$$\alpha\rho - \{1 + \lambda(\varepsilon_0)(m + \lambda_0/2)^{-1}(1 + \varepsilon_0)^{-1}\}A((m + \lambda_0/2)\rho) = 0, \tag{23}$$

and (ii) follows.

The second part (i) is an immediate consequence of the explicit form of the integral operator A. ■

Main result

Let us consider the problem of the existence of solution x to the differential inclusion

$$-\frac{d^2x}{dt^2} \in \mathcal{F}(t,x) \tag{25}$$

in the class of function s $x \in W^{2,1} \cap W_0^{1,2}$ and therefore x satisfies the boundary conditions

$$x(0) = 0 = x(\pi). \tag{26}$$

Let us impose the conditions (H1) (H2) and (H3) on the right hand side $\mathcal{F}(t,x)$ and let us assume that operator L_m, where m is "Lipschitz constant" of the multifunction $\mathcal{F}(t,\cdot)$ satisfies (H_+). The solution set $\mathcal{R}$ is the set of all x such that (24) is fulfiled almost everywhere in T with (25) on the boundary of T. The main result in this paper is the following:

THEOREM 2. *Let us assume that for the multifunction $\mathcal{F}(t,x)$ (H1), (H2), (H3) and (H_+) hold. Then there exists a positive weight $\rho(t)$ in T such that the set of solutions to the problem* (24) *with* (25) *is a retract of the space $W_0^{2,1}(\rho)$.*

Proof. Let ρ be the function given by the Lemma 4. Let us consider the space $L^1(\rho) \supset L^1$. Denote by

$$\mathcal{K}(u) = \{v \in L^1(\rho) : v(t) \in \mathcal{F}(t, A(u)(t)) \text{ a.e. in } T\}.$$

We shall prove that $\mathcal{K} : L^1(\rho) \to \operatorname{dec} L^1(\rho)$ is a contraction.

First, let us observe that the sets $\mathcal{K}(u) \neq \emptyset$. Indeed, let v be a measurable selection of multifunction $t \mapsto \mathcal{F}(t, A(u)(t))$. The existence of v follows from the Kuratowski and Ryll–Nardzewski Theorem. The hypothesis (H2) implies

$$\operatorname{dist}(v(t), \mathcal{F}(t,0)) \leq m(t)|A(u)(t)|$$

for a.e. $t \in T = [0;\pi]$. Then from (H3) follows an estimate

$$|v(t)|\rho(t) \leq a(t)\rho(t) + m(t)\rho(t)|A(|u|)(t)|$$

and further $v \in L^1(\rho)$.

Second, for the contractivity of the map $u \mapsto \mathcal{K}(u)$, let us fix u_1, u_2 and $v_1 \in \mathcal{K}(u_1)$. Let $v_2(t) \in \mathcal{F}(t, A(u_2)(t))$ be a measurable selection such that $|v_1(t) - v_2(t)| \leq (m(t)|A(u_1 - u_2)(t)|$ a.e. in T. Hence together with (H2) we have

$$\begin{aligned} \|v_1 - v_2\|_\rho &= \int_0^\pi |v_1(t) - v_2(t)|\rho(t)\,dt \\ &\leq \int_0^\pi (m)(t)\rho(t)|A(u_1 - u_2)(t)|\,dt \\ &= \int_0^\pi A((m)\rho)(t)|u_1(t) - u_2(t)|\,dt. \end{aligned} \tag{27}$$

Now, from our specific construction of the weight ρ (cf. Lemma 4), we have that the right-hand side of (27) can be estimated by $\int_0^\pi \alpha\rho(t)|u_1(t) - u_2(t)|\,dt = \alpha\|u_1 - u_2\|_\rho$, and this is nothing but the contractivity of $\mathcal{K}$. We are in the position of the B–C–F Theorem which implies that the set $\mathrm{Fix}(\mathcal{K})$ of fixed points of multifunction $\mathcal{K}(u)$ is a retract of the space $L^1(\rho)$. Let $\phi : L^1(\rho) \to \mathrm{Fix}(\mathcal{K})$ be the retraction. Then the map $r : W_0^{2,1}(\rho) \to \mathcal{R}_\mathcal{F}$ given by

$$r(x) = A\Big(\phi\Big(-\frac{d^2x}{dt^2}\Big)\Big)$$

is the retraction from the main theorem. ∎

References

[1] G. Bartuzel and A. Fryszkowski, *On existence of solutions for inclusions* $\frac{d^2u}{dt^2} \in F(x, \nabla u)$, In R. März, editor, *Proc. of the Fourth Conf. on Numerical Treatment of Ordinary Differential Equations*, pages 1–7, Sektion Mathematik der Humboldt Universität zu Berlin, Berlin, Sep. 1984.

[2] F. Berezin and M. Shubin, *Schrödinger Equation*, Moscow Univ. Publ., Moscow, 1983.

[3] F. D. Blasi and G. Pianigiani, *Solution sets of boundary value problems for nonconvex differential inclusion*, Oct. 1992, preprint n 115.

[4] A. Bressan, A. Cellina and A. Fryszkowski, *A class of absolute retracts in spaces of integrable functions*, Proc. AMS 112, 413–418, 1991.

[5] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1955.

[6] R. Dautray and J. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer, Berlin, 1988.

[7] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Wiley, New York, 1958.
[8] A. Fryszkowski, *Continuous selections for a class of nonconvex multivalued maps*, Studia Math. 76 (1983), 163–174.
[9] P. Hartman, *Ordinary Differential Equations*, John Wiley, New York, 1964.
[10] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci. 13 (1965), 397–403.
[11] B. Levitan and I. Sargasyan, *Sturm–Liouville and Dirac Operators*, Mir, Moskva, 1988.
[11] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York, 1979.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Plac Politechniki 1
00-661 WARSZAWA, POLAND

Received August 21st, 1995.