DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 4 1995

Jerzy Grabowski

APPROXIMATE SOLUTION OF A NON-LINEAR INTEGRAL EQUATION IN A COMPLEX DOMAIN

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction and assumptions

This article is a generalization of the paper [1] and deals with an approximation method for solution of integral equation. This method consists in replacing the examined equation by a system of numerical equations (see [3], [4]).

We apply this method to the equation

$$\varphi(t) = \int_L \frac{K[t,\tau,\varphi(\tau)]}{|\tau - t|^\alpha}\, d\tau + N[t,\varphi(t)] \quad (t \in L), \tag{1}$$

with respect to function φ. We prove the uniqueness of the solutions of (1) and the uniqueness of the corresponding numerical equation and give an error estimation of the approximate solution.

We admit the following assumptions:

I. Let

$$L = \bigcup_{k=1}^{p} L_k \subset Z, \tag{2}$$

where Z denotes an open complex plane and each of the set L_k, $k = 1,\dots,p$, is a given ordinary, smooth, open or closed directed arc (in the usual sense). The arc L_k $(k = 1,\dots,p)$ is discribed by

$$L_k : z = z_k(s), \quad l_{k,1} \le s \le l_{k,2}, \tag{3}$$

where $s - l_{k,1}$ denotes the length of the arc $z_k(l_{k,1}), z_k(s) \subset L_k$ with the beginning at $z_k(l_{k,1}) = x_k(l_{k,1}) + i \cdot y_k(l_{k,1})$ and the end at $z_k(s) = x_k(s) + i \cdot y_k(s)$.

We assume that the end points of the interval $\langle l_{k,1}, l_{k,2}\rangle$, $k = 1,\dots,p$, satisfy the condition

$$l_{1,1} \geq 0, \quad l_{j,1} < l_{j,2} < l_{j+1,1} < l_{j+1,2} \tag{4}$$

for $j = 1,\dots,p-1$. Assume that the function

$$z : \Big(\bigcup_{k=1}^{p} \langle l_{k,1}; l_{k,2}\rangle\Big) \to Z \tag{5}$$

defined by

$$z(s) = z_k(s) \tag{6}$$

for $s \in \langle l_{k,1}, l_{k,2}\rangle$, $k = 1,\dots,p$, is "one to one" in the set $\bigcup_{k=1}^{p}(l_{k,1}, l_{k,2})$.

II. α is a given real number satisfying the condition

$$0 < \alpha < 1. \tag{7}$$

III.1. Functions $K = K(t,\tau,u) : L \times L \times Z \to Z$ and $N = N(t,u) : L \times Z \to Z$, satisfy Lipschitz condition with respect to variable u, i.e. there exist positive constants k_K and k_N such for arbitrary elements $t \in L$, $\tau \in L$, $u \in Z$ and $u' \in Z$ the following conditions are satisfied

$$|K(t,\tau,u) - K(t,\tau,u')| \leq k_K \cdot |u - u'|, \tag{8}$$

$$|N(t,u) - N(t,u')| \leq k_N \cdot |u - u'|. \tag{9}$$

These functions are continuous with respect to $t = z(s_t) \in L$, $\tau = z(s_\tau) \in L$, i.e. for every $u \in Z$ functions $K[z(s_t), z(s_\tau), u]$, $N[z(s_t), u]$ are continuous with respect to

$$(s_t, s_\tau) \in \Big(\bigcup_{k=1}^{p} \langle l_{k,1}; l_{k,2}\rangle\Big) \times \Big(\bigcup_{k=1}^{p} \langle l_{k,1}; l_{k,2}\rangle\Big), \quad s_t \in \bigcup_{k=1}^{p} \langle l_{k,1}; l_{k,2}\rangle.$$

III.2. The following inequality holds

$$0 < \gamma < 1, \tag{10}$$

where

$$\gamma = k_K \cdot b + k_N, \tag{11}$$

$$b = \sup_{t\in L}\Big(\sum_{k=1}^{p} \int_{l_{k,1}}^{l_{k,1}} \frac{ds}{|z_k(s) - t|^\alpha}\Big). \tag{12}$$

III.3. For every $R > 0$ there exist positive constants $M(K,R)$ and $M(N,R)$ such that for arbitrary elements $t \in L$, $\tau \in L$, and $u \in \{u \in Z : |u| \leq R\}$ the following inequalities hold

$$|K(t,\tau,u)| \leq M(K,R), \tag{13}$$

$$|N(t,u)| \leq M(N,R). \tag{14}$$

From the assumptions III.1 and III.3 we infer that for every $R > 0$ and $\delta > 0$ we have

$$(15)\quad \omega(K,R,\delta) = \sup_{|u|\le R}[\sup_{\tau\in L}(\sup_{|s-s'|\le\delta}|K[z(s),\tau,u] - K[z(s'),\tau,u]|)] < +\infty$$

and

$$(16)\qquad \omega(N,R,\delta) = \sup_{|u|\le R}(\sup_{|s-s'|\le\delta}|N[z(s),u] - N[z(s'),u]|) < +\infty$$

satisfying the conditions

$$(17)\qquad \lim_{\delta\to 0+}\omega(K,R,\delta) = 0,$$

$$(18)\qquad \lim_{\delta\to 0+}\omega(N,R,\delta) = 0.$$

2. Existence and uniqueness of the solution of the equation (1)

Let $C(L)$ denote a set of functions $\varphi : L \to Z$ defined as follows: $\varphi \in C(L)$ iff the function $\varphi\circ z$, being a superposition of the functions φ and z, is continuous at each point of the set $\bigcup_{k=1}^{p}\langle l_{k,1}; l_{k,2}\rangle$. The set $C(L)$ is a linear space with the usual addition and multiplication by a complex number. Denote the norm of $\varphi \in C(L)$ by

$$(19)\qquad \|\varphi\| = \sup_{t\in L}|\varphi(t)|$$

and let $\|\varphi - g\|$ be the distance between $g, \varphi \in C(L)$. So the set $C(L)$ is a Banach space.

For an arbitrary function $\varphi \in C(L)$ let

$$(20)\qquad (A\varphi)(t) = \int_L \frac{K[t,\tau,\varphi(\tau)]}{|\tau - t|^\alpha}\,d\tau + N[t,\varphi(t)] \quad (t \in L).$$

Note that if $\varphi \in C(L)$, and $g \in C(L)$ and the assumptions I, II and III.1 are satisfied, then $A\varphi \in C(L)$, $Ag \in C(L)$ and

$$\|A\varphi - Ag\| \le \gamma\cdot\|\varphi - g\|,$$

where γ is given by (11).

Basing on the Banach–Cacciopoli theorem (see [2], p. 197–199), we have the following

THEOREM 1. *If the assumptions* I, II, III.1 *and* III.2 *are satisfied, then the equation* (1) *has in* $C(L)$ *exactly one solution* φ. *This solution can be obtained by taking limit of the sequence of successive approximations*

$$(21)\qquad \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_m, \ldots$$

uniformly convergent on the set L, *where*

$$\varphi_{m+1}(t) = \int_L \frac{K[t,\tau,\varphi_m(\tau)]}{|\tau - t|^\alpha}\, d\tau + N[t,\varphi_m(t)] \quad (t \in L) \tag{22}$$

for $m = 0,1,2,\ldots$, *and* $\varphi_0 \in C(L)$ *is arbitrary but fixed.*

3. Approximate equation and its solution

Let $n_1,\ldots,n_p$ be fixed positive integers. Consider a partition of the set $\bigcup_{k=1}^{p}\langle l_{k,1}; l_{k,2}\rangle$ into intervals $p_{k,1} = \langle s_{k,0}; s_{k,1}\rangle$, $p_{k,i} = (s_{k,i-1}; s_{k,i}\rangle$, $i = 2,3,\ldots,n_k, k = 1,\ldots,p$, having no common points, defined by the conditions

$$l_{k,1} = s_{k,0} < s_{k,1} < s_{k,2} < \ldots < s_{k,n_k-1} < s_{k,n_k=l_{k,2}}, \qquad k = 1,\ldots,p.$$

Denote

$$\mathcal{L} = \bigcup_{k=1}^{p}\langle l_{k,1}; l_{k,2}\rangle = \bigcup_{k=1}^{p}\bigcup_{i=1}^{n_k} p_{k,i},$$

$$n = n_1 + n_2 + \ldots + n_p,$$

$$\delta_n = \max_{k\in\{1,\ldots,p\}}[\max_{i\in\{1,\ldots,n_k\}}(s_{k,i} - s_{k,i-1})].$$

Let $C_n(\mathcal{L})$ denote the set of all functions $\psi_n : \mathcal{L} \to Z$ defined by

$$\psi_n(s) = \sum_{k=1}^{p}\sum_{i=1}^{n_k}\chi_{ki}(s)\cdot\left(\frac{s_{k,i} - s}{s_{k,i} - s_{k,i-1}}\cdot z_{k,i-1} + \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}}\cdot z_{k,i}\right), \tag{24}$$

where $z_{k,j} \in Z$ for $j = 0,1,\ldots,n_k$, $k = 1,\ldots,p$ and

$$\chi_{k,i}(s) = \begin{cases} 1, & \text{if } s \in p_{k,i} \\ 0, & \text{if } s \in \mathcal{L} - p_{k,i} \end{cases}$$

$(i = 1,\ldots,n_k,\ k = 1,\ldots,p)$.

The set $C_n(\mathcal{L})$ is a complex Banach space with the norm

$$\|\psi_n\|_n = \sup_{s\in\mathcal{L}}|\psi_n(s)|$$

and usual operations.

As the approximate equation of (1) we take the equation

$$\psi_n(s) = \sum_{k=1}^{p}\sum_{i=1}^{n_k}\chi_{k,i}(s)\left\{\frac{s_{k,i} - s}{s_{k,i} - s_{k,i-1}}\cdot\right. \tag{25}$$

$$\cdot\left\{\sum_{j=1}^{p}\int_{l_{j,1}}^{l_{j,2}}\frac{K[z(s_{k,i-1}), z(s_\tau), \psi_n(s_\tau)]\cdot z'(s_\tau)}{|z(s_\tau) - z(s_{k,i-1})|^\alpha}\, ds_\tau +\right.$$

$$+ N[z(s_{k,i-1}), \psi_n(s_{k,i-1})]\Big\} + \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}} \cdot$$

$$\cdot \Big\{ \sum_{j=1}^{p} \int_{l_{j,1}}^{l_{j,2}} \frac{K[z(s_{k,i}), z(s_\tau), \psi_n(s_\tau)] \cdot z'(s_\tau)}{|z(s_\tau) - z(s_{k,i})|^\alpha} ds_\tau +$$

$$+ N[z(s_{k,i}), \psi_n(s_{k,i})]\Big\}\Big\}$$

with respect to function $\psi_n \in C_n(\mathcal{L})$. If the function ψ_n is given by (24) then equation (25) is equivalent to the following system of numerical equations

$$z_{k,i} = \sum_{j=1}^{p} \int_{l_{j,1}}^{l_{j,2}} \frac{K[z(s_{k,i}), z(s_\tau), \psi_n(s_\tau)] \cdot z'(s_\tau)}{|z(s_\tau) - z(s_{k,i})|^\alpha} ds_\tau +$$
$$+ N[z(s_{k,i}), \psi_n(s_{k,i})], \quad i = 0, 1, \ldots, n_k, \ k = 1, \ldots, p$$

with unknowns complex numbers $z_{k,i}$, $i = 0, 1, \ldots, n_k$, $k = 1, \ldots, p$, where in place of $\psi_n(s)$ we have to put the right-hand side of the equality (24).

For an arbitrary function $\psi_n \in C_n(\mathcal{L})$ denote by $(A_n\psi_n)(s)$ the right-hand side of the equality (25). Let us note that if $\psi_n \in C_n(\mathcal{L})$, $h_n \in C_n(\mathcal{L})$ and the conditions I, II, III.1, are satisfied then $A_n\psi_n \in C_n(\mathcal{L})$, $A_nh_n \in C_n(\mathcal{L})$ and

$$|(A_n\psi_n)(s) - (A_nh_n)(s)| \le$$

$$\le \sum_{k=1}^{p} \sum_{i=1}^{n_k} \chi_{k,i}(s) \cdot \Big[\frac{s_{k,i} - s}{s_{k,i} - s_{k,i-1}} \cdot (k_K \cdot b + k_N) \cdot \|\psi_n - h_n\|_n +$$

$$+ \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}} \cdot (k_K \cdot b + k_N) \cdot \|\psi_n - h_n\|_n \Big] =$$

$$\gamma \cdot \|\psi_n - h_n\|_n \cdot \Big(\sum_{k=1}^{p} \sum_{i=1}^{n_k} \chi_{k,i}(s) \Big) = \gamma \cdot \|\psi_n - h_n\|_n.$$

Hence

$$\|A_n\psi_n - A_nh_n\|_n \le \gamma \cdot \|\psi_n - h_n\|_n.$$

Basing on the above mentioned Banach-Cacciopoli theorem we have the following

THEOREM 2. *If the assumptions* I, II, III.1 *and* III.2, *are satisfied then the equation* (25) *has in* $C_n(\mathcal{L})$ *exactly one solution* ψ_n, *being the limit of the sequence of successive approximations*

$$\psi_{n,0}, \psi_{n,1}, \psi_{n,2}, \ldots, \psi_{n,m}, \ldots \tag{26}$$

uniformly convergent in the set $\mathcal{L}$, where

$$(27)\quad \psi_{n,m+1}(s) = (A_n\psi_{n,m})(s) = \sum_{k=1}^{p}\sum_{i=1}^{n_k}\chi_{k,i}(s)\cdot\left\{\frac{s_{k,i}-s}{s_{k,i}-s_{k,i-1}}\cdot\right.$$

$$\left\{\sum_{j=1}^{p}\int_{l_{j,1}}^{l_{j,2}}\frac{K[z(s_{k,i-1}),z(s_\tau),\psi_{n,m}(s_\tau)]\cdot z'(s_\tau)}{|z(s_\tau)-z(s_{k,i-1})|^{\alpha}}ds_\tau+\right.$$

$$\left.+N[z(s_{k,i-1}),\psi_{n,m}(s_{k,i-1})]\right\}+\frac{s-s_{k,i-1}}{s_{k,i}-s_{k,i-1}}\cdot$$

$$\cdot\left\{\sum_{j=1}^{p}\int_{l_{j,1}}^{l_{j,2}}\frac{K[z(s_{k,i}),z(s_\tau),\psi_{n,m}(s_\tau)]\cdot z'(s_\tau)}{|z(s_\tau)-z(s_{k,i})|^{\alpha}}ds_\tau+\right.$$

$$\left.\left.+N[z(s_{k,i}),\psi_{n,m}(s_{k,i})]\right\}\right\}$$

for $m = 0,1,2,\ldots$, and $\psi_{n,0}\in C_n(\mathcal{L})$ is an arbitrary but fixed element.

4. Error estimation of the approximate solution

Let the assumptions I, II, III.1, III.2 be satisfied and let the first elements φ_0 and $\psi_{n,0}$ of sequences (21) and (26) satisfy the condition

$$(28)\quad \bigvee_{c\in Z}\bigwedge_{s\in\mathcal{L}}(\varphi_0[z(s)] = \psi_{n,0}(s) = c).$$

Because of uniform convergence of these sequences we have that there exists a number $R_n > 0$, such that for arbitrary $s\in\mathcal{L}$ and $m\in\{0,1,2,\ldots\}$ the following inequalities hold:

$$(29)\quad |\varphi_m[z(s)]|\le R_n,$$

$$(30)\quad |\psi_{n,m}(s)|\le R_n.$$

Then error estimation of the approximate solution is given by the following

THEOREM 3. *Let the assumptions* I, II, III.1, III.2, III.3 *and conditions* (22), (27)-(30) *be satisfied. Then*

$$(31)\quad \sup_{s\in\mathcal{L}}|\psi_n(s)-\varphi[z(s)]|\le$$

$$\frac{1}{1-\gamma}\cdot\frac{1}{1-k_N}\cdot[d(\delta_n)\cdot M(K,R_n)+b\cdot\omega(K,R_n,\delta_n)+\omega(N,R_n,\delta_n)],$$

where

(32) $d(\delta_n) =$

$$\max_{\substack{i\in\{1,\dots,n_k\}\\ k\in\{1,\dots,p\}}}\left[\sup_{s',s''\in p_{k,i}}\left(\sum_{j=1}^{p}\int_{l_{j,1}}^{l_{j,2}}\left|\frac{1}{|z(s)-z(s')|^\alpha}-\frac{1}{|z(s)-z(s'')|^\alpha}\right|ds\right)\right].$$

Proof. To prove Theorem 3 we give estimation of $\sup_{s\in\mathcal{L}}|\psi_{n,m}(s)-\varphi_m[z(s)]|$.

For this aim we use (22), (27) and the equalities

$$\frac{K[z(s_{k,l}),z(s_\tau),\psi_{n,m}(s_\tau)]}{|z(s_\tau)-z(s_{k,l})|^\alpha}-\frac{K[z(s),z(s_\tau),\varphi_m(z(s_\tau))]}{|z(s_\tau)-z(s)|^\alpha}=$$
$$=\frac{K[z(s_{k,l}),z(s_\tau),\psi_{n,m}(s_\tau)]-K[z(s),z(s_\tau),\psi_{n,m}(s_\tau)]}{|z(s_\tau)-z(s_{k,l})|^\alpha}+$$
$$+\frac{K[z(s),z(s_\tau),\psi_{n,m}(s_\tau)]-K[z(s),z(s_\tau),\varphi_m(z(s_\tau))]}{|z(s_\tau)-z(s_{k,l})|^\alpha}+$$
$$+\left(\frac{1}{|z(s_\tau)-z(s_{k,l})|^\alpha}-\frac{1}{|z(s_\tau)-z(s)|^\alpha}\right)\cdot K[z(s),z(s_\tau),\varphi_m(z(s_\tau))],$$
$$N[z(s_{k,l}),\psi_{n,m}(s_{k,l})]-N[z(s),\varphi_m(z(s))]=$$
$$=N[z(s_{k,l}),\psi_{n,m}(s_{k,l})]-N[z(s),\psi_{n,m}(s_{k,l})]+$$
$$+N[z(s),\psi_{n,m}(s_{k,l})]-N[z(s),\psi_{n,m}(s)]+$$
$$+N[z(s),\psi_{n,m}(s)]-N[z(s),\varphi_m(z(s))]$$

for $l=i-1,i$, $i=1,2,\dots,n_k$, $k=1,2,\dots,p$. Basing on the conditions (8)–(16) and on the equality

$$\sum_{k=1}^{p}\sum_{i=1}^{n_k}\chi_{k,i}(s)=1$$

for $s\in\mathcal{L}$, we have the following estimation

$$|\psi_{n,m+1}(s)-\varphi_{m+1}[z(s)]|\le b\cdot\omega(K,R_n,\delta_n)+$$
$$+d(\delta_n)\cdot M(K,R_n)+\omega(N,R_n,\delta_n)+(k_K\cdot b+k_N)\cdot\sup_{s\in\mathcal{L}}|\psi_{n,m}(s)-\varphi_m[z(s)]|+$$
$$+k_N\cdot\sum_{k=1}^{p}\sum_{i=1}^{n_k}\chi_{k,i}(s)\cdot\left[\frac{s_{k,i}-s}{s_{k,i}-s_{k,i-1}}\cdot|\psi_{n,m}(s_{k,i-1})-\psi_{n,m}(s)|+\right.$$
$$\left.+\frac{s-s_{k,i-1}}{s_{k,i}-s_{k,i-1}}\cdot|\psi_{n,m}(s_{k,i})-\psi_{n,m}(s)|\right].$$

If $s\in p_{k,i}$, $i=1,2,\dots,n_k$, $k=1,\dots,p$, then in virtue of (24) we have

$$\psi_{n,m}(s_{k,i-1})-\psi_{n,m}(s)=$$

$$= z_{k,i-1} - \frac{s_{k,i} - s}{s_{k,i} - s_{k,i-1}} \cdot z_{k,i-1} - \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}} \cdot z_{k,i} =$$

$$= \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}} \cdot (z_{k,i-1} - z_{k,i}) = \frac{s - s_{k,i-1}}{s_{k,i} - s_{k,i-1}} \cdot [\psi_{n,m}(s_{k,i-1}) - \psi_{n,m}(s_{k,i})],$$

$$\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s) = \frac{s_{k,i} - s}{s_{k,i} - s_{k,i-1}} \cdot [\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s_{k,i-1})].$$

Hence

$$\begin{aligned} \text{(33)} \qquad & \sup_{s \in \mathcal{L}} |\psi_{n,m+1}(s) - \varphi_{m+1}[z(s)]| \leq \\ & \leq \gamma \cdot \sup_{s \in \mathcal{L}} |\psi_{n,m}(s) - \varphi_m[z(s)]| + b \cdot \omega(K, R_n, \delta_n) + d(\delta_n) \cdot M(K, R_n) + \\ & + \omega(N, R_n, \delta_n) + k_N \cdot \max_{\substack{i \in \{1,\ldots,n_k\} \\ k \in \{1,\ldots,p\}}} |\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s_{k,i-1})| \end{aligned}$$

for $m = 0, 1, 2, \ldots$.

Now we shall give the estimation of

$$\max_{\substack{i \in \{1,\ldots,n_k\} \\ k \in \{1,\ldots,p\}}} |\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s_{k,i-1})|.$$

Taking into account (27), we have

$$\begin{aligned} & \psi_{n,m+1}(s_{k,i}) - \psi_{n,m+1}(s_{k,i-1}) = \\ & = \sum_{j=1}^{p} \int_{l_{j,1}}^{l_{j,2}} \Bigg\{ \frac{K[z(s_{k,i}), z(s_\tau), \psi_{n,m}(s_\tau)] - K[z(s_{k,i-1}), z(s_\tau), \psi_{n,m}(s_\tau)]}{|z(s_\tau) - z(s_{k,i-1})|^\alpha} + \\ & + \left(\frac{1}{|z(s_\tau) - z(s_{k,i})|^\alpha} - \frac{1}{|z(s_\tau) - z(s_{k,i-1})|^\alpha} \right) \cdot \\ & \cdot K[z(s_{k,i}), z(s_\tau), \psi_{n,m}(s_\tau)] \Bigg\} z'(s_\tau)\, ds_\tau + \\ & + N[z(s_{k,i}), \psi_{n,m}(s_{k,i})] - N[z(s_{k,i-1}), \psi_{n,m}(s_{k,i})] + \\ & + N[z(s_{k,i-1}), \psi_{n,m}(s_{k,i})] - N[z(s_{k,i-1}), \psi_{n,m}(s_{k,i-1})]. \end{aligned}$$

Hence we have the inequality

$$\begin{aligned} & \max_{\substack{i \in \{1,\ldots,n_k\} \\ k \in \{1,\ldots,p\}}} |\psi_{n,m+1}(s_{k,i}) - \psi_{n,m+1}(s_{k,i-1})| \leq \\ & \leq b \cdot \omega(K, R_n, \delta_n) + d(\delta_n) \cdot M(K, R_n) + \omega(N, R_n, \delta_n) + \\ & + k_N \cdot \max_{\substack{i \in \{1,\ldots,n_k\} \\ k \in \{1,\ldots,p\}}} |\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s_{k,i-1})| \end{aligned}$$

for $m = 0,1,2,\ldots$. By means of (28) and $0 < k_N < 1$ we have

$$(34) \qquad \max_{\substack{i\in\{1,\ldots,n_k\}\\ k\in\{1,\ldots,p\}}} |\psi_{n,m}(s_{k,i}) - \psi_{n,m}(s_{k,i-1})| \leq$$
$$\leq \frac{1}{1-k_N}\cdot[d(\delta_n)\cdot M(K,R_n) + b\cdot\omega(K,R_n,\delta_n) + \omega(N,R_n,\delta_n)]$$

for $m = 0,1,2,\ldots$. Basing on (33) and (34) we have

$$\sup_{s\in\mathcal{L}} |\psi_{n,m+1}(s) - \varphi_{m+1}[z(s)]| \leq \gamma\cdot\sup_{s\in\mathcal{L}} |\psi_{n,m}(s) - \varphi_m[z(s)]| +$$
$$+\frac{1}{1-k_N}\cdot[d(\delta_n)\cdot M(K,R_n) + b\cdot\omega(K,R_n,\delta_n) + \omega(N,R_n,\delta_n)]$$

for $m = 0,1,2\ldots$. Hence, because of (28) and the assumption (10) we obtain the inequality

$$\sup_{s\in\mathcal{L}} |\psi_{n,m}(s) - \varphi_m[z(s)]| \leq$$
$$\leq \frac{1}{1-\gamma}\cdot\frac{1}{1-k_N}\cdot[d(\delta_n)\cdot M(K,R_n) + b\cdot\omega(K,R_n,\delta_n) + \omega(N,R_n,\delta_n)]$$

for $m = 0,1,2,\ldots$. Passing to the limit in the last inequality we get (31), what completes the proof.

Remark. It is possible to admit that in the inequalities (29), (30) we have

$$R_n = |c| + \frac{1}{1-\gamma}\cdot[b\cdot\sup_{(t,\tau)\in L\times L} |K(t,\tau,c)| + \sup_{t\in L}|N(t,c)| + |c|],$$

where c is a complex constant from condition (28). Indeed, we have

$$\|\varphi_m\| = \|\varphi_0 + \varphi_1 - \varphi_0 + \varphi_2 - \varphi_1 + \ldots + \varphi_m - \varphi_{m-1}\| \leq$$
$$\leq \||\varphi_0\| + \|\varphi_1 - \varphi_0\| + \|\varphi_2 - \varphi_1\| + \ldots \|\varphi_m - \varphi_{m-1}\| \leq$$
$$\leq |c| + \|\varphi_1 - \varphi_0\| + \gamma\cdot\|\varphi_1 - \varphi_0\| + \ldots + \gamma^{m-1}\cdot\|\varphi_1 - \varphi_0\| \leq$$
$$\leq |c| + \frac{1}{1-\gamma}\cdot\|\varphi_1 - \varphi_0\| \leq |c| + \frac{1}{1-\gamma}\cdot(\|\varphi_1\| + \|\varphi_0\|) \leq$$
$$\leq |c| + \frac{1}{1-\gamma}\cdot[b\cdot\sup_{(t,\tau)\in L\times L} |K(t,\tau,c)| + \sup_{t\in L}|N(t,c)| + |c|],$$

and similarly

$$\|\psi_{n,m}\|_n \leq |c| + \frac{1}{1-\gamma}\cdot[b\cdot\sup_{(t,\tau)\in L\times L} |K(t,\tau,c)| + \sup_{t\in L}|N(t,c)| + |c|].$$

for $m = 1,2,\ldots$ and an arbitrary, defined in p. 3 partition of the set $\bigcup_{k=1}^{p}\langle l_{k,1}; l_{k,2}\rangle$.

If R is an arbitrary, fixed positive number and $c \in \{u \in Z : |u| \leq R\}$, one can admit that

$$R_n = R + \frac{1}{1-\gamma} \cdot [b \cdot M(K, R) + M(N, R) + R].$$

If $\lim_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} d(\delta_n) = 0$. Hence we obtain

COROLLARY. *If the assumptions* I, II, III.1, III.2, III.3 *are satisfied and* $\lim_{n\to\infty} \delta_n = 0$, *then* $\lim_{n\to\infty} \sup_{s\in\mathcal{L}} |\psi_n(s) - \varphi[z(s)]| = 0$.

References

[1] J. Grabowski, W. Leksiński, *O przybliżonym rozwiązaniu pewnego równania całkowego*, Preprinty Instytutu Matematyki Politechniki Warszawskiej, Nr 59, November 94.

[2] W. Pogorzelski, *Integral equations and their Applications*, vol. 1, PWN Warszawa 1966.

[3] П. Бенерджи, П. Баттерфилд, *Методы граничных елементов в прикладных науках*, Москва, 1984.

[4] В. К. Дзядык, *Аппроксимационные методы решения дифференциальных и интегральных уравнений*, Киев, Наукова Думка 1988.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-660 WARSZAWA, POLAND

Received June 30, 1995.