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A GOURSAT-TYPE PROBLEM FOR A HIGH ORDER PARTIAL DIFFERENTIAL EQUATION

Dedicated to Professor Janina Wolska-Bochenek

0. Introduction

Goursat-type boundary value problems for hyperbolic partial differential equations of orders greater than two in rectangular domains have been examined in papers [1], [2], [4] and [9]–[11] (see also the references therein). One of the most fundamental assumptions in all these papers is that the curves considered in the problem do not intersect one another except one vertex of the rectangle. In this paper we consider a Goursat-type problem in the case when the said curves additionally meet at the opposite vertex of the rectangle which makes the problem overdetermined and hence much harder. We examine it by using the method given by Fichera [5] and then applied by the present author [3] in a more complicated case. Let us observe that also the differential operator appearing in the considered equation and the boundary conditions dealt with in the paper are more general than those in the earlier papers (cf. Remark 1.1 in the sequel).

To the best of our knowledge,the present problem has not been examined so far.

1. The Problem and the assumptions

Let $\mathbb{Y}$ be a Banach space with norm $\|\cdot\|$, and N the set of all positive integers. We consider two numbers $p, q \in N$ (in the sequel it will be assumed that $p \leq q$; it is easily seen how the argument should be modified in the opposite case).

Let r be a positive divisor of p, and set $k = \frac{p}{r}$. We denote by $\mathbf{P}$ the rectangle $[0,1]\times[0,\sigma]$, where $0 < \sigma < \infty$, and we introduce the class $\mathfrak{K}$ of all functions $u : \mathbb{P} \to \mathbb{Y}$ possessing continuous derivatives $D_x^\alpha D_y^\beta u$ (where $D_x^\alpha =$

$\frac{\partial^\alpha}{\partial x^\alpha}$; $D_y^\beta = \frac{\partial^\beta}{\partial y^\beta}$) for $\alpha = 0,1,\ldots,p$; $\beta = 0,1,\ldots,q$. We also consider the set $(\mathbb{S})$ of $2r$ curves $\Gamma_1,\ldots,\Gamma_r;\widetilde{\Gamma}_1,\ldots,\widetilde{\Gamma}_r$, of equations $y = f_i(x)$; $x = h_i(y)$ $(i = 1,2,\ldots,r)$ respectively, where $f_i : [0,1] \to [0,\sigma]$; $h_i : [0,\sigma] \to [0,1]$.

The aim of this paper is to examine the following boundary value problem $(\mathfrak{G})$:

Find a solution u to the equation

$$D_x^p D_y^q u(x,y) = 0 \tag{1.1}$$

in $\mathbb{P}$ (that is a function $u \in \mathfrak{K}$ satisfying (1.1) at each point $(x,y) \in \mathbb{P}$), fulfilling on $(\mathbb{S})$ the following system of $p+q$ boundary conditions

$$\begin{aligned} &(a) \quad L^{jr} D_y^{q-p} u[x, f_i(x)] = M_{i,j}(x), \\ &(b) \quad L^{jr} D_y^{q-p} u[h_i(y), y] = N_{i,j}(y), \\ &(c) \quad L^s u[h_r(y), y) = \overline{N}_s(y), \end{aligned} \tag{1.2}$$

where $L = D_x D_y$; $j = 0,1,\ldots,k-1$; $i = 1,2,\ldots,r$; $s = 0,1,\ldots,q-p-1$ (we set $(0,1,\ldots,t) := \emptyset$ when $t < 0$).

Remark 1.1. Let us consider the particular case $p = q$ (in which equation (1.1) takes the form $L^p u = 0$ and is called the polyvibrating equation of Mangeron (see [7], [10], [11]). As a consequence, condition (c) in (1.2) is deleted. If no two of the curves $\Gamma_1,\ldots,\Gamma_r$ and $\widetilde{\Gamma}_1,\ldots,\widetilde{\Gamma}_r$ intersect in $\mathbb{P}\setminus\{\mathbb{O}\}$, where $\mathbb{O}(0,0)$, then the $(\mathfrak{G})$-problem is identical with that in [9] (for the homogeneous partial differential equation; the same refers to the sequel of this Remark), and in the subcase $r = 1$ with the one in [10], [11] while in the subcase $r = p$ with that in [1], [2]. If $r = 1$ and the curves considered pass through the points $\mathbb{O}$ and $M(1,\sigma)$ and do not intersect elsewhere, then the $(\mathfrak{G})$-problem coincides with that examined in [3]. Let us observe, however, that the boundary value problem for equation (1.1) dealt with in [4] cannot be obtained from problem $(\mathfrak{G})$.

We make the following assumptions

I. The functions f_i and h_i $(i = 1,2,\ldots,r)$ are of class C^q, strictly increase and satisfy the conditions

$$\begin{aligned} &f_i(0) = h_i(0) = 0;\ f_i(1) = h_i^{-1}(1) = \sigma;\ f_{\mu-1}(x) < f_\mu(x); \\ &h_{\mu-1}(y) < h_\mu(y);\ f_r(x) < h_r^{-1}(x) \end{aligned} \tag{1.3}$$

$(x \in (0,1);\ y \in (0,\sigma);\ \mu = 2,3,\ldots,r)$;

$$\min(\overset{*}{f}_1, \widehat{h}_r) > 0;\ g_0 := \max(\overset{*}{f}_r \overset{*}{h}_r, (\widehat{f}_r \widehat{h}_r)^{-1}) < (r+1)^{\frac{-2r}{\varkappa_0}} \tag{1.4}$$

where $\overset{*}{f} = f_i'(0)$; $\overset{*}{h}_i = h_i'(0)$; $\widehat{f}_i = f_i'(1)$; $\widehat{h}_i = h_i'(\sigma)$ and $\varkappa_0 \in (0,1]$;

$$(1.5)\quad \min_{2\le i\le r}(\overset{*}{f}_i - \overset{*}{f}_{i-1}) > [e(\varepsilon)]^{-1}\overset{*}{f}_r;\ \min_{2\le i\le r}(\overset{*}{h}_i - \overset{*}{h}_{i-1}) > [e(\varepsilon)]^{-1}\overset{*}{h}_r,$$

$$(1.6)\quad \min_{2\le i\le r}(\widehat{f}_i^{-1} - \widehat{f}_{i-1}^{-1}) > [e(\varepsilon)]^{-1}\widehat{f}_r^{-1};\ \min_{2\le i\le r}(\widehat{h}_i^{-1} - \widehat{h}_{i-1}^{-1}) > [e(\varepsilon)]^{-1}\widehat{h}_r^{-1},$$

when $r \ge 2$, where

$$(1.7)\qquad e(\varepsilon) = (r+1)(1+\varepsilon)$$

with ε being any number such that

$$(1.8)\qquad 0 < \varepsilon < [(r+1)^r g_0]^{\frac{1}{1-r}} - 1.$$

It is clear that by Assumption I above, the curves $\Gamma_1, \dots, \Gamma_r$ and $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_r$ pass through the points $\mathbb{O}$ and M, and have no other common points.

II. The function $M_{i,j} : [0,1] \to \mathbb{Y}$ and $N_{i,j} : [0,\sigma] \to \mathbb{Y}$ ($i = 1,2,\dots,r$; $j = 0,1,\dots,k-1$) are of class C^{q-jr}, respectively, and satisfy the following conditions*)

$$(1.9)\qquad \begin{aligned} &\|M_{i,j}^{(m)}(x)\| \le \text{const}[\min(x, 1-x)]^{p+q-(2j-1)r-m-2+\varkappa_0},\\ &\|N_{i,j}^{(m)}(y)\| \le \text{const}[\min(y, \sigma-y)]^{p+q-(2j-1)r-m-2+\varkappa_0} \end{aligned}$$

$((x,y) \in \Omega;\ i = 1,2,\dots,r;\ m = 0,1,\dots,q-jr;\ j = 0,1,\dots,k-1).$

III. The functions $\overline{N}_s : [0,\sigma] \to \mathbb{Y}$ ($s = 0,1,\dots,q-p-1$) are of class $C^{\max(0,p-s)}$, respectively.

EXAMPLE 1.1. We give an example of the functions satisfying Assumption I.

Let $r = 2$ and $\sigma > 3^{\frac{4}{\varkappa_0}}$. One can verify that

$$\begin{aligned} f_1(x) &= \frac{\sigma}{e^3-1}(e^{3x}-1);\ f_2(x) = \frac{1}{e-1}(e^x-1),\\ h_1(y) &= \frac{1}{e^{3\sigma}-1}(e^{3y}-1);\ h_2(y) = \frac{\sigma}{e^\sigma-1}(e^y-1), \end{aligned}$$

satisfy all the requirements of the said Assumption.

2. Auxiliary theorems

In this section we give some lemmas.

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*) Here and in the sequel, const denotes a positive constant.

LEMMA 2.1. *If $u : \mathbb{P} \to \mathbb{Y}$ satisfies*

$$
(2.1)\qquad u(x,y) = \begin{cases} \text{(a)} & \sum_{m=1}^{p} [y^{q-p+m-1}\widehat{\phi}(x) + x^{m-1}\widehat{\psi}_m(y)] + \\ & \qquad + \sum_{m=1}^{q-p} y^{m-1}\widehat{\omega}_m(x) \\ \text{(b)} & \sum_{m=1}^{p} [(\sigma - y)^{q-p+m-1}\overset{*}{\phi}(x) + (1-x)^{m-1}\overset{*}{\psi}_m(y)] + \\ & \qquad + \sum_{m=1}^{q-p} (\sigma - y)^{m-1}\overset{*}{\omega}_m(x) \end{cases}
$$

$((x, y) \in \Omega)$, where $\widehat{\phi}, \overset{*}{\phi}_m : [0,1] \to \mathbb{Y}$; $\widehat{\psi}_m, \overset{*}{\psi}_m : [0,\sigma] \to \mathbb{Y}$ and $\widehat{\omega}_\beta, \overset{*}{\omega}_\beta : [0,1] \to \mathbb{Y}$ $(m = 1,2,\dots,p;\ \beta = 1,2,\dots,q-p)$ are functions of class C^p, C^q and C^p, respectively, then u is a solution of equation (1.1) in $\mathbb{P}$.

Conversely, if u is a given solution of equation (1.1) in $\mathbb{P}$, then there are function $\widehat{\Phi}_\nu, \overset{*}{\Phi}_\nu : [0,1] \to \mathbb{Y}$, $\widehat{\Psi}_\nu, \overset{*}{\Psi}_\nu : [0,\sigma] \to \mathbb{Y}$ and $\widehat{\omega}_\beta, \overset{*}{\omega}_\beta : [0,1] \to \mathbb{Y}$ $(\nu = 0,1,\dots,p-1;\ \beta = 1,2,\dots,q-p)$ of class $C^{p-\nu}, C^{p-\nu}$ and C^p, respectively, such that

$$
(2.2)\qquad \begin{aligned}
\widehat{\phi}_m(x) &= \delta_{1m}\widehat{\Phi}_0(x) + (1-\delta_{1m}) \int_0^x \frac{(x-\xi)^{m-2}}{(m-2)!}\widehat{\Phi}_{m-1}(\xi)\,d\xi, \\
\overset{*}{\phi}_m(x) &= \delta_{1m}\overset{*}{\Phi}_0(x) + (1-\delta_{1m}) \int_x^1 \frac{(x-\xi)^{m-2}}{(m-2)!}\overset{*}{\Phi}_{m-1}(\xi)\,d\xi, \\
\widehat{\psi}(y) &= \delta_{1m}\left[\delta_{pq}\widehat{\Psi}_0(y) + (1-\delta_{pq}) \int_0^y \frac{(y-\eta)^{q-p-1}}{(q-p-1)!}\widehat{\Psi}_0(\eta)\,d\eta\right] + \\
&\quad + (1-\delta_{1m}) \int_0^y \frac{(y-\eta)^{q-p+m-2}}{(q-p+m-2)!}\widehat{\Psi}_{m-1}(\eta)\,d\eta, \\
\overset{*}{\psi}_m(y) &= \delta_{1m}\left[\delta_{pq}\overset{*}{\Psi}_0(y) + (1-\delta_{pq}) \int_y^\sigma \frac{(\eta-y)^{q-p-1}}{(q-p-1)!}\overset{*}{\Psi}_0(\eta)\,d\eta\right] + \\
&\quad + (1-\delta_{1m}) \int_y^\sigma \frac{(\eta-y)^{q-p+m-2}}{(q-p+m-2)!}\overset{*}{\psi}_{m-1}(\eta)\,d\eta
\end{aligned}
$$

$(m = 1,2,\dots,p$ and $\delta_{\nu\mu}$ is the Kronecker delta) and that equalities (2.1) are satisfied, respectively.

We omit a straightforward proof of this lemma.

LEMMA 2.2. *There is a sufficienty small number* δ_1 $(0 < \sigma_1 \le \min(1,\delta))$ *such that the inequalities*

$$(2.3)\quad \begin{aligned} &\text{(a)} && f_i(x) - f_j(x) > [e(\varepsilon)]^{-1}(\sigma - f_1(x)) > \\ &&& > [e(\varepsilon)]^{-1}(1-\varepsilon_0)\widehat{f}_1(1-x) \\ &\text{(b)} && h_i(y) - h_j(y) > [e(\varepsilon)]^{-1}(1 - h_1(y)) > \\ &&& > [e(\varepsilon)]^{-1}(1-\varepsilon_0)\widehat{h}_1(\sigma - y) \\ &\text{(c)} && f_j^{-1}(y) - f_i^{-1}(y) > [e(\varepsilon)]^{-1}(1 - f_r^{-1}(y)) > \\ &&& > [e(\varepsilon)]^{-1}(1-\varepsilon_0)\widehat{f}_r^{-1}(\sigma - y) \\ &\text{(d)} && h_j^{-1}(x) - h_i^{-1}(x) > [e(\varepsilon)]^{-1}(\sigma - h_r^{-1}(x)) > \\ &&& > [e(\varepsilon)]^{-1}(1-\varepsilon_0)\widehat{h}_r^{-1}(1 - x) \end{aligned}$$

$(1 \le j < i \le r;\ r \ge 2)$ *hold good for* $x \in (1-\delta_1, 1)$ *and* $y \in (\sigma - \delta_1, \sigma)$, *respectively*, ε_0 *being any number such that*

$$(2.4)\qquad 0 < \varepsilon_0 < 1 - [(r+1)^r g_0^{\varkappa_0}]^{\frac{1}{2(3q+\varkappa_0)}}.$$

Proof. The proof, being similar for the remaining inequalities, will be given only for (2.3)(c).

Introducing the auxiliary function

$$(2.5)\qquad F_{i,j}(y) := f_j^{-1}(y) - f_i^{-1}(y) - e(\varepsilon)(1 - f_r^{-1}(y))$$

and using the equality $F_{i,j}(\sigma) = 0$, we get

$$F_{i,j}(y) = [(f_i^{-1\prime})(\eta) - (f_j^{-1\prime})(\eta) - e(\varepsilon)(f_r^{-1\prime})(\eta)](\sigma - y),$$

where $\eta = y + \vartheta(\sigma - y)$; $\vartheta \in (0,1)$.

Set $\varepsilon_* = \frac{1}{2}[\min\limits_{2\le i\le r}(\widehat{f}_i^{-1} - \widehat{f}_{i-1}^{-1}) - e(\varepsilon)\widehat{f}_r^{-1}][2\widehat{f}_r^{-1} + e(\varepsilon)\widehat{f}_r^{-1}]^{-1}$ (let us observe that by (1.4), (1.5) we have $\varepsilon_* > 0$).

It follows from Assumption **I** that

$$(2.6)\qquad (1-\varepsilon_*)\widehat{f}_\alpha^{-1} < (f_\alpha^{-1\prime})(\eta) < (1+\varepsilon_*)\widehat{f}_\alpha^{-1}$$

$(\alpha = 1,2,\ldots,r)$ provided that $\sigma - \delta_1 < y < \sigma$ with $\delta_1 = \delta_1(\varepsilon_*) \in (0,\sigma)$ being sufficiently small.

Thus, we can assert that

$$(2.7)\quad \begin{aligned} F_{i,j}(y) &> [\widehat{f}_i^{-1} - \widehat{f}_j^{-1} - e(\varepsilon)\widehat{f}_r^{-1} - \varepsilon_*[\widehat{f}_i^{-1} + \widehat{f}_j^{-1} + e(\varepsilon)\widehat{f}_r^{-1}]](\sigma - y) > \\ &> [\min_{2\le i\le r}(\widehat{f}_i^{-1} - \widehat{f}_{i-1}^{-1}) - e(\varepsilon)\widehat{f}_r^{-1} - \varepsilon_*[2\widehat{f}_r^{-1} + e(\varepsilon)\widehat{f}_r^{-1}]](\sigma - y) > \\ &> \frac{1}{2}[\min_{2\le i\le r}(f_i^{-1} - \widehat{f}_{i-1}^{-1}) - e(\varepsilon)\widehat{f}_r^{-1}](\delta - y) > 0, \end{aligned}$$

whence we conclude that the first of inequalities (2.3)(c) is valid. The second one follows from it and from relation (2.6) (with the replacement of ε_* by ε_0).

Now, let us introduce the following notation

$$(2.8)\qquad \begin{cases} z_{\vec{k}(2s)}(x) = h_{k_{2s}} \circ f_{k_{2s-1}} \circ z_{\vec{k}(2s-2)}(x) & \text{for } s \geq 2; \\ z_{\vec{k}(2)}(x) = h_{k_2} \circ f_{k_1}(x); \end{cases}$$

$$(2.9)\qquad \begin{cases} \mathring{z}_{\vec{k}(2s-1)}(x) = f_{k_{2s-1}} \circ z_{\vec{k}(2s-2)}(x) & \text{for } s \geq 2; \\ \mathring{z}_{\vec{k}(1)}(x) = f_{k_1}(x); \end{cases}$$

$$(2.10)\qquad \begin{cases} \mu_{\vec{k}(2s)}(x) = f_{k_{2s}}^{-1} \circ h_{k_{2s-1}}^{-1} \circ \mu_{\vec{k}(2s-2)} & \text{for } s \geq 2; \\ \mu_{\vec{k}(2)}(x) = f_{k_2}^{-1} \circ h_{k_1}^{-1}(x); \end{cases}$$

$$(2.11)\qquad \begin{cases} \mathring{\mu}_{\vec{k}(2s-1)}(x) = h_{k_{2s-1}}^{-1} \circ \mu_{\vec{k}(2s-2)}(x) & \text{for } s \geq 2; \\ \mathring{\mu}_{\vec{k}(1)}(x) = h_{k_1}^{-1}(x), \end{cases}$$

where $\vec{k}(m) = (k_m, k_{m-1}, \ldots, k_1)$ for $m \in N$ with $1 \leq k_\nu \leq r$ for $\nu = 1, 2, \ldots, m$.

The functions $z_{\vec{k}(2s)}(x)$ and $\mathring{z}_{\vec{k}(2s)}(x)$ were introduced in [1], [2]. The function $\mu_{\vec{k}(2s)}(x)$ is a generalization of the functions $\mu(x)$ and $\mu_j(x)$ considered in [5] and [3], respectively. The function $\mathring{\mu}_{\vec{k}(2s-1)}(x)$ has not been dealt with so far.

LEMMA 2.3. *The following relations*

$$(2.12)\qquad z_{\vec{k}(2s)} \rightrightarrows 0 \quad \textit{on } [0,1),$$

$$(2.13)\qquad \mu_{\vec{k}(2s)} \rightrightarrows \quad \textit{on } (0,1]$$

hold good, when $s \to \infty$, *with* $\rightrightarrows$ *denoting the almost-uniform convergence.*

Proof. The validity of (2.12) follows from Lemma 4 in [2].

In order to prove (2.13), let us observe that by Assumption I and definition (2.10), we have

$$(2.14)\qquad \underline{\mu}^s(x) \leq \mu_{\vec{k}(2s)}(x) \leq \overline{\mu}^s(x)$$

($x \in (0,1]$), where $\underline{\mu}(x) = f_r^{-1} \circ h_r^{-1}(x)$; $\overline{\mu}(x) = f_1^{-1} \circ h_1^{-1}(x)$ (we set $\underline{\mu}^s(x) = \underline{\mu} \circ \underline{\mu}^{s-1}(x)$; $\overline{\mu}(x) = \overline{\mu}^s \circ \overline{\mu}^{s-1}(x)$ for $s \in N$; $s \geq 2$).

Using Lemma 2 in [3], we can assert that $\underline{\mu}^s \rightrightarrows 1$ on $(0,1]$ and $\overline{\mu}^s \rightrightarrows 1$ on $(0,1]$ when $s \to \infty$, whence and from (2.14) it follows that relation (2.13) is valid, as required.

LEMMA 2.4. *If δ_2 (where $0 < \delta_2 \leq \delta_1$) is a sufficiently small number then the inequalities*

$$\left|\frac{d}{dx} z_{\vec{k}(2s)}(x)\right| \leq (1+\varepsilon_0)^{2s} \prod_{\nu=1}^{s} \check{h}_{k_{2\nu}} \check{f}_{k_{2\nu-1}} \leq (1+\varepsilon_0)^{2s} g_0^s \tag{2.15}$$

(where $x \in [0, \delta_2]$);

$$\left|\frac{d}{dx} \mu_{\vec{k}(2s)}(x)\right| \leq (1+\varepsilon_0)^{2s} \prod_{\nu=1}^{s} \widehat{f}_{k_{2\nu}}^{-1} \widehat{h}_{k_{2\nu-1}}^{-1} \leq (1+\varepsilon_0)^{2s} g_0^s \tag{2.16}$$

(where $x \in [1-\delta_2, 1]$) are valid.

Proof. Relation (2.15) follows from (51) in [2]. The estimate (2.16) is a consequence of definition (2.10), inequality (2.6) with ε_* replaced by ε_0, and an analogous inequality for the function h_α^{-1}.

LEMMA 2.5. *The following inequalities*

$$\left|\frac{d^m}{dx^m} z_{\vec{k}(2s)}(x)\right| \leq \text{const} s^{q(m-1)} (1+\varepsilon_0)^{2s} g_0^s \tag{2.17}$$

(where $x \in [0, \delta_2]$);

$$\left|\frac{d^m}{dx^m} \mu_{\vec{k}(2s)}(x)\right| \leq \text{const} s^{q(m-1)} (1+\varepsilon_0)^{2s} g_0^s \tag{2.18}$$

(where $x \in [1-\delta_2, 1]$) hold good for $m = 2, 3, \ldots, q$.

The validity of the Lemma follows from Lemma 2.4 and the formula for m-th derivative of a composite function (cf. [8], Remark).

3. Solution of the Problem

We shall find sufficient conditions for the existence of a solution to problem ($\mathfrak{G}$) and give a formula for this solution. Our method will be an adaptation of those used in [5] and [9] (cf. also [1]–[3] and [6], Chap. II).

Let $\sigma_0 \in (0, \sigma)$ be arbitrarily fixed and denote $x_i = f_i^{-1}(\sigma_0)$; $\widetilde{x}_i = h_i(\sigma_0)$ ($i = 1, 2, \ldots, r$). We introduce the rectangles $\Delta = [0, \widetilde{x}_r] \times [0, \sigma_0]$ and $\Omega = [\widetilde{x}_r, 1] \times [\sigma_0, \sigma]$ (see Fig. 1).

We are going to consider problem ($\mathfrak{G}$) in the said domains, successively, beginning with Ω.

Writing the boundary conditions (1.2)(a), (b) in the form

$$L^{jr} D_y^{q-p} u[f_i^{-1}(y), y] = M_{i,j} \circ f_i^{-1}(y)$$
$$L^{jr} D_y^{q-p} u[x, h_i^{-1}(x)] = N_{i,j} \circ h_i^{-1}(x)$$

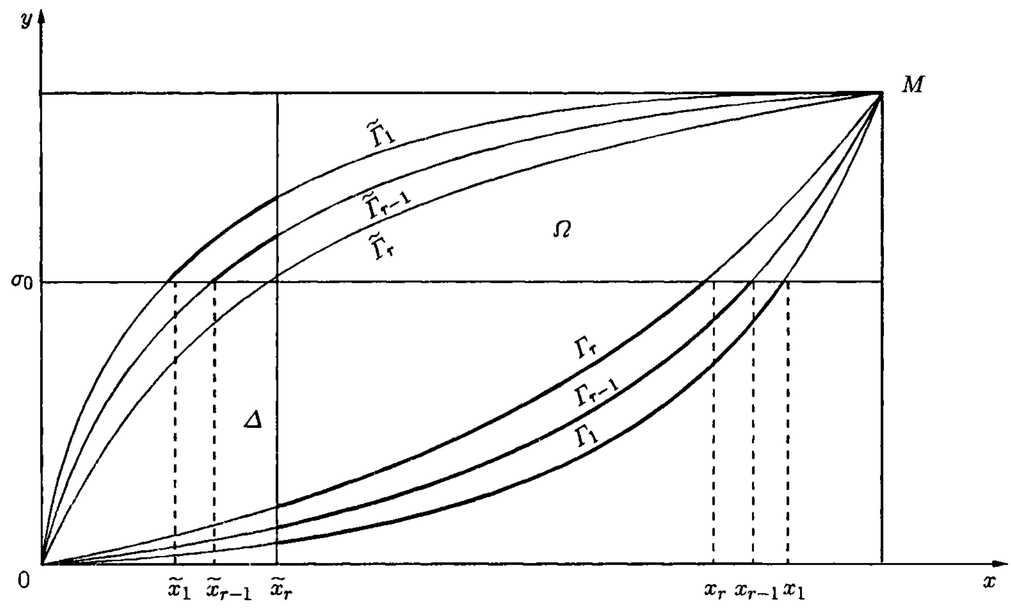

Fig. 1

($(x, y) \in \Omega$; $i = 1, 2, \ldots, r$; $j = 0, 1, \ldots, k-1$, and imposing them on the solution of equation (1.1) (cf. (2.1)(b)), we get the following system of differential-functional equations (for simplicity we omit $*$ over ϕ_m and ψ_m)

$$
\begin{aligned}
&\sum_{m=jr+1}^{p} \{(-1)^{q-p} b_{m,j}^{1,q-p} (\sigma - y)^{m-jr-1} \phi_m^{(jr)} \circ f_i^{-1}(y) + \\
&\qquad + b_{m,j}^{1,0} (1 - f_i^{-1}(y))^{m-jr-1} \breve{\psi}_m^{(jr)}(y)\} = (-1)^{jr} M_{i,j} \circ f_i^{-1}(y), \\
(3.1)\qquad &\sum_{m=jr+1}^{p} \{b_{m,j}^{1,q-p} (\sigma - h_i^{-1}(x))^{m-jr-1} \phi_m^{(jr)}(x) + \\
&\qquad + b_{m,j}^{1,0} (1 - x)^{m-jr-1} \breve{\psi}^{(jr)} \circ h_i^{-1}(x)\} = (-1)^{jr} N_{ij} \circ h_i^{-1}(x),
\end{aligned}
$$

where $\breve{\psi}_m = \psi_m^{(q-p)}$ and

$$
(3.1') \qquad b_{m,j}^{1,z} = \frac{(z + m - 1)!}{(z + m - jr - 1)!}
$$

($z = q - p, 0$), which, by using an argument analogous to that in [2], pp. 258–260, can be for $x \in [\widetilde{x}_r, 1)$; $y \in [\sigma_0, \sigma)$ equivalently transformed to the form (cf. [9], p. 219)

$$(3.2)\quad \begin{aligned} &\phi^{(jr)}_{jr+\alpha}(x)=\overset{*}{V}_{jr+\alpha}(x)+\\ &\quad+b^{2,q-p}_{\alpha,j}(-1)^{q-p}\sum_{\nu,i=1}^{r}\frac{(\nu+jr-1)!}{(\nu-1)!}\overset{*\alpha}{G}_{\nu,i}(x)\check{\psi}^{(jr)}_{jr+\nu}\circ h_i^{-1}(x),\\ &\check{\psi}^{(jr)}_{jr+\alpha}(y)=\overset{*}{W}_{jr+\alpha}(y)+\\ &\quad+b^{2,0}_{\alpha,j}(-1)^{q-p}\sum_{\nu,i=1}^{r}\frac{(q-p+\nu+jr-1)!}{(q-p+\nu-1)!}\overset{**\alpha}{G}_{\nu,i}(y)\phi^{(jr)}_{jr+\nu}\circ f_i^{-1}(y)\end{aligned}$$

$(x\in[\widetilde{x}_r,1);\ y\in[\sigma_0,\sigma);\ \alpha=1,2,\dots,r;\ j=0,1,\dots,k-1)$, where

$$(3.2')\qquad b^{2,z}_{\alpha,j}=\frac{(z+\alpha-1)!}{(z+\alpha+jr-1)!}$$

$(z=q-p,0)$;

$$(3.3)\quad \begin{aligned} &\overset{*}{V}_{jr+\alpha}=\\ &=(-1)^{q-p+\alpha-1}b^{2,q-p}_{\alpha,j}\sum_{\nu=1}^{r}\overset{*}{w}_\nu(x)\overset{*}{e}{}^{\alpha}_{\nu}(x)\{(-1)^{jr}N_{\nu,j}\circ h_\nu^{-1}(x)+\\ &-\sum_{m=(j+r)r+1}^{p}<(-1)^{q-p}b^{1,q-p}_{m,j}(\sigma-h_i^{-1}(x))^{m-jr-1}\phi^{(jr)}_m(x)+\\ &+b^{1,0}_{m,j}(1-x)^{m-jr-1}\check{\psi}^{(jr)}_m\circ h_i^{-1}(x)>\},\end{aligned}$$

$$(3.4)\quad \begin{aligned} &\overset{*}{W}_{jr+\alpha}(y)=\\ &=(-1)^{\alpha-1}b^{2,0}_{\alpha,j}\sum_{\nu=1}^{r}\overset{**}{w}_\nu(y)\overset{**}{e}{}^{\alpha}_{\nu}(y)\{(-1)^{jr}M_{\nu,j}\circ f_\nu^{-1}(y)+\\ &-\sum_{m=(j+1)r+1}^{p}<(-1)^{q-p}b^{1,q-p}_{m,j}(\sigma-y)^{m-jr-1}\phi^{(jr)}_m\circ f_i^{-1}(y)+\\ &+b^{1,0}_{m,j}(1-f_i^{-1}(y))^{m-jr-1}\check{\psi}^{(jr)}(y)>\};\end{aligned}$$

$$(3.5)\quad \begin{aligned} &\overset{*\alpha}{G}_{\nu,i}(x)=(-1)^\alpha\overset{*}{w}_i(x)\overset{*}{e}{}^{\alpha}_{i}(x)(1-x)^{\nu-1}\\ &\overset{**\alpha}{G}_{\nu,i}(y)=(-1)^\alpha\overset{**}{w}_i(y)\overset{**}{e}{}^{\alpha}_{i}(y)(\sigma-y)^{\nu-1}\end{aligned}$$

with

$$(3.6)\qquad \overset{*}{w}_i(x)=\begin{cases}\displaystyle\prod_{\substack{t=1\\ t\neq i}}^{r}[h_i^{-1}(x)-h_t^{-1}(x)]^{-1} & \text{for } r\geq 2\\ 1 & \text{for } r=1\end{cases}$$

(3.7) $\overset{*}{e}{}_i^{\alpha}(x) =$

$$= \begin{cases} \sum\limits_{1<t_1<\ldots<t_{r-\alpha}\le r} \prod\limits_{\beta=1}^{r-\alpha}[\sigma - h_\beta^{-1}(x)] & \text{for } \alpha = 1,2,\ldots,r-1 \text{ when } r \geq 2 \\ 1 & \text{for } \alpha = r, \end{cases}$$

$(t_1 \neq i, \ldots, t_{r-\alpha} \neq i)$, $\overset{**}{w}_i(y)$ and $\overset{**}{e}{}_i^{\alpha}(y)$ being given by formulae (3.6), (3.7), respectively, with the replacement of $h_i^{-1}(x)$ by $f_i^{-1}(y)$ $(i = 1,2,\ldots,r)$ and σ by 1.

In what follows we shall use the following notation

$$\overset{*}{\Xi}{}^{\alpha}_{\vec{\nu}(2n)\vec{k}(2n)}(x) = \prod_{\gamma=1}^{n} \overset{*}{G}{}^{\nu_{2\gamma-2}}_{\nu_{2\gamma-1},k_{2\gamma-1}} \circ \mu_{\vec{k}(2\gamma-2)}(x) \times \times \prod_{\gamma=1}^{n} \overset{**}{G}{}^{2\gamma-1}_{\nu_{2\gamma},k_{2\gamma}} \circ \overset{*}{\mu}_{\vec{k}(2\gamma-1)}(x) \tag{3.8}$$

(where $\nu_0 := \alpha$; $\mu_{\vec{k}(0)}(x) := x$);

$$\overset{*}{Q}_{jr+\alpha}(x) = \overset{*}{V}_{jr+\alpha}(x) + (-1)^{q-p}\frac{(q-p+\alpha-1)!}{(q-p+\alpha+jr-1)!} \times \times \sum_{\nu,\mu=1}^{r} \frac{(\nu+jr-1)!}{(\nu-1)!} \overset{*}{G}{}^{\alpha}_{\nu,\mu}(x)\overset{*}{W}_{jr+\nu} \circ h_\mu^{-1}(x). \tag{3.9}$$

PROPOSITION 3.1. *The set of functions $\phi_1,\ldots,\phi_p$; $\psi_1,\ldots,\psi_p$ given by the formulae*

(3.10) $\phi_{jr+\alpha}(x) = \overset{*}{\phi}_{jr+\alpha}(x) :=$

$$:= \begin{cases} (1-\delta_{0j} \int\limits_x^1 \frac{(\xi - x)^{jr-1}}{|jr-1|!}\overset{*}{S}_{j,\alpha}(\xi)\,d\xi + \delta_{0j}\overset{*}{S}_{0,\alpha}(x) & \textit{for } x \in [\tilde{x}_r, 1) \\ 0 & \textit{for } x = 1; \end{cases}$$

(3.11) $\psi_{jr+\alpha}(y) = \overset{*}{\psi}_{jr+\alpha}(y) :=$

$$:= \begin{cases} (1-\delta_{0j}) \int\limits_y^\sigma \frac{(\eta - y)^{q-p+jr-1}}{|q-p+jr-1|!}\overset{**}{S}_{j,\alpha}(\eta)\,d\eta + \\ +\delta_{0,j}\Big\{(1-\delta_{0,q-p})\Big(\int\limits_y^\sigma \frac{(\eta-y)^{q-p-1}}{|q-p-1|!}\overset{**}{S}_{0,\alpha}(\eta)\cdot d\eta + \delta_{0,q-p}\overset{**}{S}_{0,\alpha}(y)\Big\} \\ \qquad \textit{for } y \in [\sigma_0, \sigma) \\ 0 \qquad \textit{for } y = \sigma \end{cases}$$

($\alpha = 1,2,\dots,r$; $j = 0,1,\dots,k-1$; $0(\pm\infty) := 0$), *where**)

$$(3.12)\qquad \overset{*}{S}_{j,\alpha}(x) = \overset{*}{Q}_{jr+\alpha}(x) + \sum_{n=1}^{\infty} \overset{*}{a}_n^{j,\alpha}(x);$$

$$(3.13)\qquad \overset{**}{S}_{j,\alpha}(y) = \overset{*}{W}_{jr+\alpha}(y) + \sum_{\nu,\mu=1}^{r} \overset{**\alpha}{G}_{\nu,\mu}(y)\Big[\overset{*}{V}_{jr+\nu} \circ f_\mu^{-1}(y) + \\ + \sum_{n=1}^{\infty} \overset{*}{a}_n^{j,\alpha} \circ f_\mu^{-1}(y)\Big]$$

with

$$(3.14)\qquad \overset{*}{a}_n^{j,\alpha}(x) = \sum_{\substack{\nu_1,\dots,\nu_{2n}=1 \\ k_1,\dots,k_{2n}=1}}^{r} (-1)^{jr} b_{\alpha,j}^{2,q-p} \frac{(q-p+\nu_{2n}+jr-1)!}{(q-p+\nu_{2n}-1)!} \times \\ \times \overset{*}{\Xi}_{\vec{\nu}(2n),\vec{k}(2n)}(x) \overset{*}{Q}_{jr+\nu_{2n}} \circ \mu_{\vec{k}(2n)}(x)$$

($x \in [\widetilde{x}_r, 1)$; $y \in [\sigma_0, \sigma)$) is a solution to system (3.2) in the set $[\widetilde{x}_r, 1) \times [\sigma_0, \sigma)$. The functions $\overset{*}{\phi}_m$ and $\overset{*}{\psi}_m$ ($m = 1,2,\dots,p$) are of class C^q. It is the only solution of system (3.2) in the class $\overset{*}{\mathfrak{K}}$ of all sets of functions ϕ_m and ψ_m ($m = 1,2,\dots,p$) possesing continuous derivatives of the orders up to and including $\left[\frac{m-1}{r}\right] r$ and $\left[\frac{m-1}{r}\right] r + q - p$, respectively (where $[\;]$ is the greatest integer function) and such that the relations

$$(3.15)\qquad \begin{cases} \|\phi_m^{(l)}(x)\| \le \mathrm{const}(1-x)^{q+2p-2m-l+\varkappa_0} \\ \|\psi_m^{(t)}(y)\| \le \mathrm{const}(\sigma-y)^{q+2p-2m-t+\varkappa_0} \end{cases}$$

($(x,y) \in \Omega$; $m = 1,2,\dots,p$; $t = q-p+l$; $l = 0,1,\dots,\left[\frac{m-1}{r}\right] r$) hold good.

Proof. We shall use the inequalities

$$(3.16)\qquad |\overset{*}{e}_i^{\alpha}(x)| \le C_*(\sigma - h_r^{-1}(x))^{r-\alpha},$$

$$(3.17)\qquad |\overset{*}{w}_i^{\alpha}(x)| \le [e(\varepsilon)]^{r-1}(\sigma - h_r^{-1}(x))^{1-r},$$

(where $C_* = r^{r-1}$; $x \in (1-\delta_1, 1)$), resulting from (3.6), (3.7) and Lemma 2.2.

From (3.5), (3.16) and (3.17) it follows that

$$(3.18)\qquad |\overset{*\alpha}{G}_{\nu,i}(x)| \le C_*[e(\varepsilon)]^{r-1}(\sigma - h_r^{-1}(x))^{1-\alpha}(1-x)^{\nu-1}$$

($x \in (1-\delta_1, 1)$).

---

*) We point out that the iteration process used in deriving formulae (3.12)-(3.14) is different from those in papers [2] (see p. 262) and [9] (see p. 223).

In a similar way we get

$$(3.19)\qquad |\overset{**}{G}{}^{\alpha}_{\nu,i}(y)| \le C_*[e(\varepsilon)]^{r-1}(1-f_r^{-1}(y))^{1-\alpha}(\sigma-y)^{\nu-1}$$

$(y \in (\sigma-\sigma_1, \sigma))$.

Let us observe that, by Lemma 2.3, there is a number $n_0 \in N$ such that for $n \in N$; $n > n_0$ and $x \in [\widetilde{x}_r, 1]$ the relation $\mu_{\vec{k}(2n)}(x) \in (1-\delta, 1]$ holds good, where $0 < \delta \le \delta_2$ (cf. Lemma 2.4).

We shall first consider system (3.2) for $j = k-1$ (i. e. $jr+\alpha = p-r+\alpha$).

It is clear that in this case the function $\overset{*}{Q}_{p-r+\alpha}$ (cf. (3.3), (3.4) and (3.9)) does not depend on the functions ϕ_m and ψ_m $(m = 1, 2, \ldots, p)$. One can also prove by using relations (1.9) and (3.16)–(3.18) that

$$(3.20)\qquad \|\overset{*}{Q}_{p-r+\alpha}(x)\| \le \text{const}(1-x)^{q-p+3r-2\alpha+\varkappa_0}$$

$(x \in (1-\delta, 1)$; $\alpha = 1, 2, \ldots, r)$.

Basing on (2.16), (3.8), (3.14), and (3.18)–(3.20), we have for $n > n_0$ the following sequence of inequalities

$$(3.21)\quad \|\overset{*}{a}{}_n^{k-1,\alpha}(x)\| \le$$

$$\le \text{const}\{C_*[e(\varepsilon)]^{r-1}\}^n \sum_{\substack{\nu_1,\ldots,\nu_{2n}=1\\ k_1,\ldots,k_{2n}=1}} \frac{(1-x)^{\nu_1-1}}{(\sigma-h_r^{-1}(x))^{\alpha-1}} \times$$

$$\times \frac{(\sigma-\overset{*}{\mu}_{\vec{k}(1)}(x))^{\nu_2-1}}{(1-f_r^{-1}\circ\overset{*}{\mu}_{\vec{k}(1)}(x))^{\nu_1-1}} \frac{(1-\mu_{\vec{k}(2)}(x))^{\nu_3-1}}{(\sigma-h_r^{-1}\circ\mu_{\vec{k}(2)}(x))^{\nu_2-1}} \cdots \times$$

$$\times \frac{(1-\mu_{\vec{k}(2n-2)}(x))^{\nu_{2n-1}-1}}{(\sigma-h_r^{-1}\circ\mu_{\vec{k}(2n-2)}(x))^{\nu_{2n-2}-1}} \frac{(\sigma-\overset{*}{\mu}_{\vec{k}(2n-1)}(x))^{\nu_{2n}-1}}{(1-f_r^{-1}\circ\overset{*}{\mu}_{\vec{k}(2n-1)}(x))^{\nu_{2n-1}-1}} \times$$

$$\times \|\overset{*}{Q}_{p-r+\nu_{2n}}\circ\mu_{\vec{k}(2n)}(x)\| \le \text{const}\{[re(\varepsilon)]^{r-1}(1-\varepsilon_0)^{2(1-r)}\}^n \times$$

$$\times \sum_{\substack{\nu_1,\ldots,\nu_{2n}=1\\ k_1,\ldots,k_{2n}=1}}^{r} \prod_{s=1}^{2n-1} g_0^{1-\nu_s}(\sigma-\overset{*}{\mu}_{\vec{k}(2n-1)}(x))^{q-p+3r-\nu_{2n}-1+\varkappa_0}(1-x)^{1-\alpha} \le$$

$$\le \text{const}\{[re(\varepsilon)]^{r-1}(1-\varepsilon_0)^{-2(3q+\varkappa_0)}g_0^{1+\varkappa_0}\}^n(1-x)^{q-p+2r-\alpha+\varkappa_0} \le$$

$$\le \text{const}[(r+1)^r(1+\varepsilon)^{r-1}g_0]^n \times$$

$$\times [(r+1)^r g_0^{\varkappa_0}(1-\varepsilon_0)^{-2(3q+\varkappa_0)}]^n(1-x)^{q-p+2r-\alpha+\varkappa_0}$$

$(\alpha = 1, 2, \ldots, r)$, where const is independent of n.

It follows from the choice of the parameters ε and ε_0 (see (1.8) and (2.4)) that $(r+1)^r(1+\varepsilon)^{r-1}g_0 < 1$; $(r+1)^r g_0^{\varkappa_0}(1-\varepsilon_0)^{-2(3q+\varkappa_0)} < 1$, whence and from (3.21) we can conclude that the series in (3.12) is (for $j = k-1$) uniformly convergent in the interval $[\widetilde{x}_r, 1)$.

One can also conclude from (3.21) (cf. [2], p. 267) that the inequality

$$\|\overset{*}{a}{}_n^{k-1,\alpha}(x)\| \leq \text{const} b^n (1-x)^{q-p+2r-\alpha+\varkappa_0}, \tag{3.22}$$

where $b \in (0,1)$, holds good for all $n \in N$; $x \in [\widetilde{x}_r, 1)$, whence and from (3.12), (3.20) it follows that

$$\|\overset{**}{S}_{k-1,\alpha}(x)\| \leq \text{const}(1-x)^{q-p+2r-\alpha+\varkappa_0} \tag{3.23}$$

($x \in [\widetilde{x}_r, 1)$; $\alpha = 1, 2, \ldots, r$).

Inequality (3.23) implies (cf. (3.10))

$$\|\overset{*(l)}{\phi}_{p-r+\alpha}(x)\| \leq \text{const}(1-x)^{q+r-\alpha-l+\varkappa_0} \tag{3.24}$$

where $l = 0, 1, \ldots, p-r$.

Differentiating the expression $\overset{*}{a}{}_n^{k-1,\alpha}(x)$ (cf. 3.14)), basing on Lemma 2.5 and on the formula quoted in the proof of that lemma, and using an argument similar to that in the derivation (3.22) above (cf. also [2], pp. 270–275), one can prove that the functions $\overset{**}{S}_{k-1,\alpha}(x)$, $\alpha = 1, 2, \ldots, r$, are of class C^q in $[\widetilde{x}_r, 1)$ and that the estimate

$$\|\overset{**(l)}{S}_{k-1,\alpha}(x)\| \leq \text{const}(1-x)^{q-p+2r-\alpha-l+\varkappa_0} \tag{3.25}$$

($x \in [\widetilde{x}_r, 1)$; $\alpha = 1, 2, \ldots, r$: $l = 1, 2, \ldots, q-p+r$) holds good, whence we can conclude that relation (3.24) is valid for $l = p-r+1, \ldots, q$.

As a consequence

$$\lim_{x \to 1_-} \overset{*(l)}{\phi}_{p-r+\alpha}(x) = 0 \tag{3.26}$$

($\alpha = 1, 2, \ldots, r$; $l = 0, 1, \ldots, q$) and hence the functions $\overset{*}{\phi}_{p-r+\alpha}$; $\alpha = 1, 2, \ldots, r$ are of class C^q in $[\widetilde{x}_r, 1]$.

By a similar argument one can prove that the functions $\psi_{p-r+\alpha}$; $\alpha = 1, 2, \ldots, r$ are of class C^q in $[\sigma_0, \sigma]$.

A direct calculation, similar to that in [2], p. 268, shows that the set of functions $\overset{*}{\phi}_{p-r+1}, \ldots, \overset{*}{\phi}_p$; $\overset{**}{\psi}_{p-r+1}, \ldots, \overset{**}{\psi}_p$ (where $\overset{**}{\psi}_{p-r+\alpha} = \overset{*(q-p)}{\psi}_{p-r+\alpha}$) is a solution to system (3.2) with $j = k-1$ in the set $[\widetilde{x}_r, 1) \times [\sigma_0, \sigma)$.

Let us observe that if the functions $\phi_{p-r+\alpha}$ and $\psi_{p-r+\alpha}$ ($\alpha = 1, 2, \ldots, r$) satisfy the system (3.2) with $j = k-1$, then for each $m \in N$ the equality

$$\phi_{p-r+\alpha}^{(p-r)}(x) = \overset{*}{Q}_{p-r+\alpha}(x) + \sum_{n=1}^{m} \overset{*}{a}{}_n^{k-1,\alpha}(x) + \rho_m^\alpha(x) \tag{3.27}$$

($x \in [\widetilde{x}_r, 1)$; $\alpha = 1, 2, \ldots, r$) holds good, where

$$\rho_m^{\alpha}(x) = \sum_{\substack{\nu_1,\ldots,\nu_{2n+2}=1\\ k_1,\ldots,k_{2n+2}=1}} (-1)^{jr} \frac{(q-p+\alpha-1)!(q-p+\nu_{2m+2}+p-r-1)!}{(q+\alpha-r-1)!(q-p+\nu_{2m+2}-1)!} \times$$
$$\times \overset{*\alpha}{\Xi}_{\vec{\nu}(2m+2),\vec{k}(2m+2)}(x)\phi_{p-r+\nu_{2m+2}} \circ \mu_{\vec{k}(2m+2)}(x).$$

The uniqueness of the solution in the class $\overset{*}{\mathfrak{K}}$ is proved by using an argument similar to that in the derivation of (3.21) and by basing on (3.27).

Thus, the proof of Proposition 3.1 is completed in the case $k = 1$ (that is $r = p$).

If $k > 1$ then we base on the results obtained above for $j = k - 1$ and use mathematical induction (cf. [9], p. 228), hence completing the proof of Proposition 3.1.

We still have to impose on function u the boundary condition (1.2)(c) for $y = [\sigma_0, \sigma]$. It is easily observed that this condition yields the following system of differential equations

$$(3.28)\quad \overset{*}{\omega}{}_{l+1}^{(l)}(x) = \overline{N}_l \circ h_r^{-1}(x) + (-1)^{l+1}\Bigg\{ \sum_{s=l+2}^{q-p} \frac{(\sigma - h_r^{-1}(x))^{s-l-1}}{(s-l-1)!} \overset{*}{\omega}{}_s^{(l)}(x) +$$
$$+ \sum_{m=m_1}^{p} C_{m,l}(\sigma - h_r^{-1}(x))^{q-p+m-l-1} \overset{*}{\phi}{}_m^{(l)}(x) +$$
$$+ \sum_{m=m_2}^{q-p} (1-x)^{m-l-1} \overset{*}{\psi}{}_m^{(l)} \circ h_r^{-1}(x) \Bigg\}$$

($x \in [\widetilde{x}_r, 1]$; $l = 0, 1, \ldots, q-p-1$; $C_{m,l} = \dfrac{(q-p+m-1)!}{(q-p-m-l-1)!}$; $m_1 = \max(1, p-q+l+1)$; $m_2 = l+1$), where the functions $\overset{*}{\phi}_m$ and $\overset{*}{\psi}_m$ are given by formulae (3.10), (3.11), respectively.

We use relation (3.28) for $l = q-p-1, \ldots, 2, 1$, successively, and hence find the functions $\overset{*}{\omega}_m(x)$ ($m = 1, 2, \ldots, q-p$) from the formula

$$(3.29)\quad \overset{*}{\omega}_{l+1} = (1-\delta_{0l}) \int_x^1 (\xi - x)^{l-1} \overset{*}{H}_l(\xi)\, d\xi + \delta_{0l}\overset{*}{H}_0(x) + \sum_{\nu=0}^{l-1} \overset{*}{A}_\nu (1-x)^\nu$$

($x \in [\widetilde{x}_r, 1]$; $l = 0, 1, \ldots, q-p-1$), where $\overset{*}{H}_l$ denotes the right-hand side expression in (3.28) and $\overset{*}{A}_\nu$ ($\nu = 0, 1, \ldots, l-1$; $l = 0, 1, \ldots, q-p-1$) are arbitrary constants.

Now, we consider problem ($\mathfrak{G}$) in the domain Δ.

Imposing on the solution of equation (1.1) (cf. (2.1)(a)) the boundary conditions (1.2)(a), (b), and proceeding analogously as in the case of Ω, we get a counterpart of the system (3.2) in the form

$$(3.30)\quad \begin{cases} \phi^{(jr)}_{jr+\alpha}(x) = V_{jr+\alpha}(x) + \\ \quad + b^{2,q-p}_{\alpha,j}(-1)^{q-p}\sum_{\nu,i=1}^{r}\frac{(\nu+jr-1)!}{(\nu-1)!}G^{\alpha}_{\nu,i}(x)\breve{\psi}^{(jr)}_{jr+\nu}\circ f_i(x), \\ \breve{\psi}_{jr+\alpha}(y) = W_{jr+\alpha}(y) + \\ \quad + b^{2,0}_{\alpha,j}(-1)^{q-p}\sum_{\nu,i=1}^{r}\frac{(q-p+\nu+jr-1)!}{(q-p+\nu-1)!}\widetilde{G}^{\alpha}_{\nu,i}(y)\breve{\phi}^{(jr)}_{jr+\nu}\circ h_i(y) \end{cases}$$

$(x \in (0, \widetilde{x}_r];\ y \in (0, \sigma_0];\ \alpha = 1, 2, \ldots, r;\ j = 0, 1, \ldots, k-1)$ with

$$(3.31)\quad V_{jr+\alpha}(x) = (-1)^{q-p+\alpha-1}b^{2,q-p}_{\alpha,j}\sum_{\nu=1}^{r} w_\nu(x)e^{\alpha}_{\nu}(x)\Big\{(-1)^{jr}M_{\nu,j}(x) + \\ - \sum_{m=(j+1)r+1}^{r} < (-1)^{q-p}b^{1,q-p}_{m,j}[f_i(x)]^{m-jr-1}\phi^{(jr)}_{m}(x) + \\ + \frac{(m-1)!}{(m-jr-1)!}x^{m-jr-1}\breve{\psi}^{(jr)}\circ f_i(x) > \Big\},$$

$$(3.32)\quad W_{jr+\alpha}(y) = (-1)^{\alpha-1}b^{2,0}_{\alpha,j}\sum_{\nu=1}^{r}\widetilde{w}_\nu(y)\widetilde{e}^{\alpha}_{\nu}(y)\Big\{(-1)^{jr}N_{\nu,j}(y) + \\ - \sum_{m=(j+1)r+1}^{p} < (-1)^{q-p}b^{1,q-p}_{m,j} + y^{m-jr-1}\phi^{(jr)}_{m}\circ h_i(y) + \\ + \frac{(m-1)!}{(m-jr-1)!}[h_i(y)]^{m-jr-1}\breve{\psi}^{(jr)}_{m}(y) > \Big\},$$

$$(3.33)\quad \begin{cases} G^{\alpha}_{\nu,i}(x) = (-1)^{\alpha}w_i(x)e^{\alpha}_i(x)x^{\nu-1} \\ \widetilde{G}^{\alpha}_{\nu,i}(y) = (-1)^{\alpha}\widetilde{w}_i(y)\widetilde{e}^{\alpha}_i(y)y^{\nu-1}, \end{cases}$$

where the functions w_i and e^{α}_i are given by formulae (3.6), (3.7) with $h^{-1}_{\mu}(x)$ $(\mu = \nu, t)$ and $\sigma - h^{-1}_{\beta}(x)$ replaced by $f_\mu(x)$ and $f_\beta(x)$, respectively (the functions $\widetilde{w}_i$ and $\widetilde{e}_i$ are defined by formulae analogous to those for w_i and e^{α}_i, respectively, with the replacement of $f_\tau(x)$ by $h_\tau(y)$, where $\tau = \mu, \beta)$.

The following proposition is valid (the proof of which is analogous to that of Proposition 3.1):

PROPOSITION 3.2. *Let the functions* $\phi_1, \ldots, \phi_p$; $\psi_1, \ldots, \psi_p$ *be given by the formulae*

$$(3.34)\quad \phi_{jr+\alpha}(x) = \overset{\circ}{\psi}_{jr+\alpha}(x) :=$$
$$:= \begin{cases} (1-\delta_{0j}) \int\limits_0^x \frac{(x-\xi)^{jr-1}}{|jr-1|!} S_{j,\alpha}(\xi) d\xi + \delta_{0j} S_{0,\alpha}(x) & \text{for } x \in (0, \widetilde{x}_r] \\ 0 & \text{for } x = 0; \end{cases}$$

$$(3.35)\quad \psi_{jr+\alpha}(y) = \overset{\circ}{\psi}_{jr+\alpha}(y) :=$$
$$:= \begin{cases} (1-\delta_{0j}) \int\limits_0^y \frac{(y-\eta)^{q-p+jr-1}}{|q-p+jr-1|!} \widetilde{S}_{j,\alpha}(\eta)\, d\eta + \\ +\delta_{0j} \left\{ (1-\delta_{0,q-p}) \int\limits_0^y \frac{(y-\eta)^{q-p-1}}{|q-p-1|!} \widetilde{S}_{0,\alpha}(\eta)\, d\eta + \delta_{0,q-p} \widetilde{S}_{0,\alpha}(y) \right\} \\ & \text{for } y \in (0, \sigma_0] \\ 0 & \text{for } y = 0; \end{cases}$$

($\alpha = 1, 2, \ldots, r$; $j = 0, 1, \ldots, k-1$), *where*

$$(3.36)\quad S_{j,\alpha}(x) = Q_{jr+\alpha}(x) + \sum_{n=1}^{\alpha} a_n^{j,\alpha}(x);$$

$$(3.37)\quad \widetilde{S}_{j,\alpha}(y) = W_{jr+\alpha}(y) +$$
$$+ \sum_{\nu,\mu=1}^{r} \widetilde{G}_{\nu,\mu}(y) \Big[V_{jr+\nu} \circ h_\mu(y) + \sum_{n=1}^{\alpha} a_n^{j,\alpha} \circ h_\mu(y) \Big],$$

$a_n^{j,\alpha}(x)$ *being defined by a formula analogous to* (3.14) (*together with* (3.8), (3.9)) *with the replacement of* $\overset{*\alpha}{G}_{\nu,i}(x)$, $\overset{**\alpha}{G}_{\nu,i}(y)$, $\mu_{\vec{k}(2s)}(x)$ *and* $\overset{*}{\mu}_{\vec{k}(2s-1)}(x)$ *by* $G_{\nu,i}(x)$, $\widetilde{G}^{\alpha}_{\gamma,i}(y)$, $z_{\vec{k}(2s)}(x)$ *and* $\overset{*}{z}_{\vec{k}(2s-1)} l(x)$, *respectively.*

The set of the said functions is a solution to system (3.30) in the set $(0, \widetilde{x}_r] \times (0, \sigma_0]$. The functions $\overset{\circ}{\phi}_m$ and $\overset{\circ}{\psi}_m$ ($m = 1, 2, \ldots, p$) are of class C^q. It is the only soluton of system (3.30) in the class $\overset{\circ}{\mathfrak{K}}$ of all sets of functions ϕ_m and ψ_m ($m = 1, 2, \ldots, p$) possessing continuous derivatives of the orders up to and including $\lceil \frac{m-1}{r} \rceil r$ and $\lceil \frac{m-1}{r} \rceil r + q - p$, respectively, and such that the relations

$$(3.38)\qquad \begin{cases} \|\overset{\circ(l)}{\phi}_m(x)\| \le \text{const} x^{q+2p-2m-l+\varkappa_0} \\ \|\overset{\circ(t)}{\psi}_m(y)\| \le \text{const} y^{q+2p-2m-l+\varkappa_0} \end{cases}$$

($(x, y) \in \Delta$; $m = 1, 2, \ldots, p$; $t = q - p + l$; $l = 0, 1, \ldots, \lceil \frac{m-1}{r} \rceil r$) hold good.

Having found the functions $\overset{\circ}{\phi}_m$ and $\overset{\circ}{\psi}_m$, $x \in [0, \widetilde{x}_r]$; $y \in [0, \sigma_0]$; $m = 1, 2, \ldots, p$ (cf. Proposition 3.2), we can determine, in a way analogous to that in deriving (3.29), the functions

$$(3.39)\quad \mathring{\omega}_{l+1}(x) = (1-\delta_{0l}) \int_0^x (x-\xi)^{l-1} \mathring{H}_l(\xi)\, d\xi + \delta_{0,l}\mathring{H}_0(x) + \sum_{\nu=0}^{l-1} \mathring{A}_\nu x^\nu$$

$(x \in [0, \widetilde{x}_r];\ l = 0, 1, \ldots, q-p-1)$, where

$$(3.40)\quad \mathring{H}_l(x) = \overline{N}_l \circ h_r^{-1}(x) - \Big\{ \sum_{s=l+2}^{q-p} \frac{[h_r^{-1}(x)]^{s-l-1}}{(s-l-1)!} \mathring{\omega}_s^{(l)}(x) +$$

$$+ \sum_{m=m_1}^{p} C_{m,l} [h_r^{-1}(x)]^{q-p+m-l-1} \mathring{\phi}_m^{(l)}(x) + \sum_{m=m_2}^{q-p} x^{m-l-1} \mathring{\psi}_m^{(l)} \circ h_r^{-1}(x) \Big\}$$

and $\mathring{A}_\nu$ $(\nu = 0, 1, \ldots, l-1;\ l = 0, 1, \ldots, q-p-1)$ are arbitrary constants.

In the sequel we shall use the following notation

$$(3.41)\quad \mathring{\chi}_m(x) = \begin{cases} \mathring{\omega}_m(x) & \text{for } m = 1, 2, \ldots, q-p \\ \mathring{\phi}_{m-q+p}(x) & \text{for } m = q-p+1, \ldots, q \end{cases}$$

$(x \in [0, \widetilde{x}_r];\ m = 1, 2, \ldots, q)$;

$$(3.42)\quad \overset{*}{\chi}_\mu(x) = \begin{cases} \overset{*}{\omega}_\mu(x) & \text{for } \mu = 1, 2, \ldots, q-p \\ \overset{*}{\phi}_{\mu-q+p}(x) & \text{for } \mu = q-p+1, \ldots, q \end{cases}$$

$(x \in [\widetilde{x}_r, 1];\ \mu = m, m+1, \ldots, q;\ m = 1, 2, \ldots, q)$;

$$(3.43)\quad \widetilde{\chi}_m(x) = (-1)^{m+1} \sum_{\mu=m}^{q} \binom{\mu-1}{m-1} \sigma^{\mu-m} \overset{*}{\chi}_\mu(x)$$

$(x \in [\widetilde{x}_r, 1];\ m = 1, 2, \ldots, q)$, and

$$(3.44)\quad \widetilde{\psi}_m(y) = (-1)^{m+1} \sum_{\mu=m}^{p} \binom{\mu-1}{m-1} \overset{*}{\psi}_\mu(y)$$

$(y \in [\sigma_0, \sigma];\ m = 1, 2, \ldots, p)$, where $\mathring{\phi}_\nu, \mathring{\psi}_\nu, \mathring{\omega}_\beta$ and $\overset{*}{\phi}_\nu, \overset{*}{\psi}_\nu, \overset{*}{\omega}_\beta$ $(\nu = 1, 2, \ldots, p;\ \beta = 1, 2, \ldots, q-p)$ are the functions given by formulae (3.34), (3.35), (3.39) and (3.10, (3.11), (3.29), respectively.

It follows from Propositions 3.1 and 3.2 that the function

$$(3.45)\quad u(x,y) = \sum_{m=1}^{q} y^{m-1} \chi_m(x) + \sum_{m=1}^{p} x^{m-1} \psi_m(y)$$

$((x,y) \in \mathbf{P})$, where

$$(3.46)\quad \chi_m(x) = \begin{cases} \mathring{\chi}_m(x) & \text{for } 0 \le x \le \widetilde{x}_r \\ \widetilde{\chi}_m(x) & \text{for } \widetilde{x}_r \le x \le 1 \end{cases}$$

$(m = 1, 2, \ldots, q)$;

$$\psi_m(y) = \begin{cases} \mathring{\psi}_m(y) & \text{for } 0 \le y \le \sigma_0 \\ \tilde{\psi}_m(y) & \text{for } \sigma_0 \le y \le \sigma \end{cases} \tag{3.47}$$

$(m = 1, 2, \ldots, p)$, satisfies the boundary conditions (1.2) on the parts of $\Gamma_1, \ldots, \Gamma_r$ and $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_r$ contained in $\Delta \cup \Omega$.

Evidently, $u \in \mathfrak{K}$, and hence (cf. Lemma 2.1) is a solution to equation (1.1), provided that the equalities

$$\mathring{\chi}_m^{(\nu)}(\tilde{x}_r) = \tilde{\chi}_m^{(\nu)}(\tilde{x}_r); \quad \mathring{\psi}_s^{(\mu)}(\sigma_0) = \tilde{\psi}_s^{(\mu)}(\sigma_0) \tag{3.48}$$

$(m = m_1, \ldots, p;\ s = s_1, \ldots, p$ with $m_1 = \max(1, \mu + 1 - q + p);\ s_1 = \nu + 1)$ hold good for $\nu = 0, 1, \ldots, p;\ \mu = 0, 1, \ldots, q$.

Let us observe that we still have to impose on the function u the following requirements:

1° The conditions (1.2)(a) are to be satisfied on the parts of $\Gamma_1, \ldots, \Gamma_r$ marked on Fig. 1;

2° The conditions (1.2)(b) should be fulfilled on the parts of $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{r-1}$ marked on the said figure.

Imposing the aforementioned requirements on the function u (cf. (3.45)), and using (3.46), (3.47), we get the following system of equalities

$$\sum_{m=jr+1}^{p} \left\{ \frac{(q-p+m-1)!}{(q-p+m-jr-1)!}(-1)^{q-p+jr}(\sigma - f_i(x))^{m-jr-1} \overset{*}{\phi}{}_m^{(jr)}(x) + \right. \\ \left. + \frac{(m-1)!}{(m-jr-1)!} x^{m-jr-1} \mathring{\psi}_m \circ f_i(x) \right\} = M_{i,j}(x) \tag{3.49}$$

$(x \in [\tilde{x}_r, x_i];\ i = 1, 2, \ldots, r)$;

$$\sum_{m=jr+1}^{p} \left\{ \frac{(q-p+m-1)!}{(q-p+m-jr-1)!} y^{m-jr-1} \mathring{\phi}_m^{(jr)} \circ h_i(y) + \right. \\ \left. + \frac{(m-1)!}{(m-jr-1)!}(-1)^{jr}(1 - h_i(y))^{m-jr-1} \overset{*}{\psi}{}_m^{(jr)}(y) \right\} = N_{i,j}(y) \tag{3.50}$$

$(y \in [\sigma_0, h_i^{-1}(\tilde{x}_r);\ i = 1, 2, \ldots, r-1)$, where $\overset{*}{\phi}_m, \overset{*}{\psi}_m, \mathring{\phi}_m$ and $\mathring{\psi}_m$ $(m = 1, 2, \ldots, p)$ aree given by formulae (3.10), (3.11) and (3.34), (3.35), respectively.

Thus, a sufficient condition for the existence of a solution to problem $(\mathbb{P})$ is the following one

$$(\mathbb{C}) = \{(3.48)\text{–}(3.50)\} \tag{3.51}$$

consisting in the validity of relations (3.48)–(3.50).

EXAMPLE 3.1. We shall give an example of the boundary data satisfying the condition ($\mathbb{C}$).

Let $r = p = q = 2$ (as a consequence, $k = 1$, $j = 0$; $i = 1,2$, the conditons (1.2)(c) do not appear and the equalities (cf. (3.41)–(3.43)) $\mathring{\chi}_m(x) = \mathring{\chi}_m(x) = 0$ for $m = 1,2,\ldots,q-p$ hold good), and assume that the functions $M_{i,0}(x)$ and $N_{i,0}(y)$ $(i = 1,2)$ satisfy, apart from Assumption **II**, the following conditions

$$\text{(3.52)}\qquad \begin{cases} M_{1,0}(x) = 0 & \text{for } x \in [0,1]; \\ M_{2,0}(x) = 0 & \text{for } x \in [0,\tilde{x}_2] \cup [x_2, 1] \\ N_{i,0}(y) = 0 & \text{for } y \in [0, f_2(\tilde{x}_2)] \cup [\sigma_0 - \eta, \sigma] \end{cases}$$

$(i = 1,2)$, where η is a number arbitrarily fixed in the interval $(f_2(\tilde{x}_2), \sigma_0)$.

One can show by using formulae (3.10), (3.11), (3.34), (3.35) and (3.52) that all the infinite series present in equalities (3.49) and (3.50) are equal to zero (we base on the relations

$$\mu_{\vec{k}(2s)}(x) \in [x_r, 1);\ z_{\vec{k}(2s)}(x) \in (0, \tilde{x}_r] \quad \text{for } x \in [\tilde{x}_r, x_i];\ i = 1,2,$$

resulting from definitions (2.8)–(2.11), respectively) and hence the said equalities reduce to the form

$$\text{(3.53)}\qquad \sum_{\nu=1}^{2} \tilde{w}_\nu(y)(\tilde{e}^1_\nu(y) - f_i^{-1}(y))N_{\nu,0}(y) = M_{i,0} \circ f_i^{-1}(y)$$

$(y \in [f_2(\tilde{x}_2), \sigma_0];\ i = 1,2)$;

$$\text{(3.54)}\qquad \sum_{\nu=1}^{2}[w_\nu \circ h_1(y)(e^1_\nu \circ h_1(y) - y)M_{\nu,0} \circ h_1(y) + \ddot{\bar{w}}_\nu(y)(\ddot{\bar{e}}^1_\nu(y) +$$
$$- (1 - h_1(y)M_{\nu,0} \circ f_\nu^{-1}(y)] = N_{1,0}(y) \qquad (y \in [\sigma_0, h_1^{-1}(\tilde{x}_2)]).$$

Condition (3.54) is identically satisfied due to assumption (3.52).

Relations (3.53) can be treated as a system of algebraic equations with the unknows $N_{\nu,0}$ $(\nu = 1,2)$. It is easily shown that the determinant of the coefficient matrix of this system is equal to $\tilde{w}_1(y)\tilde{w}_2(y)(f_2^{-1}(y) - f_1^{-1}(y)) \cdot$ $\cdot(h_1(y) - h_2(y))$ and hence is different from zero for $y \in [f_2(\tilde{x}_2), \sigma_0]$.

Thus, given $M_{i,0}$ $(i = 1,2)$ one can find the functions $N_{\nu,0}$ $(\nu = 1,2)$, and conversely, so that conditions (3.53) hold good.

As a consequence, we can assert that if the assumptions (3.52) are satisfied then (3.49), (3.50) are true.

The validity of relations (3.48) follows directly from formulae (3.10), (3.11), (3.34), (3.35) and assumptions (3.52).

Basing on the considerations performed in this chapter, we can formulate the following theorem.

THEOREM. *If Assumptions* **I–III** *are satisfied and condition* $(\mathbb{C})$ (*cf.* (3.51)) *holds good then problem* $(\mathfrak{G})$ *has a solution of the form* (3.45) (*cf. also* (3.41)–(3.44)), *where the functions* $\overset{*}{\phi}_m, \overset{*}{\psi}_m, \overset{\circ}{\phi}_m, \overset{\circ}{\psi}_m \overset{*}{\omega}_\beta$, *and* $\overset{\circ}{\omega}_\beta$ ($m = 1, 2, \ldots, p$; $\beta = 1, \ldots, q-p$) *are given by formulae* (3.10), (3.11), (3.29), (3.34), (3.35) *and* (3.39), *respectively.*

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