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THE LIMIT PROPERTY AND PERTURBATIONS OF FUNCTIONAL SHIFTS

Dedicated to Professor Janina Wolska-Bochenek

The authors continue their earlier investigations (cf. B[1]–[5], PR[1]–[4]) on shifts induced by right invertible operators. The purpose of the present paper is to study limit properties and infinitesimal generators of families of functional shifts induced by a right invertible operator D and its perturbations. Functional shifts, introduced and studied recently by the first of the authors, generalize in a sense classical notions of translations and semigroups.

Conditions for a family of functional shifts defined on a locally bounded complete linear metric space to be a commutative semigroup (with respect to the superposition of operators as a structure operation) are established. Moreover, there are given conditions for perturbed families of functional shifts to have the limit property. If these conditions are satisfied then infinitesimal generators of perturbed families are determined by means of infinitesimal generators of the original family of functional shifts.

We shall recall some definitions and theorems (without proofs) which will be used in our subsequent considerations.

Assume that X is a linear space over the field $\mathbf{C}$ of complex numbers. Denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$, by $\mathcal{R}_D$ - the set of all right inverses of a $D \in R(X)$ and by $\mathcal{F}_D$ - the set of all *initial* operators for D, i.e.

$$\mathcal{R}_D = \{R \in L_0(X) : DR = I\},$$
$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D \text{ and } \exists_{R\in\mathcal{R}_D} FR = 0\}.$$

In the sequel we shall assume that $\ker D \neq \{0\}$, i.e. D is right invertible but not invertible. The theory of right invertible operators and its applications can be found in PR[1].

We admit that $0^0 = 1$. We also write : $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$. For a given operator $D \in R(X)$ write

$$S = \bigcup_{i=1}^{\infty} \ker D^i. \tag{1}$$

The set S is equal to the linear span $P(R)$ of all *D-monomials*

$$S = P(R) = \operatorname{lin}\{R^k z : z \in \ker D,\ k \in \mathbf{N}_0\}$$

independently of the choice of a right inverse R of D (cf. PR[1]).

In the sequel, K will stand either for a disk $K_\rho = \{h \in \mathbf{C} : |h| < \rho,\ 0 < \rho < +\infty\}$ or for the complex plane $\mathbf{C}$. Denote by $H(K)$ the space of all functions analytic on the set K. Suppose that a function $f \in H(K)$ has the following expansion

$$f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K. \tag{2}$$

Definition 1 (cf. B[2]). Suppose that $D \in R(X)$ and $\ker D \neq \{0\}$. A family $T_K = \{T_h\}_{h \in K} \subset L_0(X)$ is said to be a family of *functional shifts* for the operator D induced by a function $f \in H(K)$ if

$$T_h x = [f(hD)]x = \sum_{k=0}^{\infty} a_k h^k D^k x \quad \text{for all } h \in K,\ x \in S, \tag{3}$$

where S is defined by Formula (1).

We should point out that, by definition of S, the last sum has only a finite number of members different than zero.

Proposition 1 (cf. B[2]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $T_K = \{T_h\}_{h \in K} \subset L_0(X)$. Let $f \in H(K)$ (i.e. f is of the form* (1)*). Then the following conditions are equivalent*:

(i) *T_K is a family of functional shifts for the operator D induced by the function f;*

(ii) *$T_h R^k F = \sum_{j=0}^{k} a_j h^j R^{k-j} F$ for all $h \in K$, $k \in \mathbf{N}_0$.*

Proposition 2 (cf. B[2]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $T_{f,K} = \{T_{f,h}\}_{h \in K}$ is a family of functional shifts for the operator D induced by a function $f \in H(K)$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then*

(i) *For all* $h \in K$, $z \in \ker D$, $k \in \mathbb{N}_0$

$$T_{f,h}R^k z = \sum_{j=0}^{k} a_j h^j R^{k-j} z; \tag{4}$$

(ii) *The operators* $T_{f,h}$ $(h \in K)$ *are uniquely determined on the set* S;

(iii) *If* X *is a complete linear metric space,* $\overline{S} = X$ *and* $T_{f,h}$ *are continuous for* $h \in K$ *then* $T_{f,h}$ *are uniquely determined on the whole space*;

(iv) *For all* $h \in K$ *the operators* $T_{f,h}$ *commute on the set* S *with the operator* D.

The listed properties and other informations about shifts for right invertible operators can be found in B[1]–B[4] (cf. also PR[1]–PR[4]).

Proposition 1.2 of B[3] implies

PROPOSITION 3. *Suppose that all assumptions of Proposition* 2 *are satisfied and* $f(0) = a_0 \neq 0$. *Let*

$$F_h = f^{-1}(0)FT_{f,h} \quad \text{for } h \in K. \tag{5}$$

Then F_h *is an initial operator for* D *corresponding to the right inverse*

$$R_h = R - F_h R \quad \text{for } h \in K. \tag{6}$$

It is well-known that the set $H(K)$ is a commutative ring with the following algebraic operations

$$(f+g)(h) = f(h) + g(h), \quad (\alpha g)(h) = \alpha g(h), \quad (fg)(h) = f(h)g(h),$$

where $f,\ g \in H(K)$, $\alpha \in \mathbf{C}$, $h \in K$.

Let $T(K)$ be the set of all families of functional shifts for an operator $D \in R(X)$ induced by the members of $H(K)$, i.e.

$$T(K) = \{T_{g,K} : g \in H(K)\}. \tag{7}$$

Define the following operations

$$T_{f,K} + T_{g,K} = T_{f+g,K}, \quad \alpha T_{g,K} = T_{\alpha g,K}, \quad T_{f,K}T_{g,K} = T_{fg,K}, \tag{8}$$

where $f,\ g \in H(K)$, $\alpha \in \mathbf{C}$.

THEOREM 1 (cf. B[2]). *Suppose that* $D \in R(X)$ *and* $T(K)$ *is defined by Formula* (7). *Let* $T_S(K) = T(K)|_S$, *where* S *is defined by Formula* (1). *Then*

(i) *The set* $T_S(K)$ *is a commutative ring with the operations defined by Formulae* (8);

(ii) *The rings* $H(K)$ *and* $T_S(K)$ *are isomorphic. The mapping*

$$T : f \Longrightarrow T_{f,K}|_S$$

is a ring isomorphism of $H(K)$ *onto* $T_S(K)$.

THEOREM 2 (cf. B&PR[1]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of functional shifts for the operator D induced by a function $f \in H(K)$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Suppose, moreover, that $f(h) \neq 0$ for $h \in K$ and $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K} \in T(K)$. Then*

$$(9) \qquad R_h^n z = f(0) T_{1/f,h} R^n z \quad \text{for all } n \in \mathbf{N},\ h \in K,\ z \in \ker D,$$

where the operators R_h ($h \in K$) are defined by Formula (6).

Now we assume that X is an *F-space* over $\mathbf{C}$, i.e. a complete linear metric space over $\mathbf{C}$ according to the Banach definition. In the sequel K will stand for the disk K_ρ ($0 < \rho \leq +\infty$). Let a set $\Omega \subseteq K$ contains the origin. The function $f \in H(K)$ has the expansion (1) and $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \rho^{-1}$. Define for an operator $D \in R(X)$ the following sets (cf. B[4]):

$$(10) \qquad S_f^{(n)}(D) =$$

$$= \left\{ x \in X : \sum_{k=0}^{\infty} a_k h^k D^{k+n} x \text{ is convergent for all } h \in \Omega \right\} \quad (n \in \mathbf{N}_0);$$

$$(11) \qquad S_f(D) = S_f^{(0)}(D);$$

$$(12) \qquad S_f^{\infty}(D) = \bigcap_{n\in\mathbf{N}_0} S_f^{(n)}(D);$$

$$(13) \qquad S_\Omega(D) = \bigcap_{g\in H(K)} S_g^{\infty}(D).$$

$$(14)\ E_\Lambda = \bigcup_{\lambda\in\Lambda} E_\lambda, \quad \text{where } E_\lambda = \ker(D - \lambda I),\ \Lambda = \{\lambda \in \mathbf{C} : \lambda\Omega \subset K\}.$$

PROPOSITION 4 (cf. B[4]). *Suppose that $D \in R(X)$ and $\ker D \neq \{0\}$. Then*

$$S,\ E_\Lambda \subset S_K(D) \subset S_\Omega(D) \subset S_f^{\infty}(D) \subset S_f(D) \subset \operatorname{dom} D,$$

where the set S is defined by Formula (1).

Let X be a *locally bounded* F-space. Recall (cf. Rolewicz R[1]) that

- for a p, $0 < p \leq 1$, there is a p-homogeneous F-norm equivalent to the original one (Aoki–Rolewicz theorem);
- for every p, $0 < p < p_0 = \log 2/\log c(X)$, where $c(X)$ is the *modulus of concavity* of the space X, there is a p-homogeneous F-norm $\| \ \|$ equivalent to the original one (Rolewicz theorem).

PROPOSITION 5. *Suppose that X is an F-space with a p-homogeneous norm $\| \ \|$ $(0 < p \le 1)$, $D \in R(X)$ and* $\ker D \ne \{0\}$. *Let*

$$(15) \qquad X_1(D) = \{x \in X : \limsup_{n\to\infty} \sqrt[n]{\|D^n x\|} \le 1 \text{ if } \rho < +\infty \text{ and } \{D^n x\} \text{ is bounded if } \rho = +\infty\}.$$

Then $X_1(D) \subset S_f(D)$.

Proof. Let $x \in X_1$ and $f \in H(K)$ be arbitrarily fixed. Let $\rho_1 = \limsup_{n\to\infty} \sqrt[n]{\|D^n x\|}$ for $\rho < +\infty$. Let $M > 0$ be such that $\|D^n x\| < M$ for all $n \in \mathbf{N}_0$, $\rho = +\infty$.

For all $n \in \mathbf{N}_0$, $h \in K$ we have $\|a_n h^n D^n x\| = |a_n|^p |h|^{np} \|D^n x\|$. Hence for $h \in K$

$$\limsup_{n\to\infty} \sqrt[n]{\|a_n h^n D^n x\|} < \rho_1 \le 1 \quad \text{for } \rho < +\infty$$

and

$$\limsup_{n\to\infty} \sqrt[n]{\|a_n h^n D^n x\|} = 0 \quad \text{for } \rho = +\infty.$$

This implies that the scalar series $\sum_{n=0}^{\infty} |a_n|^p |h|^{np} \|D^n x\|$ is convergent for all h such that $|h| < \rho < +\infty$ (for all $h \in \mathbf{C}$ if $\rho = +\infty$, respectively). Since X is a complete linear metric space with an invariant metric $\mu(x, y)$ defined by the equality: $\mu(x, y) = \|x - y\|$ for all $x, y \in X$, we conclude that the series $\sum_{n=0}^{\infty} a_n h^n D^n x$ is absolutely convergent for all $h \in K$. ∎

Note 1. Suppose that all assumptions of Proposition 5 are satisfied. Then $S \subset X_1(D)$ but the inclusion $E_\Lambda \subset X_1(D)$ does not hold. However, $\{0\} \ne E_\lambda = \ker(D - \lambda I) \subset X_1(D)$, $\lambda \in \mathbf{C}$, if and only if $|\lambda| \le 1$. Moreover,

$$D[X_1(D)] \subset X_1(D); \quad R[X_1(D)] \subset X_1(D), \quad \text{where } R \in \mathcal{R}_D.$$

Indeed, let $x \in S$ be arbitrarily fixed. Then there is an $N \in \mathbf{N}$ such that $D^n x = 0$ for all $n > N$. This implies that $x \in X_1(D)$. Suppose now that $\lambda \in \mathbf{C}$. Let $x \in \ker(D - \lambda I) \setminus \{0\}$ be arbitrarily fixed. We have $D^n x = \lambda^n x$ and $\|D^n x\| = |\lambda|^{np} \|x\|$ for all $n \in \mathbf{N}$. Since $0 < p \le 1$, we conclude that $x \in X_1(D)$ if and only if $|\lambda| \le 1$, i.e. $E_\lambda \in X_1(D)$ if and only if $|\lambda| \le 1$ $(0 < \rho \le +\infty)$. Clearly, $Dx \in X_1(D)$ whenever $x \in X_1(D)$. Let $R \in \mathcal{R}_D$ be arbitrarily fixed. Since $D^n R x = D^{n-1} x$ for all $x \in X$, $n \in \mathbf{N}$, we conclude that $Rx \in X_1(D)$ for all $x \in X_1(D)$.

The following lemma, which is well-known for Banach space (cf. Hille and Phillips HP[1]), holds also for locally bounded F-spaces (cf. Rolewicz R[1]).

LEMMA 1. *Let X be a locally bounded F-space and let $D \in R(X)$. The series $\sum_{n=0}^{\infty} a_n h^n D^n x$ is absolutely convergent for all $x \in X_1(D)$, $h \in K$ to a holomorphic function $f(hD)x : K \to X$ in every concentric circle of* *radius less than ρ. The function $f(hD)x$ is strongly continuous and strongly differentiable in K, uniformly with respect to h in any compact subset of K.*

Lemma 1 immediately implies

PROPOSITION 6. *Suppose that all assumptions of Lemma 1 are satisfied. Define for $f \in H(K)$ the following families of operators:*

$$f(hD)x = \sum_{n=0}^{\infty} a_n h^n D^n x; \qquad f'(hD)x = \sum_{n=1}^{\infty} n a_n h^{n-1} D^{n-1} x$$

for all $x \in X_1(D)$, $h \in K$. Then

(i) *the families $f(hD)$ and $f'(hD)$ are well-defined, the function*

$$f'(h) = \sum_{n=1}^{\infty} n a_n h^{n-1} \qquad (h \in K)$$

is the derivative of the function f and $f' \in H(K)$;

(ii) *if $0 \in \Omega \subset K$ is a limit point of Ω then*

$$\lim_{\Omega \ni h \to 0} \frac{1}{h}[f(hD) - f(0)I]x = f'(0)Dx \qquad \text{for all } x \in X_1(D);$$

(iii) *if $\Omega \subset K$ is an open set then*

$$\frac{\mathrm{d}}{\mathrm{d}h} f(hD)x = f'(hD)Dx \qquad \text{for all } x \in X_1(D),\ h \in \Omega.$$

LEMMA 2. *Suppose that $D \in R(X)$,* $\ker D \neq \{0\}$ *and $R \in \mathcal{R}_D$. Then*

$$R(S \cup E_\Lambda) \subset S_f(D),$$

where the sets S, E_Λ, $S_f(D)$ are defined by Formulae (1), (14), (11), *respectively. Moreover,*

$$\frac{\mathrm{d}}{\mathrm{d}h} f(hD)Rx = f'(\lambda h)x \qquad \text{for all } x \in S \cup E_\Lambda,\ h \in \Omega.$$

Proof. The equality $S = P(R)$ and Proposition 4 together imply that $R(S) \subset S \subset S_K(D) \subset S_f(D)$. Let $x \in E_\Lambda \setminus S$ be arbitrarily fixed. Then there is a $\lambda \in \Lambda \setminus \{0\} \subset \mathbf{C}$ such that $x \in E_\lambda = \ker(D - \lambda I)$. For $h \in K$ we have

$$f(hD)Rx = \sum_{n=0}^{\infty} a_n h^n D^n Rx = a_0 Rx + \sum_{n=1}^{\infty} a_n h^n D^{n-1} x =$$

$$= a_0 Rx + \sum_{n=1}^{\infty} a_n h^n \lambda^{n-1} x = a_0 Rx + \lambda^{-1} \sum_{n=1}^{\infty} a_n (\lambda h)^n x =$$

$$= a_0 Rx + \frac{f(\lambda h) - f(0)}{\lambda} x.$$

This implies that $Rx \in S_f(D)$ and

$$\lim_{h_1 \to h} \frac{1}{h_1 - h}[f(h_1 D) - f(hD)]Rx = f'(\lambda h)x \quad (h \in K). \ \blacksquare$$

Similarly, as Definition 1, we have

DEFINITION 2. (cf. B[4]). A family $T_{f,\Omega} = \{T_{f,h}\}_{h \in \Omega} \subset L_0(X)$ is said to be a family of *functional shifts* for an operator $D \in R(X)$ induced by a function $f \in H(K)$ if

$$(16) \qquad T_{f,h}x = f(hD)x \quad \text{for all } h \in \Omega,\ x \in S_f(D),$$

where the operator $f(hD)$ is defined by Formula (3) and the set $S_f(D)$ is defined by Formula (11).

Note 2. (cf. B[4]). Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $T_{f,\Omega} = \{T_{f,h}\}_{h \in \Omega}$ is a family of functional shifts for D induced by a function $f \in H(K)$. If $x \in E_\lambda$, $\lambda \in \Lambda$, then

$$T_{f,h}x = f(\lambda h)x \quad \text{for all } h \in \Omega.$$

Theorem 3.1 of B[1] and Note 2 together imply

PROPOSITION 7. *Let $T_{f,\Omega}$, $T_{f',\Omega}$ be families of functional shifts for the operator $D \in R(X)$ induced by the functions f, $f' \in H(K)$, respectively. Then*

(i) *if $h_0 \in \Omega \subset K$ is a limit point of Ω then*

$$\lim_{\Omega \ni h \to h_0} \frac{1}{h_0 - h}(T_{f,h_0} - T_{f,h})x = T_{f',h_0}Dx = DT_{f',h_0}x \quad \textit{for all } x \in S \cup E_\Lambda;$$

(ii) *if $\Omega \subset K$ is an open set then*

$$\frac{\mathrm{d}}{\mathrm{d}h}T_{f,h} = T_{f',h}D = DT_{f',h} \quad \textit{on } S \cup E_\Lambda \quad \textit{for all } h \in \Omega,$$

where S, E_Λ are defined by Formulae (1), (14), *respectively.*

Note 3. (cf. B[4]). Suppose that all assumptions of Note 2 are satisfied and $T_{f,h}$ are continuous for $h \in \Omega$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then

(i) if R is continuous then

$$A_R(D) = \{x \in X : x = \sum_{n=0}^{\infty} R^n F D^n x\} \subset S_f(D);$$

(ii) if D is closed then

$$A(D) = \bigcup_{R \in \mathcal{R}_D} A_R(D) \subset S_f(D).$$

DEFINITION 3. Let $0 \in \Omega \subset K$ be the limit point of the set Ω. A family $T_\Omega = \{T_h\}_{h\in\Omega} \subset L_0(X)$ has the *limit property on a set* $Y \subset X$ if $\lim_{h\to 0} \frac{1}{h}(T_h - T_0)x$ exists for every $x \in Y$. If it is the case, then the operator A defined as

$$(17) \qquad Ax = \lim_{h\to 0} \frac{1}{h}(T_h - T_0)x \quad \text{for } x \in Y = \operatorname{dom} A$$

is said to be an *infinitesimal generator* for the family T_Ω.

Note that the infinitesimal operator A (if exists) is well-defined. Clearly, if X is a locally convex F-space, then any strongly continuous semigroup $T_\Omega = \{T_h\}_{h\in\Omega}$ of linear operators is a family with the limit property and its infinitesimal generator satisfies Condition (17).

Proposition 7 leads us to the following

COROLLARY 1. *Suppose that $T_{f,\Omega}$ is a family of functional shifts for D induced by a function $f \in H(K)$ and $0 \in \Omega$ is a limit point of Ω. Then this family has the limit property on the set $S \cup E_A$ and $f'(0)D$ is its infinitesimal generator.*

Proposition 6 implies

COROLLARY 2. *Suppose that all assumptions of Corollary* 1 *are satisfied and X is a locally bounded space F-space (in particular, a Banach space). Then the family $T_{f,\Omega}$ has the limit property on $X_1(D)$ and $f'(0)D$ is its infinitesimal generator, where the set $X_1(D)$ is defined by Formula* (15).

On the other hand we have

THEOREM 3. *Suppose that $K = \mathbf{C}$, $\Omega \subset K$ is the interior of a spinal semi-module (i.e. an additive semigroup which contains a ray from the origin and an open set intersected by this ray) containing the positive real axis and $T_{f,\Omega} = \{T_{f,h}\}_{h\in\Omega}$ is a family of functional shifts for a $D \in R(X)$ induced by a function $f \in H(\mathbf{C})$, $f \not\equiv 0$. Then $T_{f,\Omega}$ is a commutative semigroup with respect to the superposition of operators on the set $S \cup E_A$, i.e.*

$$T_{f,h_1}T_{f,h_2}x = T_{f,h_2}T_{f,h_1}x = T_{f,h_1+h_2}x \quad \textit{for all } x \in S \cup E_A,\ h_1, h_2 \in \Omega,$$

if and only if there is an $\alpha \in \mathbf{C}$ such that $f(h) = e^{\alpha h}$. If it is the case then

$$(18) \qquad \lim_{h\to 0} \frac{1}{h}(T_{f,h} - T_{f,0})x = \alpha Dx \quad \textit{for all } x \in S \cup E_A,$$

i.e. the operator αD is the infinitesimal generator of semigroup $T_{f,\Omega}$ on $S \cup E_A$.

Proof. Sufficiency has been proved in B[2], B[4]. Necessity. Suppose that $T_{f,\Omega}$ is a semigroup. It follows from properties of functional shifts considered on the set $S \cup E_A$ (cf. B[4]) that in this case the function f should satisfy the functional equation: $f(t+s) = f(t)f(s)$ for all $t,\ s \in \Omega$. It is well-known (cf. Hille and Phillips HP[1], Lemma 17.3.1) that either $f \equiv 0$ on Ω or there exists a complex number α such that $f(h) = e^{\alpha h}$ for all $h \in \Omega$. This implies that $f(h) = e^{\alpha h}$ for all $h \in \mathbf{C}$. Proposition 7(i) implies the equality (18). We therefore conclude that αD is the infinitesimal generator of the semigroup under consideration. ∎

Evidently, Theorem 3 holds also in the case when either $\Omega = K$ or $\Omega \subseteq K$ contains the interior of a spinal semi-module containing the positive real axis.

In a similar manner we obtain the following

THEOREM 4 (cf. B[4]). *Suppose that all assumptions of Theorem* 3 *are satisfied and the operator D is closed. Then the family $T_{f,\Omega}$ is a commutative semigroup with respect to the superposition of operators on the set $S_\Omega(D)$ defined by Formula* (13) *if and only if there is an $\alpha \in \mathbf{C}$ such that $f(h) = e^{\alpha h}$ for all $h \in K$.*

THEOREM 5. (cf. B[5]). *Suppose that all assumptions of Theorem* 3 *are satisfied and the operators $T_{f,h}$ $(h \in \Omega)$ and $R \in \mathcal{R}_D$ are continuous. Then $T_{f,\Omega}$ is a commutative semigroup with respect to the superposition of operators on the set $A_R(D)$ defined in Note* 3 *if and only if there is an $\alpha \in \mathbf{C}$ such that $f(h) = e^{\alpha h}$ for $h \in K$.*

Propositions 5 and 6 immediately imply the following

THEOREM 6. *Suppose that X is a locally bounded F-space and all assumptions of Theorem* 3 *are satisfied. Then $T_{f,\Omega}$ is a commutative semigroup with respect to the superposition of operators on the set $X_1(D)$ defined by Formula* (15) *if and only if there is an $\alpha \in \mathbf{C}$ such that $f(h) = e^{\alpha h}$ $(h \in K)$ and Formula* (18) *holds on $X_1(D)$. If it is the case then αD is the infinitesimal generator of the semigroup $T_{f,\Omega}$.*

Note 4. In a similar way, as in Definition 2 (Definition 1, respectively), functional shifts may be defined as operators induced by an analytic function $f : K_\rho \to \mathbf{R}$, where $K_\rho = (-\rho, \rho)$, $0 < \rho \leq +\infty$. The particular case, when $K = \mathbf{R}$, $\Omega \subseteq K$ is either $\mathbf{R}$ or $\mathbf{R}_+$, $f(h) = e^h$, has been considered in PR[1]-[4]. Clearly, also in that case results analogous to Theorems 3, 4, 5, 6 can be obtained.

For perturbed operators we get

THEOREM 7. *Suppose that $T_{f,K}$ is a family of functional shifts for $D \in R(X)$ induced by a function $f \in H(K)$, the operator $A \in L_0(X)$ maps $S \cup E_A$ into itself, $R \in \mathcal{R}_D$ and*

$$(19) \qquad T'_{f,h} = T_{f,h}(I + RA) - T_{f,0}RA \quad \textit{for all } h \in \Omega \subset K.$$

If the origin is the limit point of Ω then

$$\lim_{\Omega \ni h \to 0} (T'_{f,h} - T'_{f,0})x = f'(0)(D + A)x \quad \text{for all } x \in S \cup E_A, \tag{20}$$

i.e. the family $T'_{f,K}$ has the limit property on the set $S \cup E_A$, its infinitesimal generator is $f'(0)(D + A)$ and $T'_{f,K}$ acts in the following manner:

$$T'_{f,h}x = [f(hD) - f(0)](I + RA)x + f(0)x \quad \text{for all } x \in S \cup E_A,\ h \in \Omega.$$

Proof. Suppose that the family $T'_{f,\Omega}$ is defined by Formula (19). Then $T'_{f,0} = T_{f,0}$. Let $x \in S \cup E_A$ be arbitrarily fixed. By Proposition 7 and Lemma 2, we get

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}(T'_{f,h} - T'_{f,0})x &= \lim_{h\to 0} \frac{1}{h}[T_{f,h} - T_{f,0} + (T_{f,h} - T_{f,0})RA]x = \\ &= \lim_{h\to 0} \frac{1}{h}(T_{f,h} - T_{f,0})x + \lim_{h\to 0} \frac{1}{h}(T_{f,h} - T_{f,0})RAx = \\ &= f'(0)Dx + f'(0)DRAx = f'(0)(D + A)x.\end{aligned}$$

Let now $x \in S \cap E_A$ be arbitrarily fixed. By definition and our assumptions, for all $h \in K$ we have

$$\begin{aligned}T'_{f,h}x &= T_{f,h}(I + RA)x - T_{f,0}RAx = f(hD)(I + RA)x - f(0)RAx = \\ &= [f(hD) - f(0)](I + RA)x + f(0)x. \ \blacksquare\end{aligned}$$

Note 5. Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$, $A \in L_0(X)$ and the operator $I + RA$ is invertible. Recall that $D^o = D + A \in R(X)$ and $R^o = (I + RA)^{-1}R \in \mathcal{R}_D$ (cf. for instance PR[1]). Recall also that the sum of a closed operator and a continuous operator is again closed. So that, if D is closed and A is continuous, then D^o is closed.

Proposition 6 and Note 1 together imply the following

PROPOSITION 8. *Suppose that X is a locally bounded F-space over $\mathbf{C}$, $T_{f,\Omega}$ is a family of functional shifts induced by a function $f \in H(K)$, $0 \in \Omega \subset K$ is a limit point of Ω, $A \in L_0(X)$ maps the set $X_1(D)$ defined by Formula (15) into itself and the family $T'_{f,\Omega}$ is defined by Formula (19). Then Formula (20) holds on $X_1(D)$, i.e. $T_{f,\Omega}$ has the limit property on $X_1(D)$ and its infinitesimal generator is $f'(0)(D + A)$.*

COROLLARY 3. *Suppose that all assumptions of Theorem 7 are satisfied and $f'(0) \neq 0$. Let $T'_{f,K}$ be defined by Formula (19). Then the family $T^o_{f,K} = T_{1/f',0}T'_{f,K}$ has the limit property on $S \cup E_A$ and its infinitesimal generator is $D + A$. Moreover,*

$$T^o_{f,h}x = \frac{1}{f'(0)}\{[f(hD) - f(0)](I + RA)x + f(0)x\} \quad \text{for } x \in S \cup E_A,\ h \in K.$$

Proof. By our assumption, $1/f' \in H(K)$, which implies $f'(0) \neq 0$ and $T_{1/f',0}x = \frac{1}{f'(0)}x$ for $x \in S \cup E_A$. This, and Theorem 7 together imply that the family $T^{\circ}_{f,K}$ has the limit property on $S \cup E_A$ and that its infinitesimal generator is $\frac{1}{f'(0)}f'(0)(D+A) = D + A$. ∎

PROPOSITION 9. *Suppose that $0 \in \Omega \subset K$ is a limit point of Ω, $T_{f^{(n)},\Omega}$ is a family of functional shifts for a $D \in R(X)$ induced by a function $f^{(n)} \in H(K)$ and $f^{(n+1)}(0) \neq 0$ for an arbitrarily fixed $n \in \mathrm{N}_0$. Then the family*

$$\widetilde{T}^{(n)}_{f,\Omega} = \frac{1}{f^{(n+1)}(0)} T_{f^{(n)},\Omega} \qquad (n \in \mathrm{N}_0) \tag{21}$$

has the limit property on $S \cup E_A$ and its infinitesimal generator is D.

Proof. Let $n \in \mathrm{N}_0$ be arbitrarily fixed and let a family of operators be defined by Formula (21). Then for all $x \in S \cup E_A$ we get

$$\begin{aligned}\lim_{\Omega \ni h \to 0} \frac{1}{h}(\widetilde{T}^{(n)}_{f,h)} - \widetilde{T}^{(n)}_{f,0})x &= \lim_{\Omega \ni h \to 0} \frac{1}{h} \frac{1}{f^{(n+1)}(0)}(T_{f^{(n)},h} - T_{f^{(n)},0})x = \\ &= \frac{1}{f^{(n+1)}(0)} \lim_{\Omega \ni h \to 0} \frac{1}{h}(T_{f^{(n)},h} - T_{f^{(n)},0})x = \\ &= \frac{1}{f^{(n+1)}(0)} f^{(n+1)}(0) Dx = Dx. \ \blacksquare\end{aligned}$$

PROPOSITION 10. *Suppose that all assumptions of Proposition 9 are satisfied and the families $T_{f^{(k)},\Omega}$ $(k = 0, 1, \ldots, n;\ n \in \mathrm{N}_0)$ are given. Then*

(i) *the family*

$$T^{(n)}_{f,\Omega} = \frac{1}{f^{(n+1)}(0)} T_{f^{(n)},\Omega} D^n \tag{22}$$

has the limit property on the set $S \cup E_A$ and its infinitesimal generator is D^{n+1};

(ii) *if $\Omega \subset K$ is an open set then*

$$T^{(n)}_{f,\Omega} = \left\{ \frac{1}{f^{(n+1)}(0)} \frac{\mathrm{d}^n}{\mathrm{d}h^n} T_{f,h} \right\}_{h \in \Omega}.$$

Proof. Let $n \in \mathrm{N}_0$ be fixed and let the family $\widetilde{T}^{(n)}_{f,\Omega}$ be defined by Formula (21). Let a family of operators be defined by Formula (22).

(i) Our assumptions and Proposition 9 together imply that for all $x \in S \cup E_A$, $h \in \Omega$ we have

$$\lim_{h \to 0} \frac{1}{h}(T^{(n)}_{f,h} - T^{(n)}_{f,0})x = \lim_{h \to 0} \frac{1}{h}(\widetilde{T}_{f^{(n)},h} - \widetilde{T}_{f^{(n)},0})D^n x = D(D^n x) = D^{n+1}x.$$

(ii) Clearly, for all $h \in \Omega$,

$$T^{(n)}_{f,h} = \frac{1}{f^{(n+1)}(0)}\widetilde{T}^{(n)}_{f,h} = \frac{1}{f^{(n+1)}(0)}T_{f^{(n)},h}D^n.$$

This implies (cf. B[1])

$$T^{(n)}_{f,h} = \frac{1}{f^{(n+1)}(0)}\frac{\mathrm{d}^n}{\mathrm{d}h^n}T_{f,h}. \blacksquare$$

THEOREM 8. *Suppose that* $0 \in \Omega \subset K$ *is a limit point of* Ω, $T_{f,\Omega}$ *is a family of functional shifts for a* $D \in R(X)$ *induced by a function* $f \in H(K)$ *and* $1/f^{(j)} \in H(K)$ *for* $j = 0,\ldots,n$ ($n \in \mathrm{N}_0$). *Suppose, moreover, that* $R \in \mathcal{R}_D$, *the operators* $A_0,\ldots,A_n \in L_0(X)$ *are continuous and* $A_0,\ldots,A_n$ *map* $S \cup E_A$ *into itself. Let the families* $T^{(j)}_{f,\Omega}$ *be defined by Formula* (22) *and let*

$$A(D) = \sum_{j=0}^{n} A_j D^j. \tag{23}$$

Then

(i) *the family*

$$A_{f,\Omega}(D) = \frac{1}{f'(0)}A_o T_{f,\Omega}R + \sum_{j=1}^{n} A_j T^{(j-1)}_{f,\Omega} \tag{24}$$

has the limit property on $S \cup E_A$ *and its infinitesimal generator is* $A(D)$;

(ii) *the family* $A_{f,\Omega}(D)$ *acts in the following manner*

$$A_{f,h}(D)x = \frac{1}{f'(0)}A_0 f(hD)Rx + \sum_{j=1}^{n}\frac{1}{f^{(j)}(0)}A_j f^{(j-1)}(hD)D^{j-1}x$$

for $x \in S \cup E_A$, $h \in \Omega$.

(iii) *if the operator*

$$A(I,R) = \sum_{j=0}^{n} A_j R^{n-j} \tag{25}$$

is invertible (respectively, right invertible) then $A(D) \in R(X)$ *and* $R_{A(D)} = R^n[A(I,R)]^{-1} \in \mathcal{R}_{A(D)}$ *(respectively,* $R_{A(D)} = R^n R_A$, *where* $R_A \in \mathcal{R}_{A(I,R)}$*).*

Proof. (i). Similarly, as in the proof of Propositions 9 and 10 (cf. also the proof of Theorem 7), by Lemma 2, for all $x \in S \cup E_A$ we have

$$\lim_{h\to 0}\frac{1}{h}[A_{f,h}(D)-A_{f,0}(D)]x = A_0 DRx + \sum_{j=1}^{n} A_j D^j x = \sum_{j=0}^{n} A_j D^j x = A(D)x.$$

Point (ii) follows from our assumptions, Point (i) and Formula (16). The proof of Point (iii) can be found in PR[1], where we have used the identity: $A(D)R^n = A(I, R)$. ∎

Similarly, we can prove the following

THEOREM 10. *Suppose that* $0 \in \Omega \subset K$ *is a limit point of* Ω, $T_{f,\Omega}$ *is a family of functional shifts for a* $D \in R(X)$ *induced by a function* $f \in H(K)$ *and* $1/f^{(j)} \in H(K)$ *for* $j = 0, \ldots, n$ $(n \in \mathrm{N}_0)$. *Suppose, moreover, that* $R \in \mathcal{R}_D$, *the operators* $A_0, \ldots, A_n \in L_0(X)$ *are continuous and* $A_0, \ldots, A_n$ *map* $S \cup E_A$ *into itself. Let the families* $T^{(j)}_{f,\Omega}$ *be defined by Formula* (22) *and let*

$$A\langle D\rangle = \sum_{j=0}^{n} D^j A_j. \tag{26}$$

Then

(i) *The family*

$$A_{f,K}\langle D\rangle = \frac{1}{f'(0)} T_{f,\Omega} R A_0 + \sum_{j=1}^{n} T_{f^{(j-1)},\Omega} A_j \tag{27}$$

has the limit property on $S \cup E_A$ *and its infinitesimal generator is* $A\langle D\rangle$;

(ii) *If the operator*

$$A\langle I, R\rangle = \sum_{j=0}^{n} R^{n-j} A_j \tag{28}$$

is invertible (*respectively, right invertible*) *then* $A\langle D\rangle \in R(X)$ *and* $R_{A\langle D\rangle} = [A\langle I, R\rangle]^{-1} R^n \in \mathcal{R}_{A(D)}$ (*respectively,* $R_{A\langle D\rangle} = R_A R^n$, *where* $R_A \in \mathcal{R}_{A\langle I,R\rangle}$).

Proof. The proof of (i) is going on the same lines, as the proof of Point (i) in Theorem 9. Point (ii) can be found in PR[1], where we have used the identity: $A\langle D\rangle = D^n A\langle I, R\rangle$. ∎

COROLLARY 4. *Suppose that all assumptions of Theorem* 9 (*or Theorem* 10) *are satisfied and the operators* $A_0, \ldots, A_n$ *are stationary,* (*i.e. are commuting with* D *and* R *simultaneously*). *Then* $A\langle D\rangle = A(D)$, $A\langle I, R\rangle = A(I, R)$, *the family* $A_{f,\Omega}\langle D\rangle = A_{f,\Omega}(D)$ *defined by Formula* (24) (*or Formula* (27)) *has the limit property on the set* $S \cup E_A$ *and its infinitesimal generator is* $A(D)$.

Proof. By our assumptions, $A(D)T_{f,h} = T_{f,h}A(D)$ for all $h \in K$. Moreover, $A(D) = A\langle D\rangle$. This, and the equalities $A(D)R^n = A(I, R) = A\langle I, R\rangle$ together imply our conclusion. ∎

Observe that families of operators, which appear in Theorems 7, 8, 9, Propositions 8, 9 and Corollaries 3, 4, are not families of functional shifts if the operators $A, A_0, \dots, A_n$ are not operators of multiplication by scalars. Nevertheless, these all families have the limit property. Theorem 3 and 7 show that, in general, Theorem 4.1 and Corollary 4.1 in PR[3] (on perturbations of functional shifts in the case $f(h) = e^h$) do not hold.

We shall give now some examples of functional shifts.

EXAMPLE 1. Let $K = \Omega = \mathbf{C}$, $X = H(\mathbf{C})$ (with the topology of uniform convergence on compact sets). Let $D = \frac{\mathrm{d}}{\mathrm{d}t}$. Then $\overline{S} = X$. Let

$$f(h) = \sum_{k=0}^{n} a_k h^k e^{\alpha_k h},$$

where $a_0, \dots, a_n, \alpha_0 = 0, \alpha_1, \dots, \alpha_n \in \mathbf{C} \setminus \{0\}$ and $\alpha_0, \dots, \alpha_n$ are not necessarily commensurable. Clearly, for all $x \in X$

$$(e^{hD}x)(t) = \sum_{n=0}^{\infty} \frac{h^n}{n!} x^{(n)}(t) = x(t+h).$$

Thus functional shifts induced by the function f are of the form:

$$(T_{f,h}x)(t) = \sum_{k=0}^{n} a_k h^k x^{(k)}(t + \alpha_k h) \quad \text{for } x \in H(\mathbf{C}),\ t, h \in \mathbf{C}.$$

Indeed, Formula (8) implies that

$$(T_{f,h}x)(t) = \sum_{k=0}^{n} a_k h^k D^k \Big(\sum_{n=0}^{\infty} \frac{(\alpha_k h)^n}{n!} D^n x \Big) = \sum_{k=0}^{n} a_k h^k x^{(k)}(t + \alpha_k h).$$

In particular, if

$$f(h) = a \cos \alpha h + b \sin \beta h, \quad (a, b \in \mathbf{C},\ \alpha, \beta \in \mathbf{R}),$$

then

$$(T_{f,h}x)(t) = \frac{1}{2}[(a - ib)x(t + i\alpha h) + (a + ib)(x(t - i\alpha h))].$$

EXAMPLE 2. Let X and D be defined as in Example 1. Then $R = \int_0^t \in \mathcal{R}_D$ and the operators $I - \lambda R$ are invertible for all $\lambda \in \mathbf{C}$. Let $A = R$. Clearly, A satisfies all assumptions of Theorem 4. For all $x \in X$ we have $(R^2 x)(t) = \int_0^t (t-s)x(s)ds$ and $T_{f,0}x = f(0)x = a_0 x$. The family $T'_{f,K}$ defined by Formula (1.17) is then of the form

$$\begin{aligned} &(T_{f,h}x)(t) = \\ &= \{[T_{f,h} + (T_{f,K} - T_{f,0})R^2]x\}(t) = \\ &= [f(hD)x + f(hD)R^2 x - f(0)R^2 x](t) = \end{aligned}$$

$$= \Big[\sum_{k=0}^{\infty} e^{\alpha_k hD} D^k(x + R^2x) - a_0 R^2 x\Big](t) =$$

$$= \Big[a_0 R^2 x + a_1 h e^{\alpha_1 hD} Rx + \sum_{k=2}^{\infty} e^{\alpha_k hD}(D^2 x + D^{k-2}x - a_0 R^2 x\Big](t) =$$

$$= a_1 h \int_0^{t+\alpha_1 h} x(s)ds] - \sum_{k=2}^{\infty} a_k h^k [x^{(k)}(t+\alpha_k h) + \alpha_k h) + x^{(k-2)}(t+\alpha_k h)].$$

The infinitesimal operator for the family $T'_{f,K}$ is

$$f'(0)(D+A) = a_1\Big(\frac{\mathrm{d}}{\mathrm{d}t} + \int_0^t\Big).$$

If $a_1 = f'(0) \neq 0$ then the family $T^{\circ}_{f,K} = a_1^{-1}T'_{f,K}$ has the limit property and its infinitesimal operator is $D + A = \frac{\mathrm{d}}{\mathrm{d}t} + \int_0^t$.

EXAMPLE 3. Let X and D be defined as in Example 1. Assume that the function $f \in X$ can be represented in the form: $f = e^g$, where $g(h) = e^{\alpha h} \in X$, $\alpha \in \mathbf{C}$. Then functional shifts induced by the function f are of the form

$$(T_{f,h}x)(t) = \sum_{n=0}^{\infty} \frac{x(t+\alpha nh)}{n!} \quad \text{for } x \in X,\ t,h \in \mathbf{C}.$$

EXAMPLE 4. Let X and D be defined as in Example 1. Assume that the function $f \in X$ can be represented in the form: $f = e^g$, where $g(h) = h^2$, i.e. $f(h) = e^{h^2}$. Then functional shifts induced by the function f are of the form

$$(T_{f,h}x)(t) = \sum_{n=0}^{\infty} \frac{h^{2n}}{n!} x^{(2n)}(t) \quad \text{for } x \in X,\ t,h \in \mathbf{C}.$$

In particular,

$$e^{h^2D^2}\cos t = \frac{1}{2}[\cos(t-h)+\cos(t+h)]; \quad e^{h^2D^2}\sin t = \frac{1}{2}[\sin(t-h)-\sin(t+h)].$$

EXAMPLE 5. Let $X = H(K)$, where $K = \{h \in \mathbf{C} : |h| < 1\}$. Let $Dx = \frac{\mathrm{d}}{\mathrm{d}t}x + dx$, where $d \in K\setminus\{0\}$. Clearly, $D \in R(X)$. By an easy induction, we get

$$D^n x = \sum_{k=0}^{n} \binom{n}{k} d^{n-k} x^{(k)} \quad \text{for } x \in X,\ n \in \mathbf{N}.$$

Let $f(h) = \frac{1}{1-h}$ for $h \in K$. Then functional shifts induced by the function

f are of the form

$$T_{f,h}x = \frac{1}{1-dh} f\left(\frac{h}{1-dh}\frac{\mathrm{d}}{\mathrm{d}t}\right)x \quad \text{for } x \in X,\ h \in K.$$

Indeed, for all $x \in X$ and $h \in K$ we have

$$T_{f,h}x = f(hD)x = \sum_{n=0}^{\infty} h^n D^n x = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} d^{n-k}x^{(k)} =$$

$$= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} h^n d^{n-k}\right)x^{(k)} = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty}\binom{m+k}{k} h^{m+k}d^m\right)x^{(k)} =$$

$$= \sum_{k=0}^{\infty}\left[\sum_{m=0}^{\infty} h^m d^m\right]h^k x^{(k)} = \sum_{k=0}^{\infty}(1-dh)^{-(k+1)}h^k x^{(k)} =$$

$$= \frac{1}{1-dh}\sum_{k=0}^{\infty}\left(\frac{h}{1-dh}\right)^k x^{(k)} = \frac{1}{1-dh} f\left(\frac{h}{1-dh}\frac{\mathrm{d}}{\mathrm{d}t}\right)x.$$

EXAMPLE 6. Let X be the space of all sequences $\{x_n\}$ such that $x_n \in \mathbf{C}$ for $n \in \mathbf{N}$. Let D be the operator of the *forward* shift: $D\{x_n\} = \{x_{n+1}\}$. It is easy to verify that $D \in R(X)$ and that the operator R defined as follows: $R\{x_n\} = \{x_{n-1}\}$, where $x_{n-k} = 0$ if $k \geq n$, is a right inverse of D. Moreover, the space X equipped with the topology of coordinatewise convergence is an F-space and $X = A_R(D)$ (cf. PR[2]). Let

$$f(h) = \sum_{k=1}^{n} a_k e^{h^k} \quad \text{where } a_1, \ldots, a_n \in \mathbf{C}.$$

Then functional shifts induced by the function f are of the form

$$T_{f,h}\{x_n\} = \sum_{k=1}^{n} a_k \sum_{m=0}^{\infty}\frac{h^m}{m!}\{x_{n+km}\} \quad \text{for } \{x_n\} \in X.$$

References

Z. Binderman

B[1] *Cauchy integral formula induced by right invertible operators*, Demonstratio Math. 25 (1992), 671–690.

B[2] *Functional shifts induced by right invertible operators*, Math. Nachrichten 157 (1992), 211–214.

B[3] *Applications of sequential shifts to an interpolation problem*, Collect. Math. 44 (1993), 49–59.

B[4] *A unified approach to shifts induced by right invertible operators*, Math. Nachrichten 161 (1993), 239–252.

B[5] *On summation formulas induced by functional shifts of right invertible operators*, Demonstratio Math. 28 (1995), 301–314.

Z. Binderman and D. Przeworska-Rolewicz

B&PR[1] *Almost quasinilpotent right inverses*, in: Different Aspects of Differentiability '95. Proc. Conf. Warszawa, 1995 (to appear).

E. Hille and R. S. Phillips

HP[1] *Functional Analysis and Semigroups*, AMS Colloquium Publications, v. 31, Providence, R. I., 1957 (reprinted 1985).

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PR[1] *Algebraic Analysis*, PWN — Polish Scientific Publishers and D. Reidel Publish. Comp., Warszawa-Dordrecht, 1988.

PR[2] *Spaces of D-paraanalytic Elements*, Dissertationes Math. 302, Warszawa, 1990.

PR[3] *Generalized Bernoulli Operator and Euler-Maclaurin Formula*, in: Advances of Optimization, Proc. The 6th French-German Conf. on Optimization, Lambrecht (FRG), June 2–9, 1991. Lecture Notes in Economics and Math. Systems, 382. Springer-Verlag, Berlin-Heidelberg-New York, 1992, 355–368.

PR[4] *True Shifts*, J. Math. Analysis Appl. 170 (1992), 27–48.

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R[1] *Metric Linear Spaces*, 2nd extended edition. PWN — Polish Scientific Publishers and D. Reidel Publish. Comp., Warszawa-Dordrecht, 1985.

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