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A CLASS OF MULTI-DIMENSIONAL NONLINEAR VOLTERRA EQUATIONS OF CONVOLUTION TYPE

Cordially dedicated to Professor Janina Wolska-Bochenek

1. Introduction

Recently by A.L. Bukhgeim [2] and by J. Janno and the author [5] existence theorems for globally defined solutions to some classes of one-dimensional nonlinear Volterra equations of convolution type or with a convolution majorant have been derived. The proofs use the contraction principle in spaces C and L_p with weighted norms.

Equations of this kind arise in the theory of inverse problems for identifying memory kernels in viscoelasticity and heat transfer [2], [3].

In the present paper such existence theorems are given for solutions in spaces C and L_p with mixed norms to a related class of equations in n dimension.

Besides an application to a first kind nonlinear Volterra equation of auto-convolution type is briefly discussed.

2. Main result

We deal with an operator equation in R^n, $n \geq 1$, of the type

$$u(x) + G_0 u(x) + K[G_1 u, G_2 u](x) = g(x), \tag{1}$$

where $x = (x_1, \dots, x_n) \in D = \prod_{i=1}^n (0, X_i)$, $0 < X_i < \infty$, $y = (y_1, \dots, y_n) \in D$,

$$K[f_1, f_2](x) = \int_0^{x_n} \dots \int_0^{x_1} k(x,y) f_1(y) f_2(x-y) dy_1 \dots dy_n. \tag{2}$$

The solution u is sought in $C(\overline{D})$ or in the Lebesque space $L_P(D)$, $P = (p_1, \dots, p_n)$, $1 \leq P \leq \infty$ (i.e., $1 \leq p_i \leq \infty$, $i = 1, \dots, n$) with mixed norm which is obtained after taking succesively the p_1-norm in x_1, the p_2-norm in $x_2, \ldots$ the p_n-norm in x_n (cp. [1]). In particular, for $1 \le P \le \infty$ (i.e., $1 \le p_i < \infty$, $i = 1, \ldots, n$) we have

$$\|u\|_P = \Big(\int_0^{X_n} \cdots \Big(\int_0^{X_2} \Big(\int_0^{X_1} |u(x)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} \Big) dx_2 \Big)^{\frac{p_3}{p_2}} \ldots dx_n \Big)^{\frac{1}{p_n}}.$$

Further G_j, $j = 0, 1, 2$, are operators from spaces $L_P(D)$ to $L_{P_j}(D)$, where $P_0 = P$.

In the spaces $L_P(D)$, $1 \le P \le \infty$, and in $C(\overline{D})$ with $P = \infty$ we introduce the equivalent weighted norms

$$\|u\|_{P,\sigma} := \|e^{-\sigma|x|}u\|_P, \quad \sigma > 0,$$

where $|x| = \sum_{i=1}^n x_i$, which satisfy the relations

$$\|u\|_{P,\sigma} \le \|u\|_P \le e^{\sigma|X|}\|u\|_{P,\sigma}, \tag{3}$$

where $|X| = \sum_{i=1}^n X_i$.

Further we define the following set of functions $\mathfrak{M} = \{M \in R_+^2 \to R_+,\ M$ non-decreasing in its arguments$\}$.

Our main result is the following

THEOREM 1. *Let* $1 \le P \le \infty$, $1 < P_1, P_2 \le \infty$ *with* $1/P_1 + 1/P_2 < 1 + 1/P$ *and let*

$$G_0 \in (L_P(D) \to L_P(D)),\ G_j \in (L_P(D) \to L_{P_j}(D)),\ j = 1, 2,$$

satisfy Lipschitz conditions of the form

$$\|G_0u - G_0v\|_{P,\sigma} \le \lambda(\sigma)M_0(\|u\|_{P,\sigma}, \|v\|_{P,\sigma})\|u - v\|_{P,\sigma}, \tag{4}$$

$$\|G_ju - G_jv\|_{P,\sigma} \le M_j(\|u\|_{P,\sigma}, \|v\|_{P,\sigma})\|u - v\|_{P,\sigma},\ j = 1, 2, \tag{5}$$

for all $\sigma \ge \sigma_0 > 0$, *where* $M_j \in \mathfrak{M}$, $j = 0, 1, 2$, *and* λ *is a decreasing continuous function with* $\lambda(\sigma) \to 0$ *as* $\sigma \to \infty$.

Let further k *be a measurable function on* $D \times D$ *satisfying the inequality*

$$|k(x, y)| \le k_0(x)k_1(y)k_2(x - y), \tag{6}$$

where

$$\|k_0\|_{R_0}, \|k_1\|_{R_1}, \|k_2\|_{R_2} < \infty,$$

and $R_0, R_1, R_2 \ge 1$ *with* $R_0 \ge P$,

$$\frac{1}{P} - \frac{1}{R_0} < 1,\ \frac{1}{P_j} + \frac{1}{R_j} < 1, \quad j = 1, 2, \tag{7}$$

$$\frac{1}{R_0} + \frac{1}{R_1} + \frac{1}{R_2} < 1 + \frac{1}{P} - \left(\frac{1}{P_1} + \frac{1}{P_2}\right). \tag{8}$$

Then equation (1) has a unique solution $u \in L_P(D)$ for any $g \in L_P(D)$. *The same statement holds in space $C(\overline{D})$ with $P = \infty$ if $G_0 \in (C(\overline{D}) \to C(\overline{D}))$, $G_j \in (C(\overline{D}) \to L_{P_j}(D))$, $1 < P_j \leq \infty$, $j = 1,2$, and $k(x,y) = k_0(x)k_1(y)k_2(x-y)$, where $k_0 \in C(\overline{D})$, $k_j \in L_{P_j}(D)$, $j = 1,2$.*

COROLLARY 1. *The solution u of (1) depends (locally Lipschitz) continuously in the norm $\|\cdot\|_P$ on the data g. Namely, there holds the estimation*

$$(9)\quad \|u_1 - u_2\|_P \leq \overline{M}(|X|, \|G_1u_1\|_{P_1}, \|G_2u_2\|_{P_2}, \|u_1\|_P, \|u_2\|_P) \cdot \|g_1 - g_2\|_P$$

for the solutions $u = u_j$ of (1) with $g = g_j$, $j = 1,2$, where $\overline{M} \in R_+^5 \to R_+$ is a non-decreasing function in its arguments.

COROLLARY 2. *The statements of Theorem 1 hold true for equation*

$$u + G_0u + F(K[G_1u, G_2u]) = g, \tag{10}$$

where G_j, $j = 1,2$, and K as in the theorem and $F \in (L_P(D) \to L_P(D))$ satisfies the assumptions $F0 = 0$ and

$$\|Fu - Fv\|_{P,\sigma} \leq M(\|u\|_{P,\sigma}, \|v\|_{P,\sigma})\|u - v\|_{P,\sigma} \tag{11}$$

for $\sigma \geq \sigma_0 > 0$ with $M \in \mathfrak{M}$. Moreover, the statements hold true for equation (1) with finite sum of operators $K_k[G_{1,k}u, G_{2,k}u]$, $k = 1, \ldots, n$, where P is the same for all $k = 1, \ldots, n$.

Before proving Theorem 1 and Corollaries 1,2 in the next section, we give some preparations for the proof.

At first we state Young's inequality in the weighted norms. Let $1 \leq P, Q, R \leq \infty$ satisfy the relation $1/P + 1/Q = 1 + 1/R$. If $u \in L_P(D)$, $v \in L_Q(D)$, then $u * v \in L_R(D)$ and

$$\|u * v\|_{R,\sigma} \leq \|u\|_{P,\sigma}\|v\|_{Q,\sigma}, \quad \sigma \geq 0, \tag{12}$$

where $*$ denotes convolution, i.e.

$$(u * v)(x) = \int_0^{x_n} \ldots \int_0^{x_1} u(y)v(x-y)dy_1 \ldots dy_n.$$

This follows from the relation $e^{-\sigma|x|}(u*v) = (e^{-\sigma|x|}u)*(e^{-\sigma|x|}v)$ and Young's inequality in the mixed norms (cp. [1], Th. 1).

Further, by Hölder's inequality for $1 \leq R \leq P$ we have the estimations

$$\|u\|_{R,\sigma} = \|e^{-\sigma|x|}u\|_R \leq \|1\|_Q\|u\|_{P,\sigma}$$

and

$$\|u\|_{R,\sigma} \leq \|e^{-\sigma|x|}\|_Q\|u\|_P, \ 1/P + 1/Q = 1/R,$$

i.e., for $1 \le R \le P$ there hold the inequalities

(13) $$\|u\|_{R,\sigma} \le A(Q)\|u\|_{P,\sigma}, \ 1/P + 1/Q = 1/R,$$

where

$$A(Q) = \prod_{i=1}^{n} X_i^{1/q_i},$$

and

(14) $$\|u\|_{R,\sigma} \le B(Q,\sigma)\|u\|_P, \quad 1/P + 1/Q = 1/R,$$

where

$$B(Q,\sigma) = B_0(Q)\left(\frac{1}{\sigma}\right)^{\beta(q)},$$

$$B_0(Q) = \prod_{i=1}^{n}\left(\frac{1}{q_i}\right)^{1/q_i}, \quad \beta(q) = \sum_{i=1}^{n}\frac{1}{q_i}$$

with $(1/q_i)^{1/q_i}$ defined as 1 if $q_i = \infty$.

With the help of the inequalities (12)-(14) we estimate the operator K.

LEMMA 1. *Let* $1 \le P \le \infty$, $1 < P_1, P_2 \le \infty$ *with* $1/P_1 + 1/P_2 < 1 + 1/P$ *and* $f_j \in L_{P_j}$, $j = 1,2$. *Further let the assumptions* (6)-(8) *of Theorem 1 be fulfilled. Then there hold the estimations*

(15) $$\|K[f_1, f_2]\|_{P,\sigma} \le K_1\|f_1\|_{P_1,\sigma}\|f_2\|_{P_2,\sigma}$$

and

(16) $$\|K[f_1, f_2]\|_{P,\sigma} \le K_2(\sigma)\|f_k\|_{P_k}\|f_j\|_{P_j,\sigma} \quad (k,j = 1,2; k \ne j)$$

where K_1 *is a positive constant depending on the parameters* P, P_j, $j = 1,2$, $R_l, l = 0,1,2$, *and* $X_i, i = 1,\ldots,n$, *only, and* K_2 *is a positive function of* σ *and the parameters* P, P_j, R_l *continuous and decreasing in* σ *with* $K_2(\sigma) \to 0$ *as* $\sigma \to \infty$ *for fixed values of the parameters.*

Proof. By (6), Hölder's inequality and Young's inequality (12) we have

$$\begin{aligned}\|K[f_1, f_2]\|_{P,\sigma} &\le \|k_0\|_{R_0}\|k_1|f_1| * k_2|f_2|\|_{S_0,\sigma} \\ &\le \|k_0\|_{R_0}\|k_1 f_1\|_{Q_1,\sigma}\|k_2 f_2\|_{Q_2,\sigma} \le K_0\|f_1\|_{S_1,\sigma}\|f_2\|_{S_2,\sigma}\end{aligned}$$

with

$$K_0 = \|k_0\|_{R_0}\|k_1\|_{R_1}\|k_2\|_{R_2},$$

where $1/R_0 + 1/S_0 = 1/P$ with $S_0 \geq P \geq 1$ since $R_0 \geq P$, $1/Q_1 + 1/Q_2 = 1 + 1/S_0$, i.e.

$$\frac{1}{Q_1} + \frac{1}{Q_2} = 1 + \frac{1}{P} - \frac{1}{R_0} \tag{17}$$

with $Q_j \geq 1$, $j = 1, 2$, and $1/R_j + 1/S_j = 1/Q_j$, i.e.

$$\frac{1}{S_j} = \frac{1}{Q_j} - \frac{1}{R_j}, \quad j = 1, 2, \tag{18}$$

with $Q_j \leq R_j$, $j = 1, 2$. From (17) and (18) there follows

$$\frac{1}{S_1} + \frac{1}{S_2} = 1 + \frac{1}{P} - \left(\frac{1}{R_0} + \frac{1}{R_1} + \frac{1}{R_2}\right). \tag{19}$$

We now choose positive S_j, $j = 1, 2$, satisfying the relation (19) and the inequalities

$$\frac{1}{P_j} < \frac{1}{S_j} \leq 1 - \frac{1}{R_j}, \quad j = 1, 2, \tag{20}$$

so that the conditions $1 \leq Q_j \leq R_j$ and $1 \leq S_j < P_j$, $j = 1, 2$, are fulfilled. In view of (7), (8) such a choice of S_j, $j = 1, 2$, is always possible, namely by

$$\begin{aligned} \max\left\{\frac{1}{P_1}, \frac{1}{P} - \frac{1}{R_0} - \frac{1}{R_1}\right\} &< \frac{1}{S_1} \\ &< \min\left\{1 - \frac{1}{R_1}, 1 + \frac{1}{P} - \frac{1}{P_2} - \left(\frac{1}{R_0} + \frac{1}{R_1} + \frac{1}{R_2}\right)\right\} \end{aligned} \tag{21}$$

for S_1 and correspondingly for S_2 with (19).

Finally, applying the inequalities (13) and (14) with $R = S_j$, $P = P_j$, to $\|f_j\|_{S_j,\sigma}$, we obtain

$$\|f_j\|_{S_j,\sigma} \leq A(\Delta_j)\|f_j\|_{P_j,\sigma}$$

and

$$\|f_j\|_{S_j,\sigma} \leq B(\Delta_j, \sigma)\|f_j\|_{P_j},$$

where $\Delta_j > 0$ is given by $1/\Delta_j = 1/S_j - 1/P_j$. This yields the estimations (15), (16) with

$$K_1 := K_0 \prod_{j=1}^{2} A(\Delta_j)$$

and

$$K_2(\sigma) := K_0 \max[A(\Delta_2)B(\Delta_1, \sigma), A(\Delta_1)B(\Delta_2, \sigma)].$$

3. Proof of Theorem 1

The proof follows the lines of the corresonding proof in [5].

a. By the assumptions $G_0 \in (L_P(D) \to L_P(D))$, $G_j \in (L_P(D) \to L_{P_j}(D))$, $j = 1, 2$, and Lemma 1 there is

$$G_0 u + K[G_1 u, G_2 u] \in (L_P(D) \to L_P(D)) \tag{22}$$

for the spaces $L_P(D)$, $1 \leq P \leq \infty$. For the space $C(\overline{D})$ there hold $G_0 \in (C(\overline{D}) \to C(\overline{D}))$, $G_j \in C(\overline{D}) \to L_{P_j}(D))$, $1 < P_j \leq \infty$, and

$$K[G_1 u, G_2 u] = k_0 \cdot (k_1 G_1 u * k_2 G_2 u),$$

where $k_0 \in C(\overline{D})$ and

$$k_1 G_1 u * k_2 G_2 u \in C(\overline{D}) \quad \text{as } u \in C(\overline{D}),$$

since $k_1 G_1 u \in L_{Q_1}$, $k_2 G_2 u \in L_{Q_2}$ with $1/Q_j = 1/P_j + 1/R_j < 1$, $j = 1, 2$, and

$$\frac{1}{Q_1} + \frac{1}{Q_2} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{P_1} + \frac{1}{P_2} < 1$$

by (7), (8) with $P = R_0 = \infty$ (cp. [1], 10, Th. 1). Hence also

$$G_0 u + K[G_1 u, G_2 u] \in (C(\overline{D}) \to C(\overline{D})).$$

b. At first we consider the auxiliary equation

$$f + G_0 f = g. \tag{23}$$

By contraction principle we show the existence of a solution to (23) in the ball $B_{\rho,\sigma}(g) = \{f : \|f - g\|_{P,\sigma} \leq \rho\}$, where $\rho = 2\|G_0 g\|_P$ and $\sigma \geq \sigma_0$ is chosen as a solution of the equation

$$\lambda(\sigma) M_0(\rho + \|g\|_P, \rho + \|g\|_P) = \varepsilon \tag{24}$$

with some $\varepsilon \in (0, 1/2]$. Due to the assumptions on λ a solution σ of (24) exists for any sufficiently small positive ε. Further, by (4) and (24) for the operator $A_0 f := g - G_0 f$ of (23) in $B_{\rho,\sigma}(g)$ we have the estimates

$$\begin{aligned}\|A_0 f_1 - A_0 f_2\|_{P,\sigma} &= \|G_0 f_1 - G_0 f_2\|_{P,\sigma} \leq \lambda(\sigma) M_0(\|f_1\|_{P,\sigma}) \|f_1 - f_2\|_{P,\sigma} \\ &\leq \lambda(\sigma) M_0(\rho + \|g\|_{P,\sigma}, \rho + \|g\|_{P,\sigma}) \|f_1 - f_2\|_{P,\sigma} \leq \varepsilon \|f_1 - f_2\|_{P,\sigma},\end{aligned}$$

so that A_0 is a contraction. Moreover,

$$\begin{aligned}\|A_0 f - g\|_{P,\sigma} = \|G_0 f\|_{P,\sigma} &\leq \|G_0 f - G_0 g\|_{P,\sigma} + \|G_0 g\|_{P,\sigma} \\ &\leq \lambda(\sigma) M_0(\rho + \|g\|_{P,\sigma}, \|g\|_{P,\sigma}) \|f - g\|_{P,\sigma} + \|G_0 g\|_{P,\sigma} \\ &\leq (\varepsilon + 1/2)\rho \leq \rho,\end{aligned}$$

so that A_0 maps $B_{\rho,\sigma}(g)$ into itself.

c. Now we are going to show that a unique solution of (1) exists in the ball $B_{\rho,\sigma}(f) = \{u : \|u - f\|_{P,\sigma} \leq \rho\}$ with some ρ, σ, also using the contraction principle. Equation (1) writes $u = Au$ with the operator $Au := g - G_0u - K[G_1u, G_2u]$. In view of (23) we have

$$\begin{aligned} f - Au &= K[G_1u, G_2u] + G_0u - G_0f \\ &= K[G_1u - G_1f, G_2u - G_2f] + K[G_1f, G_2u - G_2f] \\ &\quad + K[G_1u - G_1f, G_2f] + K[G_1f, G_2f] + G_0u - G_0f. \end{aligned}$$

Making use of the inequalities (15), (16) and (3), we obtain

$$\begin{aligned} \|f - Au\|_{P,\sigma} \leq\ & K_1\|G_1u - G_1f\|_{P_1,\sigma}\|G_2u - G_2f\|_{P_2,\sigma} \\ &+ K_2(\sigma)\|G_1f\|_{P_1}\|G_2u - G_2f\|_{P_2,\sigma} \\ &+ K_2(\sigma)\|G_2f\|_{P_2}\|G_1u - G_1f\|_{P_1,\sigma} \\ &+ K_2(\sigma)\|G_1f\|_{P_1}\|G_2f\|_{P_2} + \|G_0u - G_0f\|_{P,\sigma}. \end{aligned}$$

Further, by the assumptions (4), (5) we have

$$\begin{aligned} \|f - Au\|_{P,\sigma} \leq\ & K_1M_1(\|f\|_{P,\sigma} + \|u - f\|_{P,\sigma}, \|f\|_{P,\sigma}) \\ & \times M_2(\|f\|_{P,\sigma} + \|u - f\|_{P,\sigma}, \|f\|_{P,\sigma})\|u - f\|^2_{P,\sigma} \\ &+ K_2(\sigma)[\|G_1f\|_{P_1}M_2(\|f\|_{P,\sigma} + \|u - f\|_{P,\sigma}, \|f\|_{P,\sigma}) \\ &+ \|G_2f\|_{P_2}M_1(\|f\|_{P,\sigma} + \|u - f\|_{P,\sigma}, \|f\|_{P,\sigma})]\|u - f\|_{P,\sigma} \\ &+ K_2(\sigma)\|G_1f\|_{P_1}\|G_2f\|_{P_2} \\ &+ \lambda(\sigma)M_0(\|f\|_{P,\sigma} + \|u - f\|_{P,\sigma}, \|f\|_{P,\sigma})\|u - f\|_{P,\sigma}. \end{aligned}$$

Now we choose $\rho_1 > 0$, $\sigma_1(\rho) \geq \sigma_0$ such that

$$\begin{aligned} & K_1M_1(\|f\|_P + \rho, \|f\|_P)M_2(\|f\|_P + \rho, \|f\|_P)\rho \leq \alpha, \quad \alpha \in (0,1), \\ & K_2(\sigma)[\|G_1f\|_{P_1}M_2(\|f\|_P + \rho, \|f\|_P) + \|G_2f\|_{P_2}M_1(\|f\|_P + \rho, \|f\|_P)]\rho \\ & \quad + K_2(\sigma)\|G_1f\|_{P_1}\|G_2f\|_{P_2} + \lambda(\sigma)M_0(\|f\|_P + \rho, \|f\|_P)\rho \leq (1 - \alpha)\rho \end{aligned}$$

provided $\rho \leq \rho_1$ and $\sigma \geq \sigma_1(\rho)$. On account of $M_j \in \mathfrak{M}$, $j = 0, 1, 2$, then

$$\|f - Au\|_{P,\sigma} \leq \rho \tag{25}$$

if $u \in B_{\rho,\sigma}(f)$, $\rho \leq \rho_1$ and $\sigma \geq \sigma_1(\rho)$. I.e., A maps the ball $B_{\rho,\sigma}(f)$ into itself.

For the difference of the operator A we write

$$\begin{aligned}Au_2 - Au_1 &= K[G_1u_1, G_2u_1] - K[G_1u_2, G_2u_1] + G_0u_1 - G_0u_2 \\ &= K[G_1u_1 - G_1u_2, G_2u_1 - G_2f] \\ &\quad + K[G_1u_1 - G_1u_2, G_2f] + K[G_1u_2 - G_1f, G_2u_1 - G_2u_2] \\ &\quad + K[G_1f, G_2u_1 - G_2u_2] + G_0u_1 - G_0u_2.\end{aligned}$$

Estimating as above, for $u_1, u_2 \in B_{\rho,\sigma}(f)$, we obtain

$$\begin{aligned}&\|Au_1 - Au_2\|_{P,\sigma} \\ &\quad \leq \{M_1(\|f\|_{P,\sigma} + \rho, \|f\|_{P,\sigma} + \rho)[K_1M_2(\|f\|_{P,\sigma} + \rho, \|f\|_{P,\sigma})\rho \\ &\qquad + K_2(\sigma)\|G_2f\|_{P_2}] + M_2(\|f\|_{P,\sigma} + \rho, \|f\|_{P,\sigma} + \rho) \\ &\qquad \times [K_1M_1(\|f\|_{P,\sigma} + \rho, \|f\|_{P,\sigma})\rho + K_2(\sigma)\|G_1f\|_{P_1}] \\ &\qquad + \lambda(\sigma)M_0(\|f\|_{P,\sigma} + \rho, \|f\|_{P,\sigma} + \rho)\}\|u_1 - u_2\|_{P,\sigma}, \quad \sigma \geq \sigma_0.\end{aligned}$$

We choose $\rho_2 > 0$, $\sigma_2 \geq \sigma_0$ such that

$$\begin{aligned}&M_1(\|f\|_P + \rho, \|f\|_P + \rho)[K_1M_2(\|f\|_P + \rho, \|f\|_P)\rho + K_2(\sigma)\|G_2f\|_{P_2}] \\ &+ M_2(\|f\|_P + \rho, \|f\|_P + \rho)[K_1M_1(\|f\|_P + \rho, \|f\|_P)\rho + K_2(\sigma)\|G_1f\|_{P_1}] \\ &+ \lambda(\sigma)M_0(\|f\|_P + \rho, \|f\|_P + \rho) \leq \mu < 1\end{aligned}$$

provided $\rho \leq \rho_2$, $\sigma \geq \sigma_2$. Then

$$\|Au_1 - Au_2\|_{P,\sigma} \leq \mu\|u_1 - u_2\|_{P,\sigma} \tag{26}$$

if $u_1, u_2 \in B_{\rho,\sigma}(f)$ and $\rho \leq \rho_2$, $\sigma \geq \sigma_2$.

The estimations (25), (26) show that the operator A is a contraction in $B_{\rho,\sigma}(f)$ with $\rho \leq \rho_3 = \min\{\rho_1, \rho_2\}$, $\sigma \geq \sigma_3(\rho) = \max\{\sigma_1(\rho), \sigma_2\}$. Hence equation (1) has a unique solution in every ball $B_{\rho,\sigma}(f)$ with $\rho \leq \rho_3$, $\sigma \geq \sigma_3(\rho)$.

d. It remains to prove the uniqueness of the solution in the whole space $L_P(D)$, $1 \leq P \leq \infty$ (inclusively $C(\overline{D})$ for $P = \infty$). For this aim let u be an arbitrary solution of (1) in $L_P(D)$. From equations (1) and (23) we have

$$u - f = G_0f - G_0u - K[G_1u, G_2u].$$

Estimating the right-hand side by means of (4) and (16) with (3), we deduce the inequality

$$\begin{aligned}\|u - f\|_{P,\sigma} \leq \lambda(\sigma)&M_0(\|f\|_P, \|u\|_P)\|u - f\|_{P,\sigma} \\ &+ K_2(\sigma)\|G_1u\|_{P_1}\|G_2u\|_{P_2}.\end{aligned}$$

Since $\lambda(\sigma), K_2(\sigma) \to 0$ as $\sigma \to \infty$, this implies

$$\|u - f\|_{P,\sigma} \to 0 \quad \text{as } \sigma \to \infty.$$

This means, any solution $u \in L_P(D)$ of equation (1) belongs to some ball $B_{\rho,\sigma}(f)$ with $\rho \leq \rho_3$ and sufficiently large $\sigma \geq \sigma_3(\rho)$, in which the uniqueness of the solution has already been shown.

Theorem 1 is completely proved.

To prove Corollary 1 let $u_j \in L_P(D)$ be the solution of (1) for right-hand side $g_j \in L_P(D)$, $j = 1, 2$. Then

$$\begin{aligned} u_1 - u_2 &= g_1 - g_2 + G_0 u_2 - G_0 u_1 \\ &\quad + K[G_1 u_2, G_2 u_2] - K[G_1 u_1, G_2 u_1] \\ &= g_1 - g_2 + G_0 u_2 - G_0 u_1 \\ &\quad + K[G_1 u_2 - G_1 u_1, G_2 u_2] - K[G_1 u_1, G_2 u_1 - G_2 u_2]. \end{aligned}$$

Hence by (4), (16) and (5) we have

$$\begin{aligned} \|u_1 - u_2\|_{P,\sigma} \leq \|g_1 - g_2\|_{P,\sigma} &+ \{\lambda(\sigma) M_0(\|u_1\|_{P,\sigma}, \|u_2\|_{P,\sigma}) \\ &+ K_2(\sigma)[\|G_1 u_1\|_{P_1} M_2(\|u_1\|_{P,\sigma}, \|u_2\|_{P,\sigma}) \\ &+ \|G_2 u_2\|_{P_2} M_1(\|u_1\|_{P,\sigma} \|u_2\|_{P,\sigma}]\} \|u_1 - u_2\|_{P,\sigma} \end{aligned}$$

from which taking σ sufficiently large and using $M_j \in \mathfrak{M}$, $j = 0, 1, 2$, and (3) the estimation (9) follows.

The proof of Corollary 2 is obvious.

Remark. As it can be seen from the proof, the assertions of Theorem 1 hold true for general equations of form (1) with G_j, $j = 0, 1, 2$, as in the theorem and with a bilinear operator $K \in (L_{P_1}(D) \times L_{P_2}(D) \to L_P(D))$ which fulfills the estimations (15), (16).

4. Volterra equation of first kind

We apply Theorem 1 to the first kind two-dimensional Volterra equation of auto-convolution type

$$\int_0^{x_2} \int_0^{x_1} u(y) v(x - y) dy_1 \, dy_2 = f(x), \tag{27}$$

where $x = (x_1, x_2) \in D = (0, X_1) \times (0, X_2)$, $y = (y_1, y_2) \in D$, $f \in C_1(\overline{D})$ with $f_{x_1, x_2} \in C(\overline{D})$ or $L_P(D)$, $P = (p_1, p_2)$, $1 < p_1 p_2 \leq \infty$, and $v \in C(\overline{D}) \cap C_2(D)$ is the solution of the Darboux problem (cf. [4], Chap. 14)

$$Lv(x_1, x_2) \equiv v_{x_1 x_2} + a(x) v_{x_1} + b(x) v_{x_2} + c(x) v = F(x, u(x)) \text{ in } D \tag{28}$$

with the initial conditions

(29) $v(x_1,0) = \varphi(x_1)$ in $0 \le x_1 \le X_1 \quad v(0,x_2) = \psi(x_2)$ in $0 \le x_2 \le X_2$.

We assume that the coefficients $a, b, c \in C(\overline{D})$ and the initial data φ, ψ are absolutely continuous with derivatives $\varphi' \in L_r(0,X_1)$, $\psi' \in L_r(0,X_2)$, $r > 1$, and $\varphi(0) = \psi(0) = 1$. Further, the function F is continuous and satisfies a (uniform in x) Lipschitz condition with respect to u.

The solution v of (28), (29) can be represented by the Riemann function R of L (cf. [4], p. 394). Namely, we have

$$v(x) = \varphi(x_1) + \psi(x_2) - 1 + \int_0^{x_2} \int_0^{x_1} R(x,y)G(y,u(y))dy_1\, dy_2, \tag{30}$$

where

$$\begin{aligned} G(x,u) &= F(x,u) - h(x) \\ h(x) &= a(x)\varphi'(x_1) + b(x)\psi'(x_2) + c(x)\{\varphi(x_1) + \psi(x_2) - 1\}. \end{aligned} \tag{31}$$

Differentiating (27) with respect to x_1 and x_2, we obtain the following second kind integral equation of form (1)

$$\begin{aligned} u(x) &+ \int_0^{x_1} u(y_1,x_2)\varphi'(x_1 - y_1)dy_1 + \int_0^{x_2} u(x_1,y_2)\psi'(x_2 - y_2)dy_2 \\ &+ \int_0^{x_2} \int_0^{x_1} u(y)[F(x-y,u(x-y)) - h(x-y) - H[u](x-y)]dy_1\, dy_2 = g(x), \end{aligned} \tag{32}$$

where $g(x) = f_{x_1x_2}(x)$ and

$$\begin{aligned} H[u](x) = c(x) &\int_0^{x_2} \int_0^{x_1} R(x,\xi)G(\xi,u(\xi))d\xi_1\, d\xi_2 \\ &+ a(x)\Big[\int_0^{x_2} R(x_1,x_2;x_1,\xi_2)G(x_1,\xi_2;u(x_1,\xi_2)) \\ &+ \int_0^{x_2} \int_0^{x_1} R_{x_1}(x,\xi)G(\xi,u(\xi))d\xi_1\, d\xi_2\Big] \\ &+ b(x)\Big[\int_0^{x_1} R(x_1,x_2;\xi_1,x_2)G(\xi_1,x_2;u(\xi_1,x_2)) \\ &+ \int_0^{x_2} \int_0^{x_1} R_{x_1}(x,\xi)G(\xi,u(\xi))d\xi_1\, d\xi_2\Big]. \end{aligned} \tag{33}$$

Equation (32) is equivalent to (27) if the compatibility conditions

$$(34) \qquad f(x_1,0) = 0 \text{ in } 0 \le x_1 \le X_1, \quad f(0,x_2) = 0 \text{ in } 0 \le x_2 \le X_2$$

are fulfilled.

To equation (32) Theorem 1 can be applied yielding the following

THEOREM 2. *Under the above assumptions about the data $a, b, c, F, \varphi, \psi$ integral equation (27) has a unique solution $u \in C(\overline{D})$ or $u \in L_P(D)$, $P = (p_1, p_2)$, $1 < p_1, p_2 \le \infty$ for any $f \in C_1(\overline{D})$ with $f_{x_1x_2} \in C(\overline{D})$ or $f_{x_1x_2} \in L_P(D)$, respectively, satisfying the compatibility conditions* (34).

Sketch of proof. In view of the continuity of the Riemann functon R and its derivatives R_{x_1}, R_{x_2} (see [4], p. 394) and the assumed continuity and Lipschitz condition of F the assumptions on the operators G_j, $j = 1,2$, in Theorem 1 are fulfilled with $P_j = P$, and the assumptions on the kernel k with $R_0 = R_1 = R_2 = \infty$. It remains to show that the linear operator G_0 defined by

$$(35) \quad G_0 u(x) = \int_0^{x_1} u(y_1, x_2)\chi(x_1 - y_1)dy_1 = \int_0^{x_1} u(x_1 - y_1, x_2)\chi(y_1)dy_1$$

for given $\chi \in L_r(0, X_1)$, $r > 1$, (and the corresponding one with x_1, y_1 and x_2, y_2 changed) is an operator in $C(\overline{D})$ and $L_P(D)$ and fulfills a Lipschitz condition of form (4). But the first property is obvious by the second integral representation in (35). And the second property follows from the estimations

$$\begin{aligned}
\|G_0 u\|_{\infty,\sigma} &= \operatorname*{ess\,sup}_{x \in D} |e^{-\sigma(x_1+x_2)} \int_0^{x_1} u(y_1, x_2)\chi(x_1 - y_1)dy_1| \\
&\le \operatorname*{ess\,sup}_{x \in D} \int_0^{x_1} e^{-\sigma(x_2+y_1)}|u(y_1, x_2)|e^{-\sigma(x_1-y_1)}|\chi(x_1 - y_1)|dy_1 \\
&\le \|u\|_{\infty,\sigma} \int_0^{X_1} e^{-\sigma\xi_1}|\chi(\xi_1)|d\xi_1 \\
&\le \|u\|_{\infty,\sigma}\|\chi\|_r\|e^{-\sigma\xi_1}\|_s \le \|\chi\|_r \left(\frac{1}{s\sigma}\right)^{1/s} \|u\|_{\infty,\sigma}
\end{aligned}$$

by applying Hölder's inequality to the integral over χ, where $1/s = 1 - 1/r > 0$ and $\|\cdot\|_r$ is the norm in $L_r(0, X_1)$, and

$$\|G_0 u\|_{P,\sigma} = \Big(\int_0^{X_2} \Big(\int_0^{X_1} e^{-\sigma p_1(x_1+x_2)} \times \Big| \int_0^{x_1} u(y_1,x_2)\chi(x_1-y_1)dy_1\Big|^{p_1} dx_1\Big)^{\frac{p_2}{p_1}} dx_2\Big)^{\frac{1}{p_2}}$$

$$\leq \Big(\int_0^{X_2} e^{-\sigma p_2 x_2} \Big(\int_0^{X_1} \Big(\int_0^{x_1} e^{-\sigma y_1} |u(y_1,x_2)| e^{-\sigma(x_1-y_1)} \times |\chi(x_1-y_1)| dy_1\Big)^{p_1} dx_1\Big)^{\frac{p_2}{p_1}} dx_2\Big)^{\frac{1}{p_2}}$$

$$\leq \Big(\int_0^{X_2} e^{-\sigma p_2 x_2} \Big(\int_0^{X_1} e^{-\sigma p_1 x_1} |u(x_1,x_2)|^{p_1} dx_1\Big]^{\frac{p_2}{p_1}} \times \Big[\int_0^{X_1} e^{-\sigma\xi_1}|\chi(\xi_1)|d\xi_1\Big)^{p_2} dx_2\Big)^{\frac{1}{p_2}}$$

$$\leq \|\chi\|_r \|e^{-\sigma\xi_1}\|_s \|u\|_{P,\sigma} \leq \|\chi\|_r \Big(\frac{1}{s\sigma}\Big)^{1/s} \|u\|_{P,\sigma}, \quad \frac{1}{r}+\frac{1}{s}=1,$$

by applying Young's inequality to the inner integral in the variable x_1 and then Hölder's inequality to the integral over χ again.

Remarks: 1. In case of constant coefficients a, b, c instead of (32) the following simpler integral equation can be used:

$$u(x) + \int_0^{x_1} u(y_1,x_2)[\varphi'(x_1-y_1) + b\varphi(x_1-y_1)]dy_1 + \int_0^{x_2} u(x_1,y_2)[\psi'(x_2-y_2) + a\psi(x_2-y_2)]dy_2 + \int_0^{x_2}\int_0^{x_1} u(y)F(x-y,u(x-y))dy_1\,dy_2 = Lf(x). \tag{36}$$

2. By extending the proof of Theorem 1, in $C(\overline{D})$ there can also be handled problems in which the initial conditions (29) have more general form

$$v_{x_1}(x_1,0) = \lambda(x_1)u(x_1,0) + \varphi_0(x_1), \tag{37}$$

$$v_{x_2}(0,x_2) = \mu(x_2)u(0,x_2) + \psi_0(x_2) \tag{38}$$

together with $v(0,0) = 1$. Here $\lambda, \varphi_0 \in L_r(0,X_1)$ and $\mu, \psi_0 \in L_r(0,X_2)$,

$r > 1$. This leads to additional one-dimensional integrals in (32) of the kind

$$\int_0^{x_1} u(y_1, x_2)\lambda(x_1 - y_1)u(x_1 - y_1, 0)dy_1 + \int_0^{x_2} u(x_1, y_2)\mu(x_2 - y_2)u(0, x_2 - y_2)dy_2. \tag{39}$$

For these integrals the related estimations in the proof of Theorem 1 can be carried out, too. Namely, for instance, there holds

$$\begin{aligned} I &:= \max_{x\in\overline{D}}\left[e^{-\sigma(x_1+x_2)}\int_0^{x_1}|\lambda(x_1-y_1)||u_1(y_1,x_2)||u_2(x_1-y_1,0)|dy_1\right] \\ &\le \max_{x\in\overline{D}}\left[e^{-\sigma x_2}\int_0^{x_1}|\lambda(x_1-y_1)|e^{-\sigma y_1}|u_1(y_1,x_2)|e^{-\sigma(x_1-y_1)} \right. \\ &\left.\times|u_2(x_1-y_1,0)|dy_1\right] \le \max_{\xi_1}[e^{-\sigma\xi_1}|u_2(\xi_1,0)|] \\ &\times\max_{x_2}[e^{-\sigma x_2}\max_{x_1}(e^{-\sigma x_1}|u_1(x_1,x_2)|)]\int_0^{X_1}|\lambda(\xi_1)|d\xi_1 \\ &\le \|\lambda\|_1\|u_1\|_{\sigma,\infty}\|u_2\|_{\sigma,\infty} \end{aligned}$$

since for the function $U_2(x_1, x_2) = e^{-\sigma(x_1+x_2)}|u_2(x_1, x_2)|$ we have $U_2(x_1, 0) \le \max_{x_2} U_2(x_1, x_2)$.

Analogously,

$$I \le \Lambda(\sigma)\|u_1\|_{\sigma,\infty}\|u_2\|_\infty$$

with

$$\Lambda(\sigma) = \int_0^{X_1}|\lambda(\xi_1)|e^{-\sigma\xi_1}d\xi_1 \to 0 \quad \text{as } \sigma\to\infty,$$

and

$$I \le \Lambda_1(\sigma)\|u_1\|_\infty\|u_2\|_{\sigma,\infty}$$

with

$$\Lambda_1(\sigma) = \max_{x_1}\int_0^{x_1}|\lambda(x_1-y_1)|e^{-\sigma y_1}dy_1 \le \|\lambda\|_r\left(\frac{1}{s\sigma}\right)^{1/s} \to 0 \text{ as } \sigma\to\infty,$$

where $1/s = 1 - 1/r > 0$.

If only one initial condition is of form (37) or (38), then one can also work in a space with mixed norm, namely with max-norm in one variable and with p-norm, $1 < p \le \infty$, in the other variable.

Moreover, we remark that taking $x_1 = 0$ and $x_2 = 0$ into equation (32) with the integrals (39) in a first step one can determine the functions $u(0, x_2)$ and $u(x_1, 0)$, respctively, from one-dimensional convolution equations, and then inserting these functions in the integrals (39) obtain these ones as additional linear integral operators in equation (32).

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