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ON THE BOUNDEDNESS OF SOME INTEGRAL OPERATORS WITH FIXED SINGULARITIES ARISING IN ELASTICITY

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

A method for solving problems of elasticity with interfaces consists in the use of contact tensors. The contact tensor is constructed as the sum of the fundamental matrix and a compensatrix such that the transmission conditions on the interface S_{00} are fulfilled. A potential on the boundary S of the whole domain having as kernel the contact tensor satisfies the differential equation and transmission conditions whereas the boundary conditions on S lead to the boundary integral equation (BIE) for the density of the potential.

If the interface S_{00} lies inside of the boundary S the BIEs are singular integral equations of the same kind as in the case of elastic homogeneous bodies and the now classical theory of Michlin is applicable [4–6].

If S_{00} touches S we have interface corners in the two-dimensional case and interface edges in the three-dimensional case. Because of the behaviour of the the contact tensor the BIEs have fixed singularities at $\overline{S_{00}} \cap S$. In the plane case the BIEs are locally considered in the neighbourhood of an interface corner of Mellin convolution type and with the Mellin technique the question of Fredholm property and asymptotics can be decided [8]. In

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the spatial case D. Mirschinka [9, 10] investigated the BIEs for bimetal heat conduction problems with perfect and non-perfect heat contact. The question of Fredholm property of the matrix boundary integral operator is connected with the invertibility of local operators acting in spaces of functions defined on the tangential half-planes to S at points of the interface edge ∂S_{00}. The local operators have fixed singularities along the boundary of the half-plane and can be written as convolution operators. The norm of these operators in weighted L_p-spaces can be estimated by the L_1-norm of the generating function.

In the present paper for the bimetal problem of elastostatics the local operators are constructed and estimates for the norm of these operators are derived.

2. The bimetal problem

We consider a domain $D \subset \mathbf{R}^3$ which is devided by the plane $E = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 0\}$ into two domains $D_1 = D^+ = \{x \in D : x_3 > 0\}$, $D_0 = D^- = \{x \in D : x_3 < 0\}$. The Lamé constants in D_i are λ_i, μ_i. We look for a solution $\mathbf{u} = \mathbf{u}(x)$ of the following boundary transmission problem:

a) The displacement $\mathbf{u} = \mathbf{u}(x)$ has to satisfy the Lamé equations

$$\mu_i \Delta \mathbf{u} + (\lambda_i + \mu_i) \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{0} \quad \text{in } D_i. \tag{1}$$

b) On the boundary $S = \partial D$ are given Dirichlet type boundary conditions

$$\mathbf{u}(x) = \mathbf{w}(x) \quad \text{for} \quad x \in S \tag{2}$$

or Neumann type boundary conditions

$$\underset{(i)}{T}(\partial_x, \mathbf{n}_x)\mathbf{u}(x) = \mathbf{P}(x) \quad \text{for } x \in S_i = (S \cap \overline{D}_i) - E. \tag{3}$$

c) On the interface $S_{00} = D \cap E$ have to be satisfied the transmission conditions

$$\{\mathbf{u}(x)\}^+ - \{\mathbf{u}(x)\}^- = \mathbf{0} \quad \text{for } x \in S_{00}, \tag{4}$$

$$\left\{ \underset{(1)}{T}(\partial_x, \mathbf{n}_x)\mathbf{u}(x) \right\}^+ - \left\{ \underset{(0)}{T}(\partial_x, \mathbf{n}_x)\mathbf{u}(x) \right\}^- = \mathbf{0} \quad \text{for } x \in S_{00}. \tag{5}$$

Here $\underset{(i)}{T}(\partial_x, \mathbf{n}_x)\mathbf{u}(x) = 2\mu_i \frac{\partial \mathbf{u}}{\partial \mathbf{n}_x} + \lambda_i \mathbf{n}_x \operatorname{div} \mathbf{u} + \mu_i [\mathbf{n}_x \times \operatorname{rot} \mathbf{u}]$ is the stress vector, $\mathbf{n}_x = (n_1(x), n_2(x), n_3(x))$ in (3) is the outer normal of S in x, $\mathbf{n}_x = (0, 0, -1)$ in (5), $\{.\}^+ = \lim_{x_3 \to +0}$, $\{.\}^- = \lim_{x_3 \to -0}$.

Let $G(x,y)$ be the contact tensor for our problem. This means

$$G(x,y) = \begin{cases} \underset{(1)}{\Gamma}(x,y) + V(x,y) & \text{for } x_3 > 0,\ y_3 > 0 \\ \underset{(0)}{\Gamma}(x,y) + V(x,y) & \text{for } x_3 < 0,\ y_3 < 0 \\ V(x,y) & \text{for } x_3 > 0,\ y_3 < 0 \\ & \text{and } x_3 < 0,\ y_3 > 0 \end{cases} \tag{6}$$

where

$$\underset{(i)}{\Gamma}(x,y) = \frac{1}{2\mu_i(\lambda_i + 2\mu_i)} \times \\ \times \left((\lambda_i + \mu_i) \frac{x_k - y_k}{|x-y|} \frac{x_j - y_j}{|x-y|} + (\lambda_i + 3\mu_i)\delta_{kj} \right) \frac{1}{|x-y|} \tag{7}$$

is the Kelvin fundamental matrix and $V(x,y)$ is the compensatrix such that the columns of $G(x,y)$ with respect to x fulfil (1) in the half-spaces $H_1 = H^+ = \{x \in \mathbf{R}^3 : x_3 > 0\}$, $H_0 = H^- = \{x \in \mathbf{R}^3 : x_3 < 0\}$, respectively, and fulfil (4), (5) on E.

Obviously the potential of the single layer

$$\mathbf{V}(x;\varphi) = \frac{1}{2\pi} \int_S G(x,y)\varphi(y)\, d_yS \tag{8}$$

and the potential of the double layer

$$\mathbf{W}(x;\varphi) = \frac{1}{2\pi} \int_S (T(\partial_y, \mathbf{n}_y)G(x,y)^T)^T \varphi(y)\, d_yS \tag{9}$$

satisfy (1), (3), (4).

If we start for the Neumann problem (3) with the Ansatz $\mathbf{u}(x) = \mathbf{V}(x;\varphi)$ we obtain the boundary integral equation for φ

$$(A_{II}\varphi)(x_0) \equiv \varphi(x_0) + \frac{1}{2\pi} \int_S K(x_0,y)\varphi(y)\, d_yS = \mathbf{P}(x_0), \quad x_0 \in S \tag{10}$$

where

$$K(x,y) = T(\partial_x, \mathbf{n}_x)G(x,y) \quad \text{for } x \in S. \tag{11}$$

If we use for the Dirichlet problem (2) the direct boundary integral method with the Ansatz

$$\mathbf{u}(x) = -\frac{1}{2}\mathbf{W}(x;\mathbf{w}) + \frac{1}{2}\mathbf{V}(x;\mathbf{P}),$$

we obtain for the unknown stress $\mathbf{P}$ the boundary integral equation

$$(A_I\mathbf{P})(x_0) \equiv -\mathbf{P}(x_0) + \frac{1}{2\pi} \int_S K(x_0,y)\mathbf{P}(y)\, d_yS \\ = \lim_{x \to x_0} T(\partial_x, \mathbf{n}_x)\mathbf{W}(x;\mathbf{w}), \quad x_0 \in S. \tag{12}$$

In (9), (11), (12) T means $\underset{(i)}{T}$ if y or x lies on S_i. In both cases (10), (12) we have the same integral operator with the kernel (11). In order to know the structure of $K(x,y)$ we have to look at $V(x,y)$. The matrix $V(x,y)$ is only published in [7]. Therefore we write down $V(x,y)$. We have for $x_3 > 0$, $y_3 < 0$

$$V(x,y) = H_1^- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial^2 U}{\partial y_3^2} \tag{13}$$

$$+ \begin{pmatrix} H_2^- \frac{\partial^2 U}{\partial x_2^2} & -H_2^- \frac{\partial^2 U}{\partial x_1 \partial x_2} & -H_3^- \frac{\partial^2 U}{\partial x_1 \partial y_3} \\ -H_2^- \frac{\partial^2 U}{\partial x_1 \partial x_2} & H_2^- \frac{\partial^2 U}{\partial x_1^2} & -H_3^- \frac{\partial^2 U}{\partial x_2 \partial y_3} \\ H_3^- \frac{\partial^2 U}{\partial x_1 \partial y_3} & H_3^- \frac{\partial^2 U}{\partial x_2 \partial y_3} & 0 \end{pmatrix}$$

$$+(H_4^- x_3 + H_5^- y_3) \begin{pmatrix} -\frac{\partial^3 U}{\partial x_1^2 \partial y_3} & -\frac{\partial^3 U}{\partial x_1 \partial x_2 \partial y_3} & \frac{\partial^3 U}{\partial x_1 \partial y_3^2} \\ -\frac{\partial^3 U}{\partial x_1 \partial x_2 \partial y_3} & -\frac{\partial^3 U}{\partial x_2^2 \partial y_3} & \frac{\partial^3 U}{\partial x_2 \partial y_3^2} \\ \frac{\partial^3 U}{\partial x_1 \partial y_3^2} & \frac{\partial^3 U}{\partial x_2 \partial y_3^2} & -\frac{\partial^3 U}{\partial y_3^3} \end{pmatrix},$$

for $x_3 < 0,\ y_3 < 0$

$$V(x,y) = H_6^- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial^2 U}{\partial x_3^2} \tag{14}$$

$$+ \begin{pmatrix} H_7^- \frac{\partial^2 U}{\partial x_2^2} & -H_7^- \frac{\partial^2 U}{\partial x_1 \partial x_2} & -H_8^- \frac{\partial^2 U}{\partial x_1 \partial x_3} \\ -H_7^- \frac{\partial^2 U}{\partial x_1 \partial x_2} & H_7^- \frac{\partial^2 U}{\partial x_1^2} & -H_8^- \frac{\partial^2 U}{\partial x_2 \partial x_3} \\ H_8^- \frac{\partial^2 U}{\partial x_1 \partial x_3} & H_8^- \frac{\partial^2 U}{\partial x_2 \partial x_3} & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} (-H_9^- x_3 - H_{10}^- y_3) \frac{\partial^3 U}{\partial x_1^2 \partial x_3} & (-H_9^- x_3 - H_{10}^- y_3) \frac{\partial^3 U}{\partial x_1 \partial x_2 \partial x_3} & (-H_9^- x_3 + H_{10}^- y_3 \frac{\partial^3 U}{\partial x_1 \partial x_3^2}) \\ (-H_9^- x_3 - H_{10}^- y_3) \frac{\partial^3 U}{\partial x_1 \partial x_2 \partial x_3} & (-H_9^- x_3 - H_{10}^- y_3) \frac{\partial^3 U}{\partial x_2^2 \partial x_3} & (-H_9^- x_3 + H_{10}^- y_3) \frac{\partial^3 U}{\partial x_2 \partial x_3^2} \\ (-H_9^- x_3 + H_{10}^- y_3) \frac{\partial^3 U}{\partial x_1 \partial x_3^2} & (-H_9^- x_3 + H_{10}^- y_3) \frac{\partial^3 U}{\partial x_2 \partial x_3^2} & (-H_9^- x_3 - H_{10}^- y_3) \frac{\partial^3 U}{\partial x_3^3} \end{pmatrix}$$

$$+H_{11}^- x_3 y_3 \begin{pmatrix} \frac{\partial^4 U}{\partial x_1^2 \partial x_3^2} & \frac{\partial^4 U}{\partial x_1 \partial x_2 \partial x_3^2} & -\frac{\partial^4 U}{\partial x_1 \partial x_3^3} \\ \frac{\partial^4 U}{\partial x_1 \partial x_2 \partial x_3^2} & \frac{\partial^4 U}{\partial x_2^2 \partial x_3^2} & -\frac{\partial^4 U}{\partial x_2 \partial x_3^3} \\ \frac{\partial^4 U}{\partial x_1 \partial x_3^3} & \frac{\partial^4 U}{\partial x_2 \partial x_3^3} & -\frac{\partial^4 U}{\partial x_3^4} \end{pmatrix}$$

where the constants are

$$H_1^- = \frac{\mu_1(\lambda_0+3\mu_0)(\lambda_1+2\mu_1)+\mu_0(\lambda_1+3\mu_1)(\lambda_0+2\mu_0)}{[\mu_0(\lambda_1+3\mu_1)+\mu_1(\lambda_1+\mu_1)][\mu_1(\lambda_0+3\mu_0)+\mu_0(\lambda_0+\mu_0)]},$$

$$H_2^- = H_1^- - \frac{2}{\mu_0+\mu_1},$$

$$H_3^- = \frac{\mu_0^2(\lambda_1+3\mu_1)-\mu_1^2(\lambda_0+3\mu_0)}{[\mu_1(\lambda_0+3\mu_0)+\mu_0(\lambda_0+\mu_0)][\mu_0(\lambda_1+3\mu_1)+\mu_1(\lambda_1+\mu_1)]},$$

$$H_4^- = -\frac{\lambda_1+\mu_1}{\mu_0(\lambda_1+3\mu_1)+\mu_1(\lambda_1+\mu_1)},$$

$$H_5^- = \frac{\lambda_0+\mu_0}{\mu_1(\lambda_0+3\mu_0)+\mu_0(\lambda_0+\mu_0)},$$

$$H_6^- = H_1^- - \frac{\lambda_0+3\mu_0}{2\mu_0(\lambda_0+2\mu_0)},$$

$$H_7^- = H_6^- + \frac{\mu_1-\mu_0}{\mu_0(\mu_0+\mu_1)},$$

$$H_8^- = H_3^-, \quad H_9^- = \frac{(\mu_0-\mu_1)(\lambda_0+\mu_0)(\lambda_0+3\mu_0)}{2\mu_0(\lambda_0+2\mu_0)[\mu_1(\lambda_0+3\mu_0)+\mu_0(\lambda_0+\mu_0)]},$$

$$H_{10}^- = H_9^-,$$

$$H_{11}^- = \frac{(\lambda_0+\mu_0)^2(\mu_1-\mu_0)}{\mu_0(\lambda_0+2\mu_0)[\mu_1(\lambda_0+3\mu_0)+\mu_0(\lambda_0+\mu_0)]}.$$

For $x_3 < 0$, $y_3 > 0$ in (13) and for $x_3 > 0$, $y_3 > 0$ in (14) the constants H_k^- have to be replaced by H_k^+. The H_k^+ follow from H_k^- by changing the indices 0, 1 in the Lamé constants. The function U is the piecewise harmonic function

$$U(x,y) = (|x_3|+|y_3|)\ln(|x_3|+|y_3|+W) - W \tag{15}$$

where

$$W = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(|x_3|+|y_3|)^2}. \tag{16}$$

If S_0, S_1 are $C^{1,\kappa}$-regular we have from $\underset{(i)}{\Gamma}(x,y)$ for $x,y \in S_i$ in the kernel $K(x,y)$ weakly singular terms and singular terms of the form

$$\frac{\mu_i}{\lambda_i+2\mu_i}\left(n_k(x)\frac{x_j-y_j}{|x-y|^3} - n_j(x)\frac{x_k-y_k}{|x-y|^3}\right).$$

From the compensatrix $V(x,y)$ we obtain kernels with fixed singularities for $x = y$ on the interface edge ∂S_{00}. If we omit the factors $n_k(x)$ and elastic constants we get in $K(x,y)$ for $x,y \in S_1$ terms of the form

$$\frac{x_3^{l_1} y_3^{l_2} (x_1 - y_1)^{l_3'} (x_2 - y_2)^{l_4'} (x_3 + y_3)^{l_5}}{W_+^{m_1^+} (x_3 + y_3 + W_+)^{m_2}} \tag{17}$$

with

$$W_+ = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}$$

and for $x \in S_1$, $y \in S_0$ terms of the form

$$\frac{x_3^{l_1} y_3^{l_2} (x_1 - y_1)^{l_3'} (x_2 - y_2)^{l_4'} (x_3 - y_3)^{l_5}}{W_-^{m_1^-} (x_3 - y_3 + W_-)^{m_2}} \tag{18}$$

with

$$W_- = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

where $0 \le l_1 \le 1$, $0 \le l_2 \le 1$, $0 \le l_3' \le 2$, $0 \le l_4' \le 2$, $0 \le l_5 \le 3$, $0 \le m_1^+ \le 7$, $0 \le m_1^- \le 5$, $0 \le m_2 \le 3$, $m_1^\pm + m_2 = l_1 + l_2 + l_3' + l_4' + l_5 + 2$.

3. The local operators with fixed singularities

Let $a = (a_1, a_2, 0) \in \partial S_{00}, \Pi_a^+$ the upper tangential half-plane to S_1 in $a, \alpha \in (0,\pi)$ the angle between Π_a^+ and E, Π_a^- the lower tangential half-plane to S_0 in $a, \beta \in (0,\pi)$ the angle between Π_a^- and E. We consider new co-ordinates

$$\begin{aligned} x_1 &= a_1 + \alpha_{11}\xi_1 + \alpha_{12}\xi_2, & y_1 &= a_1 + \alpha_{11}\eta_1 + \alpha_{12}\eta_2, \\ x_2 &= a_2 + \alpha_{21}\xi_1 + \alpha_{22}\xi_2, & y_2 &= a_2 + \alpha_{21}\eta_1 + \alpha_{22}\eta_2, \\ x_3 &= \xi_3, & y_3 &= \eta_3, \end{aligned}$$

such that the ξ_1-axis is directed tangential to ∂S_{00}. Then the kernel (18) is a linear combination of kernels of the form

$$\frac{\xi_3^{l_1} \eta_3^{l_2} (\xi_1 - \eta_1)^{l_3} (\xi_2 - \eta_2)^{l_4} (\xi_3 - \eta_3)^{l_5}}{W_-^{m_1^-} (\xi_3 - \eta_3 + W_-)^{m_2}}$$

with

$$W_- = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2}, \quad l_3 + l_4 = l_3' + l_4'.$$

Now we consider in the half-planes Π_a^+, Π_a^- co-ordinates t_1, t_2 such that for $\xi \in \Pi_a^+$

$$\xi = (t_1, t_2 \cos\alpha, t_2 \sin\alpha)$$

and for $\eta \in \Pi_a^-$

$$\eta = (\tau_1, \tau_2 \cos\beta, -\tau_2 \sin\beta).$$

Thus, to (18) is associated the local operator

(19) $$(K_{\alpha,\beta}u)(t_1,t_2) = \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty \frac{(t_2\sin\alpha)^{l_1}(\tau_2\sin\beta)^{l_2}(t_1-\tau_1)^{l_3}(t_2\cos\alpha-\tau_2\cos\beta)^{l_4}}{W_0^{m_1}(t_2\sin\alpha+\tau_2\sin\beta+W_0)^{m_2}}\times$$
$$\times(t_2\sin\alpha+\tau_2\sin\beta)^{l_5}u(\tau_1,\tau_2)\,d\tau_1\,d\tau_2$$

with

$$W_0 = \sqrt{(t_1-\tau_1)^2+t_2^2+\tau_2^2-2t_2\tau_2\cos(\alpha+\beta)},$$
$$m_1+m_2 = l_1+l_2+l_3+l_4+l_5+2$$

and to (17) is associated the local operator $K_\alpha = K_{\alpha,\alpha}$.

The function $u(t_1,t_2)$ is defined on $\mathbf{R}^2_+ = \{\mathbf{t}=(t_1,t_2) : t_1\in(-\infty,\infty),\ t_2>0\}$. We investigate the operator $K_{\alpha,\beta}$ in the space $L_{p,\gamma}(\mathbf{R}^2_+)$ with the weight function t_2^γ. The norm in that space is given by

(20) $$\|u\|_{p,\gamma} = \Big(\iint_{\mathbf{R}^2_+}|u(t_1,t_2)|^p t_2^\gamma\,dt_1dt_2\Big)^{1/p}.$$

In $\mathbf{R}^2_+$ we define a multiplication

$$\mathbf{st} = (s_1,s_2)(t_1,t_2) = (s_2t_1+s_1, s_2t_2).$$

With that product $\mathbf{R}^2_+$ becomes a non-commutative group which we denote by $\mathbf{G}$. The group $\mathbf{G}$ is isomorphic to the group of the matrices $\begin{pmatrix}1 & 0\\ t_1 & t_2\end{pmatrix}$ with the usual matrix multiplication. The group $\mathbf{G}$ is also isomorphic to the group of affine transformations on $\mathbf{R}$ if $(t_1,t_2)\in\mathbf{G}$ defines the mapping $(t_1,t_2)(t) = t_2t+t_1$. The inverse of $\mathbf{t}=(t_1,t_2)\in\mathbf{G}$ is given by $\mathbf{t}^{-1} = (-t_1/t_2, 1/t_2)$.

Beside the space $L_{p,\gamma}(\mathbf{R}^2_+)$ we consider for functions defined on $\mathbf{G}$ the space $L_p(\mathbf{G})$ with the norm

(21) $$\|u\|_{L_p(\mathbf{G})} = \Big(\int_{\mathbf{G}}|u(\mathbf{t})|^p\,d\rho(\mathbf{t})\Big)^{1/p}$$

where $d\rho(\mathbf{t}) = t_2^{-1}dt_1dt_2$ is the right Haar measure (cf. [3]).

LEMMA 1. *The mapping $\tau_{p,\gamma}$ defined by $(\tau_{p,\gamma}u)(t_1,t_2) = t_2^{\frac{1+\gamma}{p}}u(\mathbf{t})$ is an isometric isomorphism from $L_{p,\gamma}(\mathbf{R}^2_+)$ onto $L_p(\mathbf{G})$.*

Indeed, we have

$$\|\tau_{p,\gamma}u\|^p_{L_p(\mathbf{G})} = \|t_2^{\frac{1+\gamma}{p}} u(\mathbf{t})\|^p_{L_p(\mathbf{G})} = \iint\limits_{\mathbf{R}^2_+} |t_2^{\frac{1+\gamma}{p}} u(\mathbf{t})|^p t_2^{-1}\, dt_1\, dt_2$$

$$= \iint\limits_{\mathbf{R}^2_+} |u(\mathbf{t})|^p t_2^{\gamma}\, dt_1\, dt_2 = \|u\|^p_{p,\gamma}.$$

In $L_p(\mathbf{G})$ we consider the convolution

$$(22)\qquad (u * v)(\mathbf{t}) = \int\limits_{\mathbf{G}} u(\mathbf{t}\mathbf{s}^{-1})v(\mathbf{s})\, d\rho(\mathbf{s})$$

$$= \iint\limits_{\mathbf{R}^2_+} u\left(t_1 - s_1\frac{t_2}{s_2}, \frac{t_2}{s_2}\right) v(s_1, s_2)\frac{1}{s_2}\, ds_1\, ds_2.$$

For the operator $T_v u = u * v$ generated by a function v the following Lemma is valid (cf. [3, 9, 10]).

LEMMA 2. *Let $v \in L_1(\mathbf{G})$, $p \geq 1$. Then T_v is a linear bounded operator from $L_p(\mathbf{G})$ in $L_p(\mathbf{G})$ and for the norm it holds*

$$\|T_v u\|_{L_p(\mathbf{G})} \leq \|v\|_{L_1(\mathbf{G})}\|u\|_{L_p(\mathbf{G})}.$$

If the generating function v is positive then the operator norm of T_v is

$$\|T_v\| = \|v\|_{L_1(\mathbf{G})}.$$

In order to apply Lemma 2 we write $\tau_{p,\gamma}K_{\alpha,\beta}$ as a convolution operator. Therefore we substitute in (19)

$$(23)\qquad \boldsymbol{\tau} = (\tau_1, \tau_2) = \mathbf{t}\mathbf{s}^{-1} = (t_1, t_2)(s_1, s_2)^{-1} = (t_1 - s_1 t_2 s_2^{-1}, t_2 s_2^{-1}).$$

The functional determinant has the value $\frac{\partial(\tau_1,\tau_2)}{\partial(s_1,s_2)} = t_2^2 s_2^{-3}$. Then elementary calculations yield

$$(24)\quad (\tau_{p,\gamma}K_{\alpha,\beta}u)(\mathbf{t}) =$$

$$= \frac{1}{2\pi}\iint\limits_{\mathbf{R}^2_+} \frac{t_2^{\frac{1+\gamma}{p}} (s_2\sin\alpha)^{l_1}(\sin\beta)^{l_2} s_1^{l_3}(s_2\cos\alpha - \cos\beta)^{l_4}}{A^{m_1}(s_2\sin\alpha + \sin\beta + A)^{m_2}} \times$$

$$\times (s_2\sin\alpha + \sin\beta)^{l_5} u(\mathbf{t}\mathbf{s}^{-1})\frac{1}{s_2}\, ds_1\, ds_2$$

where

$$A = \sqrt{s_1^2 + s_2^2 + 1 - 2s_2\cos(\alpha + \beta)}.$$

If we observe

$$(\tau_{p,\gamma}u)(\tau) = \left(\frac{t_2}{s_2}\right)^{\frac{1+\gamma}{p}} u(\tau)$$

then (24) can be written as convolution

$$(25) \qquad (\tau_{p,\gamma}K_{\alpha,\beta}u)(\mathbf{t}) = (\tau_{p,\gamma}u * k_{p,\gamma}^{\alpha,\beta})(\mathbf{t})$$

with the generating function

$$(26) \quad k_{p,\gamma}^{\alpha,\beta}(\mathbf{s}) = = \frac{1}{2\pi}\frac{s_1^{l_3}s_2^{\frac{1+\gamma}{p}+l_1}\sin^{l_1}\alpha\sin^{l_2}\beta(s_2\cos\alpha-\cos\beta)^{l_4}(s_2\sin\alpha+\sin\beta)^{l_5}}{A^{m_1}(s_2\sin\alpha+\sin\beta+A)^{m_2}}.$$

THEOREM 1. *Let $k_{p,\gamma}^{\alpha,\beta} \in L_1(\mathbf{G})$, $p \geq 1$. Then $K_{\alpha,\beta}$ is a linear bounded operator from $L_{p,\gamma}(\mathbf{R}_+^2)$ in $L_{p,\gamma}(\mathbf{R}_+^2)$ and for the operator norm the estimate*

$$\|K_{\alpha,\beta}\| \leq \|k_{p,\gamma}^{\alpha,\beta}\|_{L_1(\mathbf{G})}$$

holds.

Proof. Let $u \in L_{p,\gamma}(\mathbf{R}_+^2)$, then $\tau_{p,\gamma}u \in L_p(\mathbf{G})$ and $\|\tau_{p,\gamma}u\|_{L_p(\mathbf{G})} = \|u\|_{p,\gamma}$. From (25) and Lemma 2 we conclude that $\tau_{p,\gamma}K_{\alpha,\beta}u \in L_p(\mathbf{G})$. Therefore $K_{\alpha,\beta}u \in L_{p,\gamma}(\mathbf{R}_+^2)$ and

$$\|K_{\alpha,\beta}u\|_{p,\gamma} = \|\tau_{p,\gamma}K_{\alpha,\beta}u\|_{L_p(\mathbf{G})} \leq \|k_{p,\gamma}^{\alpha,\beta}\|_{L_1(\mathbf{G})}\|\tau_{p,\gamma}u\|_{L_p(\mathbf{G})} = = \|k_{p,\gamma}^{\alpha,\beta}\|_{L_1(\mathbf{G})}\|u\|_{p,\gamma}.$$

Now it remains to estimate the $L_1(\mathbf{G})$-norm of the generating function $k_{p,\gamma}^{\alpha,\beta}$. Obviously we have

$$|k_{p,\gamma}^{\alpha,\beta}(\mathbf{s})| \leq \frac{1}{2\pi}\frac{s_2^{\frac{1+\gamma}{p}}(s_2\sin\alpha)^{l_1}\sin^{l_2}\beta(s_2\cos\alpha-\cos\beta)^{l_4}(s_2\sin\alpha+\sin\beta)^{l_5}}{A^{m_1+m_2-l_3}}.$$

The inequality $0 \leq s_1^2 + (s_2\cos\alpha - \cos\beta)^2$ is equivalent to the inequality

$$(s_2\sin\alpha + \sin\beta)^2 \leq A^2,$$

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and

$$(s_2\sin\alpha)^2 \leq (s_2\sin\alpha + \sin\beta)^2 \leq A^2.$$

Since $m_1 + m_2 = l_1 + l_2 + l_3 + l_4 + l_5 + 2$ we obtain

$$(27) \qquad |k_{p,\gamma}^{\alpha,\beta}(\mathbf{s})| \leq \frac{1}{2\pi}\sin^{l_2}\beta s_2^{\frac{1+\gamma}{p}}A^{-l_2-2}.$$

Because of $0 \leq l_2 \leq 1$ we calculate the $L_1(\mathbf{G})$-norm for the two cases $l_2 = 0$ and $l_2 = 1$.

For $l_2 = 0$ we have with $\mu = \frac{1+\gamma}{p}$

$$\left\| \frac{1}{2\pi} \frac{s_2^\mu}{s_1^2 + s_2^2 + 1 - 2s_2\cos(\alpha+\beta)} \right\|_{L_1(\mathbf{G})}$$

$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{s_2^\mu}{s_1^2 + s_2^2 + 1 - 2s_2\cos(\alpha+\beta)} \frac{1}{s_2}\, ds_1\, ds_2$$

$$= \frac{1}{2} \int_0^\infty \frac{s_2^{\mu-1}}{\sqrt{1 + s_2^2 - 2s_2\cos(\alpha+\beta)}}\, ds_2$$

$$= \frac{1}{2} M\left\{ \frac{1}{\sqrt{1 + s_2^2 + 2s_2\cos(\alpha+\beta-\pi)}} \right\}(\mu)$$

$$= \frac{\pi}{2} \frac{P_{\mu-1}(\cos(\alpha+\beta-\pi))}{\sin(\pi\mu)} \quad \text{if } 0 < \mu < 1.$$

The last equation follows from the table of Mellin transforms (cf. [2], 6.2.(17)). Thus, we have a finite $L_1(\mathbf{G})$-norm for

$$0 < \frac{1+\gamma}{p} < 1. \tag{28}$$

If the weight exponent $\gamma = 0$ then this condition is fulfilled for $p > 1$. For $\gamma = 0, p = 2$ ($\mu = 1/2$) the function $P_{-1/2}$ can be expressed by the complete elliptic integral of the second kind (cf. [1], 8.13.9.)

$$P_{-1/2}(\cos(\alpha+\beta-\pi)) = \frac{2}{\pi} K\left(\sin \frac{\alpha+\beta-\pi}{2} \right).$$

For $l_2 = 1$ we calculate

$$N_1 := \left\| \frac{1}{2\pi} \frac{s_2^\mu \sin\beta}{\sqrt{s_1^2 + s_2^2 + 1 - 2s_2\cos(\alpha+\beta)}^3} \right\|_{L_1(\mathbf{G})}$$

$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{s_2^\mu \sin\beta}{\sqrt{s_1^2 + s_2^2 + 1 - 2s_2\cos(\alpha+\beta)}^3} \frac{1}{s_2}\, ds_1\, ds_2$$

$$= \frac{\sin\beta}{\pi} \int_0^\infty \frac{s_2^{\mu-1}}{1 + s_2^2 - 2s_2\cos(\alpha+\beta)}\, ds_2$$

$$= \frac{\sin\beta}{\pi} M\left\{ \frac{1}{1 + s_2^2 + 2s_2\cos(\alpha+\beta-\pi)} \right\}(\mu)$$

$$= -\sin\beta \frac{\sin[(\mu-1)(\alpha+\beta-\pi)]}{\sin(\alpha+\beta-\pi)\sin(\pi\mu)} \quad \text{if} \quad 0 < \mu < 2$$

(cf. [2], 6.2.(12)). For $\alpha + \beta - \pi = 0$ we obtain for the norm the value (cf. [2], 6.2.(6))

$$N_1 = -\sin\beta \frac{\mu - 1}{\sin(\pi\mu)}.$$

For $\gamma = 0$, $p = 2$ ($\mu = 1/2$) we have

$$N_1 = \sin\beta \frac{\sin\frac{\alpha+\beta-\pi}{2}}{\sin(\alpha+\beta-\pi)} = \frac{1}{2}\,\frac{\sin\beta}{\sin\frac{\alpha+\beta}{2}}.$$

Since the condition $0 < \mu < 2$ does not restrict the condition (28) from Theorem 1 it follows

THEOREM 2. *If $p \geq 1$, $-1 < \gamma < p-1$, then $K_{\alpha,\beta}$ is a linear bounded operator from $L_{p,\gamma}(\mathbf{R}^2_+)$ in $L_{p,\gamma}(\mathbf{R}^2_+)$.*

In connection with the bimetal problem for the heat equation in [9] the operator

$$(G_\varphi v)(t_1,t_2) = \\ = \frac{1}{2\pi}\iint_{\mathbf{R}^2_+} \frac{t_2\sin\varphi}{\sqrt{(t_1-\tau_1)^2+t_2^2+\tau_2^2-2t_2\tau_2\cos\varphi}^{\,3}} v(\tau_1,\tau_2)\,d\tau_1\,d\tau_2,$$

$0 < \varphi < \pi$, appeared. This operator can be written as

$$G_\varphi v = \frac{\sin\varphi}{\sin\frac{\varphi}{2}}\, K_{\varphi/2} v = 2\cos\frac{\varphi}{2}\, K_{\varphi/2} v$$

with

$$l_1 = 1,\ l_2 = 0,\ l_3 = 0,\ l_4 = 0,\ l_5 = 0,\ m_1 = 3,\ m_2 = 0.$$

The generating function is according to (26)

$$g_{p,\gamma} = \frac{1}{2\pi}\,\frac{\sin\varphi\, s_2^{\frac{1+\gamma}{p}+1}}{\sqrt{s_1^2+s_2^2+1-2s_2\cos\varphi}^{\,3}}.$$

Since $g_{p,\gamma}$ is positive the operator norm of G_φ is equal to the $L_1(\mathbf{G})$-norm of $g_{p,\gamma}$. From the calculation of N_1it follows $\left(\mu = \frac{1+\gamma}{p}\right)$

$$\|G_\varphi\| = \|g_{p,\gamma}\|_{L_1(\mathbf{G})} = -\sin\varphi\frac{\sin[\mu(\varphi-\pi)]}{\sin(\varphi-\pi)\sin[\pi(\mu+1)]} = \frac{\sin[\mu(\pi-\varphi)]}{\sin(\pi\mu)}$$

if and only if $0 < \mu + 1 < 2$. Thus we have in accordance with [9, 10] the result that the operator G_φ is bounded in $L_{p,\gamma}(\mathbf{R}^2_+)$ if and only if $-p-1 < \gamma < p-1$, $p \geq 1$.

For $\gamma = 0$, $p = 2$ we have

$$\|G_\varphi\| = \|g_{2,0}\|_{L_1(\mathbf{G})} = \cos\frac{\varphi}{2}.$$

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