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ON A CLASS OF SINGULAR INTEGRAL EQUATIONS

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

The aim of this paper is to investigate the integral operator $C = I + A$, here I — the identity operator, and the operator A is of the form

$$(1) \qquad (Au)(\lambda) = \int_0^\infty K(\lambda,\xi)\Psi(\lambda,\xi)u(\xi)\,d\xi,$$

where $K \in L^\infty(\mathbb{R}^2_+)$, and the function $\Psi(\lambda,\xi)$ is positively homogeneous of the degree -1, i.e.

$$\tau\Psi(\tau\lambda,\tau\xi) = \Psi(\lambda,\xi), \qquad \lambda,\xi,\tau \in \mathbb{R}_+.$$

Nonhomogeneous equation $Cu = f$ appears when solving linear boundary value problems in combined domains with irregular boundary points [7, 8]. Besides, nontrivial solutions of the homogeneous equation $Cu = 0$ make it possible to construct solutions of one class of homogeneous linear boundary value problems, which play an important role in asymptotic methods theory [10].

Integral equations with the kernels represented by positively homogeneous of degree — 1 functions were investigated by many authors [see cf. 1, 9, 11, 15]. The results were eventually based on the theory of Wiener–Hopf integral equations on a semi–axis with difference kernels [2, 3, 5, 13]. The ex-

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tensive literature on the operators with fixed point singularities is reviewed in [1, 13].

Unfortunately, those results can not be immediately applied to the equation (1). The first point is that functional spaces must be different in comparison with those presented in mentioned papers, and the second point is a necessity to investigate an influence of the function $K(\lambda,\xi)$. Moreover, taking into account the fact that the function $K(\lambda,\xi)$ in (1) can be numerically found from certain recurrence procedure, see [7, 8], we need to obtain weakest conditions on $K(\lambda,\xi)$ which allows to separate properly the operator A on singular and compact parts.

2. Notation, definitions and basic facts

We denote by $L^{p,\alpha,\beta}(\mathbb{R}_+)$ a weighted Lebesgue space of functions p-summable on $\mathbb{R}_+$ with the norm $||u||$ of $u \in L^{p,\alpha,\beta}(\mathbb{R}_+)$ given by

$$||u||^{p,\alpha,\beta} = \Big(\int\limits_0^\infty |u(\xi)|^p \rho^p_{\alpha,\beta}(\xi)\xi^{-1}\, d\xi\Big)^{\frac{1}{p}}, \quad \rho_{\alpha,\beta}(\xi) = \begin{cases} \xi^\alpha, & \xi \in (0,1), \\ \xi^\beta, & \xi \in (1,\infty), \end{cases}$$

for some $\alpha,\beta \in \mathbb{R}$, $p \in [1,\infty)$. We shall also consider a space $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ of functions, having distributional derivatives $u^{(j)} \in L^{p,\alpha+j,\beta+j}(\mathbb{R}_+)$ up to the order l, with the norm

$$||u||^{p,\alpha,\beta}_{(l)} = \sum_{j=0}^{l} ||u^{(j)}||^{p,\alpha+j,\beta+j}.$$

These spaces are natural for the investigation of the operator A, because the reduction of the boundary value problems to integral equations $Cu = f$ has been justified [7, 8] in these spaces only.

Let us note that the space $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ is not the usual Sobolev space [6]. Functions from $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ and its distributional derivatives have interconnected behaviour in the neighbourhood of zero and infinity. It is evident that for any $h > 0$, $l \in N$: we have $L^{p,\frac{1}{p},\frac{1}{p}}(\mathbb{R}_+) = L^p(\mathbb{R}_+)$, $L^{p,\alpha-h,\beta+h}(\mathbb{R}_+) \subset L^{p,\alpha,\beta}(\mathbb{R}_+)$, $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+) \subset L^{p,\alpha,\beta}(\mathbb{R}_+)$, though the embeddings are not compact. Here the spaces $L^\infty(\Omega)$, $L^p(\Omega)$, ($p \in [1,\infty)$) are the usual Banach spaces with respective norms.

By $P_a : X \to X$ we shall understand the operator of multiplication by the characteristic function of a set $(0,a)$; $Q_a = I - P_a$ is the complementary projector to it in X. Here $X(= X(\mathbb{R}_+))$ is any of the defined above spaces.

We shall distinguish the following sets from the space $L^\infty(\mathbb{R}^2_+)$:

a. $\mathcal{Z}$ — the closure in $L^\infty(\mathbb{R}^2_+)$ of the set of functions in form

$$(4)\qquad \sum_{i=1}^{N} f_{1i}(\lambda) f_{2i}(\xi) f_{3i}\left(\frac{\lambda}{\xi}\right), \qquad f_{ji} \in L^\infty(\mathbb{R}_+);$$

b. $\mathcal{Z}_\delta \subset \mathcal{Z}$ $(\delta \geq 0)$ — the set of functions $f(\lambda, \xi)$, satisfying the additional conditions

$$(5)\qquad \limsup_{\lambda,\xi\to 0} (\lambda^2+\xi^2)^{-\frac{\delta}{2}} |f(\lambda,\xi)| = \limsup_{\lambda,\xi\to\infty} (\lambda^2+\xi^2)^{\frac{\delta}{2}} |f(\lambda,\xi)| = 0;$$

c. $\mathcal{Z}^*_\delta \subset \mathcal{Z}$ — the set of functions $f(\lambda,\xi)$, for which one can find $f_1, f_2 \in L^\infty(\mathbb{R}_+)$, $f_\delta \in \mathcal{Z}_\delta$ such that

$$(6)\qquad f(\lambda,\xi) = P_1(\lambda)P_1(\xi)f_1\left(\frac{\lambda}{\xi}\right) + Q_1(\lambda)Q_1(\xi)f_2\left(\frac{\lambda}{\xi}\right) + f_\delta(\lambda,\xi).$$

Set $\mathcal{Z}^*_\delta$ (as well as $\mathcal{Z}$) is not dense, of course, in $L^\infty(\mathbb{R}^2_+)$, however it is "sufficiently rich", for example $C^{k-1}(\overline{\mathbb{R}^2_+}) \subset \mathcal{Z}$ for any natural k.

Let us consider the function

$$w_{\alpha,\beta}(\lambda,\xi) = \rho_{\alpha,\beta}(\lambda)\left[\rho_{\alpha,\beta}(\xi)\rho_{\alpha,\beta}\left(\frac{\lambda}{\xi}\right)\right]^{-1}.$$

The following equality can be verified in a straightforward way

$$w_{\alpha,\beta}(\lambda,\xi) = P_1(\lambda)P_1(\xi)\rho_{0,\alpha-\beta}\left(\frac{\lambda}{\xi}\right) + Q_1(\lambda)Q_1(\lambda)\rho_{\beta-\alpha,0}\left(\frac{\lambda}{\xi}\right) + \\ + [P_1(\lambda)Q_1(\xi) + Q_1(\lambda)P_1(\xi)]\,\rho_{\beta-\alpha,\alpha-\beta}(\xi).$$

Consequently the function $w_{\alpha,\beta}(\lambda,\xi)$ depends on a difference of the parameters α,β only, so we shall write

$$w_{\alpha,\beta}(\lambda,\xi) = w_{\beta-\alpha}(\lambda,\xi).$$

It is easy to see that for any $a, b \in \mathbb{R}$, $c, \delta \in \mathbb{R}_+$

$$(7)\qquad w_a \cdot w_b = w_{a+b}, \qquad w_c \in \mathcal{Z}^*_\delta, \qquad \|w_c\|_\infty = 1.$$

Let us consider the integral operator A, determined in (1). Throughout the paper, we shall assume that $K \in L^\infty(\mathbb{R}^2_+)$, the function $\Psi(\lambda,\xi)$ is positively homogeneous of the degree -1 (see (2)) and satisfies the condition

$$(8)\qquad \Psi(\cdot,1) \in L^{1,\alpha,\beta}(\mathbb{R}_+),$$

unless otherwise is specified. By α_*, β^* we denote *inf* and *sup*, respectively, over α and β for which condition (8) is true.

R e m a r k 1. It is clear that the representation of the kernel of the integral operator A as multiplication of two functions K, Ψ is not unique.

In the fact, let a function $h \in L^\infty(\mathbb{R}_+)$ satisfy the condition $\inf_{R_+} |h| > 0$, then functions

$$K^*(\lambda,\xi) = K(\lambda,\xi)\cdot h^{\pm 1}(\lambda/\xi), \qquad \Psi^*(\lambda,\xi) = \Psi(\lambda,\xi)\cdot h^{\mp 1}(\lambda/\xi),$$

meet the conditions like as functions $K(\lambda,\xi)$, $\Psi(\lambda,\xi)$.

LEMMA 1. *Let $w_b K \in L^\infty(\mathbb{R}^2_+)$ for some $b \in \mathbb{R}$ and $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta < \beta^*$, $\beta - \alpha \geq b$, then the operator A is bounded in $L^{p,\alpha,\beta}(\mathbb{R}_+)$.*

Proof. We present the operator in the form

$$(9) \quad (Au)(\lambda) = \\ = \rho^{-1}_{\alpha,\beta}(\lambda) \int_0^\infty w_{\beta-\alpha-b}\cdot w_b K(\lambda,\xi)\cdot\Psi(\lambda,\xi)\rho_{\alpha,\beta}(\lambda/\xi)\cdot u(\xi)\rho_{\alpha,\beta}(\xi)\, d\xi.$$

Taking into account (7) and Lemma's conditions, it is sufficient to prove that the operator

$$(10) \qquad (A_* u)(\lambda) = \rho^{-1}_{\alpha,\beta}(\lambda) \int_0^\infty \Psi(\lambda,\xi)\rho_{\alpha,\beta}(\lambda/\xi)\cdot u(\xi)\rho_{\alpha,\beta}(\xi)\, d\xi,$$

is bounded. Changing arguments $\lambda = \exp(-\lambda_1)$, $\xi = \exp(-\xi_1)$, and denoting

$$(11) \qquad v(\lambda_1) = \rho_{\alpha,\beta}(e^{-\lambda_1})u(e^{-\lambda_1}), \qquad \Phi(\lambda_1) = \Psi(e^{-\lambda_1},1)\rho_{\alpha,\beta}(e^{-\lambda_1}),$$

we obtain the operator $\mathcal{A}_* : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defined as follows

$$(\mathcal{A}_* v)(\lambda_1) = \int_{-\infty}^{\infty} \Phi(\xi_1 - \lambda_1)v(\xi_1)\, d\xi_1.$$

The last operator is consequently represented by the form $\mathcal{A}_* = UA_*U^{-1}$, where by $U : L^{p,\alpha,\beta}(\mathbb{R}_+) \to L^p(\mathbb{R})$, $U^{-1} : L^p(\mathbb{R}) \to L^{p,\alpha,\beta}(\mathbb{R}_+)$ we have denoted the operators:

$$[Uu](\lambda_1) = \rho_{\alpha,\beta}(e^{-\lambda_1})u(e^{-\lambda_1}), \qquad \lambda_1 \in \mathbb{R};$$
$$[U^{-1}v](\lambda) = \rho^{-1}_{\alpha,\beta}(\lambda)v(-\ln\lambda), \qquad \lambda \in \mathbb{R}_+.$$

Because of $UU^{-1} = I$ and $\|Uu\|_{L^p(\mathbb{R})} = \|u\|_{L^{p,\alpha,\beta}(\mathbb{R}_+)}$, the operators U and U^{-1} are isometrical ones. Hence, the operator $A_* : L^{p,\alpha,\beta}(\mathbb{R}_+) \to L^{p,\alpha,\beta}(\mathbb{R}_+)$ is by definition isometrically equivalent to the operator $\mathcal{A}_* = UA_*U^{-1} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. It remains to notice that $\Phi \in L^1(\mathbb{R})$ (see (8), (10)), and the boundedness of operators in form $\mathcal{A}_*$ has been proved in [16] (see also [4, 12]).

Observe that in the conditions of Lemma 1 instead of $K \in L^\infty(\mathbb{R}^2_+)$ it is assumed a more precise inclusion $w_b K \in L^\infty(\mathbb{R}^2_+)$. Obviously in a trivial case $b \geq 0$ the second point follows from the first one. But in the case $b < 0$

the parameter b makes it possible to correct for the function Ψ the relation between the space parameters α, β, defined (8).

LEMMA 2. *Let $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta^* > \beta$, $b \in \mathbb{R}$, $\beta - \alpha \geq b$, $l \in N$, and the stronger conditions in comparison with Lemma 1 be satisfied: $w_b\lambda^s \frac{\partial^s}{\partial\lambda^s}K \in L^\infty(\mathbb{R}^2_+)$ for all $s = 0,1,2,\dots,l$, and $\Psi(\cdot,1) \in W^{1,\alpha,\beta}_{(l)}(\mathbb{R}_+)$. Then the operator $A : L^{p,\alpha,\beta}(\mathbb{R}_+) \to W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ is bounded.* (Here the derivatives of the function K is the distributional sense.)

Proof. Without loss of generality we shall prove Lemma 2 in the case $l = 1$. For this aim it is sufficient to show that the operator $\lambda\frac{\partial}{\partial\lambda}A : L^{p,\alpha,\beta}(\mathbb{R}_+) \to L^{p,\alpha,\beta}(\mathbb{R}_+)$ is bounded. Let us note that the distributional derivative of the function Au is in the form

$$\frac{\partial}{\partial\lambda}(Au)(\lambda) = \int_0^\infty \frac{\partial}{\partial\lambda}K(\lambda,\xi)\Psi(\lambda,\xi)u(\xi)\,d\xi + \int_0^\infty \frac{1}{\xi^2}K(\lambda,\xi)\Psi'\left(\frac{\lambda}{\xi},1\right)u(\xi)\,d\xi,$$

where $\Psi'(\cdot,1)$ is the distributional derivative of the function $\Psi(\cdot,1)$ with respect to the first argument. Further, we can proceed as follows

$$\lambda\frac{\partial}{\partial\lambda}(Au)(\lambda) =$$
$$= \frac{1}{\rho_{\alpha,\beta}(\lambda)}\int_0^\infty w_{\beta-\alpha-b}\cdot w_b\lambda\frac{\partial}{\partial\lambda}K\cdot\Psi(\lambda,\xi)\rho_{\alpha,\beta}(\lambda/\xi)\cdot u(\xi)\rho_{\alpha,\beta}(\xi)d\xi +$$
$$+ \frac{1}{\rho_{\alpha,\beta}(\lambda)}\int_0^\infty w_{\beta-\alpha-b}\cdot w_bK(\lambda,\xi)\cdot\Psi'(\lambda/\xi,1)\rho_{\alpha+1,\beta+1}(\lambda/\xi)\cdot u(\xi)\rho_{\alpha,\beta}(\xi)\frac{d\xi}{\xi}.$$

Now the boundedness of the operator can be verified in a similar way as in Lemma 1.

LEMMA 3. *Let $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta < \beta^*$, $b \in \mathbb{R}$, $\beta - \alpha > b$, $\delta > 0$ and the function $w_bK \in L^\infty(\mathbb{R}_+)$ satisfies the conditions* (5). *Then the operator $A : L^{p,\alpha+h,\beta-h}(\mathbb{R}_+) \to L^{p,\alpha,\beta}(\mathbb{R}_+)$ is bounded for any $0 < 2h < \min\{2\delta, \beta-\alpha-b\}$.*

Proof. Represent the operator A in the form

$$(Au)(\lambda) =$$
$$= \frac{1}{\rho_{\alpha,\beta}(\lambda)}\int_0^\infty w_{\beta-\alpha-b-2h}\cdot w_bK\gamma_h(\lambda,\xi)\cdot\Psi(\lambda,\xi)\rho_{\alpha,\beta}(\lambda/\xi)u(\xi)\rho_{\alpha+h,\beta-h}(\xi)\,d\xi,$$

where $\gamma_h(\lambda,\xi) = \rho^{-1}_{h,-h}(\lambda)\rho_{h,-h}(\lambda/\xi)$. So, in order to prove Lemma 3 it is sufficient to show that $w_bK\gamma_h \in L^\infty(\mathbb{R}^2_+)$. First of all note that since

$w_b K \in L^\infty(\mathbb{R}^2_+)$ and

$$\gamma_h(\lambda,\xi) = \begin{cases} \xi^h, & 1<\lambda<\infty,\ 0<\xi<1, \\ \xi^{-h}, & 0<\lambda<1,\ 1<\xi<\infty, \\ \lambda^{-h}\rho_{h,-h}(\lambda/\xi), & 0<\lambda,\ \xi<1, \\ \lambda^{h}\rho_{h,-h}(\lambda/\xi), & 1<\lambda,\ \xi<\infty, \end{cases}$$

it follows immediately, that $P_1(\lambda)Q_1(\xi)[\gamma_h w_b K]$, $Q_1(\lambda)P_1(\xi)[\gamma_h w_b K] \in L^\infty(\mathbb{R}^2_+)$. Consequently we need to investigate the functions $w_b K\gamma_h$ near the zero and infinity points. Consider the function $w_b K\gamma_h$ in the neighbourhood of zero ($0<\lambda$, $\xi<1$). (The other point can be handled in an analogous way.) Taking into account the condition (5) and denoting $t=\lambda/\xi$, we obtain:

$$\begin{aligned}
&\|P_1(\lambda)P_1(\xi)w_b K\gamma_h\|_\infty = \\
&\quad = \|P_1(\lambda)P_1(\xi)w_b K(\lambda^2+\xi^2)^{-\frac{\delta}{2}} \cdot P_1(\lambda)P_1(\xi)\gamma_h(\lambda^2+\xi^2)^{\frac{\delta}{2}}\|_\infty \le \\
&\quad \le C\max\{\|P_1(\lambda)P_1(\xi)P_1(t)\gamma_h(\lambda^2+\xi^2)^{\frac{\delta}{2}}\|_\infty, \\
&\qquad \|P_1(\lambda)P_1(\xi)Q_1(t)\gamma_h(\lambda^2+\xi^2)^{\frac{\delta}{2}}\|_\infty\} \le \\
&\quad \le C\max\{\|P_1(\xi)P_1(t)\xi^{\delta-h}(1+(t)^2)^{\frac{\delta}{2}}\|_\infty, \\
&\qquad \|P_1(\lambda)Q_1(t)\lambda^{\delta-h}t^{-h}(1+t^{-2})^{\frac{\delta}{2}}\|_\infty\} \le 2^{\frac{\delta}{2}}C.
\end{aligned}$$

This completes the proof.

Remark 2. If we assume stronger conditions in the Lemma 3, namely: (i) $\Psi(\cdot,1)\in W_1^{1,\alpha,\beta}(\mathbb{R}_+)$, (ii) the functions $w_b\lambda^s\frac{\partial^s}{\partial\lambda^s}K \in L^\infty(\mathbb{R}^2_+)$ $(s=0,1)$ satisfy (5), then it can be proved that the statement of Lemma 3 is true for the space $W_1^{p,\alpha,\beta}$, instead of $L^{p,\alpha,\beta}$.

To show this fact and the compactness of the inclusion $W_1^{p,\alpha,\beta}$ into $L^{p,\alpha+h,\beta-h}$ is a way to prove the following statement: *Under the conditions* (i)–(ii) *then A is a compact operator in $L^{p,\alpha+h,\beta+h}(\mathbb{R}_+)$.*

We do not prove Remark 2 but propose below (see Lemma 6) weaker conditions for the functions K,Ψ, which are sufficient for the compactness of the operator A. Namely, we shall not assume any properties of the distributional derivatives of the functions $\Psi, w_b K$, as it is in the points (i)–(ii) of Remark 2, but the conditions (5) for $w_b K$ with $\delta=0$.

The sets $\mathcal{Z},\mathcal{Z}_\delta$ play an important role. The following two Lemmas for the operator A_*, determined in (10), show that.

LEMMA 4. *Let $p\in[1,\infty)$, $\alpha>\alpha_*$, $\beta<\beta^*$, $a>0$, then the operators $P_aA_*Q_a$, $Q_aA_*P_a$ are compact ones in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$.*

Proof. Changing arguments similarly as in (11), we obtain that the operators $P_aA_*Q_a$, $Q_aA_*P_a$ are isometrically equivalent to integral operators in space $L^p(\mathbb{R}_+)$ with kernels (from $L^1(\mathbb{R})$) depending on the sum of the arguments. The compactness of such operators was proved in [3] (see also [2, 13]).

LEMMA 5. *Let* $0 < r < R < \infty$, $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta < \beta^*$. *Then the operator* $P_R Q_r A_* P_R Q_r$ *is compact in* $L^{p,\alpha,\beta}(\mathbb{R}_+)$.

Proof. First of all let us note that the operator $P_R Q_r A_* P_R Q_r$ acting in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$ is isometrically equivalent to the operator $A^*_{r,R}$ in $L^p(r,R)$,

$$(A^*_{r,R}u)(\lambda) = \int_r^R (\lambda/\xi)^{1/q}\chi(\lambda/\xi)u(\xi)d\xi/\xi, \qquad \chi(t) = \Psi(t,1)\rho_{\alpha,\beta}(t)/t,$$

here $\chi \in L^1(\mathbb{R}_+)$, $1/p + 1/q = 1$. Denote by $\mathcal{N}$ the unit ball in $L^p(r,R)$ and let $\mathcal{M} = A^*_{r,R}\mathcal{N} \in L^p(r,R)$. To prove the Lemma 5 it is sufficient to show equicontinuity of the set $\mathcal{M}$. The boundedness of $\mathcal{M}$ follows from Lemma 1.

First of all consider the case $p = 1$

$$\sup_{|z|\le h}\sup_{v\in\mathcal{M}} \|P_R Q_\delta(\lambda+z)v(\lambda+z) - v(\lambda)\|_1 \le$$

$$\le \sup_{u\in\mathcal{N}}\|u\|_1 \cdot \sup_{|z|\le h}\sup_{\xi\in<\delta,R>} \int_\delta^R \left|P_R Q_\delta\Big(\lambda+z\Big)\chi\Big(\frac{\lambda+z}{\xi}\Big) - \chi\Big(\frac{\lambda}{\xi}\Big)\right|\frac{d\lambda}{\xi} \le$$

$$\le \sup_{o\le z\le h}\sup_{\xi\in<\delta,R>}\left\{\int_\delta^{R-z}\left|\chi\Big(\frac{\lambda+z}{\xi}\Big) - \chi\Big(\frac{\lambda}{\xi}\Big)\right|\frac{d\lambda}{\xi} + \int_{R-z}^R\left|\chi\Big(\frac{\lambda}{\xi}\Big)\right|\frac{d\lambda}{\xi}\right\} +$$

$$+ \sup_{-h\le z\le 0}\sup_{\xi\in<\delta,R>}\left\{\int_{\delta-z}^R\left|\chi\Big(\frac{\lambda+z}{\xi}\Big) - \chi\Big(\frac{\lambda}{\xi}\Big)\right|\frac{d\lambda}{\xi} + \int_\delta^{\delta-z}\left|\chi\Big(\frac{\lambda}{\xi}\Big)\right|\frac{d\lambda}{\xi}\right\} \le$$

$$\le \sup_{o\le z\le h}\sup_{\xi\in<\delta,R>}\left\{2\int_0^\infty\left|\chi\Big(t+\frac{z}{\xi}\Big) - \chi(t)\right|dt + \Big(\int_{\delta/\xi}^{\delta/\xi+z/\xi} + \int_{R/\xi-z/\xi}^{R/\xi}\Big)|\chi(t)|\,dt\right\} =$$

$$= \mathcal{I}(h).$$

Using the continuity in mean of functions $\chi \in L^1(\mathbb{R}_+)$ and uniform continuity of their primitives, it can be obtained that $\mathcal{I}(h) \to 0$ when $h \to 0$, q.e.d.

Now assume that $p > 1$ and estimate the term into respective norm

$$|P_R Q_\delta(\lambda+z)v(\lambda+z) - v(\lambda)| \le$$

$$\leq P_RQ_\delta(\lambda+z)\int\limits_\delta^R\left|\left(\frac{\lambda+z}{\xi}\right)^{1/q}-\left(\frac{\lambda}{\xi}\right)^{1/q}\right|\left|\chi\left(\frac{\lambda+z}{\xi}\right)u(\xi)\right|\frac{d\xi}{\xi}+$$

$$+\int\limits_\delta^R\left(\frac{\lambda}{\xi}\right)^{1/q}\left|P_RQ_\delta(\lambda+z)\chi\left(\frac{\lambda+z}{\xi}\right)-\chi\left(\frac{\lambda}{\xi}\right)\right|\left|u(\xi)\right|\frac{d\xi}{\xi}\leq$$

further applying Hölder inequality to each integrals we obtain

$$\leq P_RQ_\delta(\lambda+z)\left|1-\left(\frac{\lambda}{\lambda+z}\right)^{1/q}\right|\left(\int\limits_\delta^R\frac{\lambda+z}{\xi}\left|\chi\left(\frac{\lambda+z}{\xi}\right)\right|\frac{d\xi}{\xi}\right)^{1/q}\times$$

$$\times\left(\int\limits_\delta^R\left|\chi\left(\frac{\lambda+z}{\xi}\right)\right||u(\xi)|^p\frac{d\xi}{\xi}\right)^{1/p}+$$

$$+\left(\int\limits_\delta^R\frac{\lambda}{\xi}\left|P_RQ_\delta(\lambda+z)\chi\left(\frac{\lambda+z}{\xi}\right)-\chi\left(\frac{\lambda}{\xi}\right)\right|\frac{d\xi}{\xi}\right)^{1/q}\times$$

$$\times\left(\int\limits_\delta^R\left|P_RQ_\delta(\lambda+z)\chi\left(\frac{\lambda+z}{\xi}\right)-\chi\left(\frac{\lambda}{\xi}\right)\right||u(\xi)|^p\frac{d\xi}{\xi}\right)^{1/p}\leq$$

$$\leq P_RQ_\delta(\lambda+z)\left|1-\left(\frac{\lambda}{\lambda+z}\right)^{1/q}\right|\|\chi\|_1^{1/q}\left(\int\limits_\delta^R\left|\chi\left(\frac{\lambda+z}{\xi}\right)\right||u(\xi)|^p\frac{d\xi}{\xi}\right)^{1/p}+$$

$$+P_RQ_\delta(\lambda+z)\left(\frac{2\lambda+z}{\lambda+z}\|\chi\|_1\right)^{1/q}\times$$

$$\times\left(\int\limits_\delta^R\left|P_RQ_\delta(\lambda+z)\chi\left(\frac{\lambda+z}{\xi}\right)-\chi\left(\frac{\lambda}{\xi}\right)\right||u(\xi)|^p\frac{d\xi}{\xi}\right)^{1/p}.$$

Now we can estimate the expression

$$\sup_{|z|\leq h}\sup_{v\in\mathcal{M}}\|P_RQ_\delta(\lambda+z)v(\lambda+z)-v(\lambda)\|_p\leq$$

$$\sup_{u\in\mathcal{N}}\|u\|_p\|\chi\|_1^{1/q}\sup_{|z|\leq h}\sup_{\lambda\in<\delta,R>}\left\{P_RQ_\delta(\lambda+z)\left|1-\left(\frac{\lambda}{\lambda+z}\right)^{1/q}\right|\times\right.$$

$$\times\left(\int\limits_\delta^R P_RQ_\delta(\lambda+z)\left|\chi\left(\frac{\lambda+z}{\xi}\right)\right|\frac{d\lambda}{\xi}\right)^{1/p}+P_RQ_\delta(\lambda+z)\left(\frac{2\lambda+z}{\lambda+z}\right)^{1/q}\times$$

$$\left.\times\left(\int\limits_\delta^R\left|P_RQ_\delta(\lambda+z)\chi\left(\frac{\lambda+z}{\xi}\right)-\chi\left(\frac{\lambda}{\xi}\right)\right|\frac{d\lambda}{\xi}\right)^{1/p}\right\}.$$

In view of the fact that the second integral has been considered, the net result holds

$$\sup_{|z|\le h}\sup_{v\in\mathcal{M}} \|P_R Q_\delta(\lambda+z)v(\lambda+z)-v(\lambda)\|_p \le$$
$$\mathrm{Const}(\delta,R)\left\{\frac{1}{q}\|\chi\|_1 h + (\|\chi\|_1\mathcal{I}(h))^{1/p}\right\},$$

which proves the statement of Lemma 5.

LEMMA 6. *Let* $p\in[1,\infty)$, $\alpha>\alpha_*$, $\beta<\beta^*$, $\beta-\alpha\ge b$, *and* $w_bK\in\mathcal{Z}_0$, *then the operator* A *is compact in* $L^{p,\alpha,\beta}(\mathbb{R}_+)$.

Proof. Fix $\varepsilon>0$. From the definition (4), (5) of the set $\mathcal{Z}_0$ it can be found certain $r,R\in\mathbb{R}_+$, and integer N, such that the following inequalities:

$$\begin{aligned}&\|\Psi(\cdot,1)\|^{1,\alpha,\beta}\|P_r(\lambda)P_r(\xi)w_bK\|_\infty<\varepsilon/3;\\ &\|\Psi(\cdot,1)\|^{1,\alpha,\beta}\|Q_R(\lambda)Q_R(\xi)w_bK\|_\infty<\varepsilon/3;\\ &\|\Psi(\cdot,1)\|^{1,\alpha,\beta}\|w_bK\ -K_N\|_\infty<\varepsilon/3;\end{aligned}\tag{12}$$

hold. Here the function K_N is determined in (4). By A_N denote the operator with the kernel function K_N instead of w_bK (see (9)).

Let us consider the operator $B=A_N-P_rA_NP_r-Q_RA_NQ_R$. It is evident from (12) that $\|A-B\|^{p,\alpha,\beta}<\varepsilon$. On the other hand the operator B can be rewritten in the form $B=P_RQ_rA_NQ_rP_R+Q_rA_NP_r+Q_RA_NP_RQ_r+Q_rP_RA_NQ_R+P_rA_NQ_r$. Further note that the function $w_{\beta-\alpha-b}K_N$ in the kernel of the operator A_N (9) appears as K_N in the form (4) also. It means that A_N is the sum of the operators, determined by the composition of bounded operators and an operator similarly as A_*. This fact together with Lemma 4 and Lemma 5 shows that the operator B (hence and A) is compact in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$.

3. Main result

Now assume that $w_bK\in\mathcal{Z}_\delta^*$. Then there exists functions K_1,K_2 and a function K_δ such that $w_bK_\delta\in\mathcal{Z}_\delta$, $\rho_{0,-b}K_1,\rho_{b,0}K_2\in L^\infty(\mathbb{R}_+)$, and the following equality (6) is true

$$K(\lambda,\xi)=P_1(\lambda)P_1(\xi)K_1(\lambda/\xi)+Q_1(\lambda)Q_1(\xi)K_2(\lambda/\xi)+K_\delta(\lambda,\xi).\tag{13}$$

Denote by

$$\Psi_1(t)=K_1(t)\Psi(t,1),\qquad \Psi_2(t)=K_2(t)\Psi(t,1).\tag{14}$$

Since $\Psi(\cdot,1)\in L^{1,\alpha,\beta}(\mathbb{R}_+)$ (as we have assumed everywhere in the paper), then

$$\Psi_1\in L^{1,\alpha,\beta-b}(\mathbb{R}_+),\qquad \Psi_2\in L^{1,\alpha+b,\beta}(\mathbb{R}_+).\tag{15}$$

Remark 3. In spite of the fact that the kernel of the integral operator A is not uniquely represented as the product of two functions K,Ψ (Remark 1), it is clear that the functions Ψ_1,Ψ_2 in (14) are uniquely defined.

Let us determine in the space $L^p(\mathbb{R})$ operators $\mathcal{C}_1, \mathcal{C}_2$

$$(\mathcal{C}_1 v)(t) = v(t) + \int_{-\infty}^{\infty} \Phi_1(t-s)v(s)\,ds,$$

$$(\mathcal{C}_2 v)(t) = v(t) + \int_{-\infty}^{\infty} \Phi_2(t-s)v(s)\,ds,$$

where $\Phi_1(t) = \Psi_1(e^{-t})e^{-\alpha t}$, $\Phi_2(t) = \Psi_2(e^{-t})e^{-\beta t}$.

By the projector P from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ we shall below understand the operator of multiplication by the characteristic function of the set $\mathbb{R}_+ \subset \mathbb{R}$. Then Q shall be the complementary projector to it in $L^p(\mathbb{R})$.

TEOREM. *Let $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta^* > \beta$, $\beta - \alpha \geq b$, and $w_b K \in \mathcal{Z}_0^*$. Then the operator C in $L^{p,\alpha,\beta}(\mathbb{R}_+)$ is isometrically equivalent with an accuracy of a compact one to the pair operator $P\mathcal{C}_1 + Q\mathcal{C}_2(\mathcal{C}_1 P + \mathcal{C}_2 Q)$ in $L^p(\mathbb{R})$.*

Proof. Using (13) and evident identities:

$$P_1(\lambda)P_1(\xi)K_1(\lambda/\xi)\rho_{\alpha,\beta}(\lambda)\rho_{\alpha,\beta}^{-1}(\xi) = P_1(\lambda)P_1(\xi)K_1(\lambda/\xi)(\lambda/\xi)^{\alpha},$$

$$Q_1(\lambda)Q_1(\xi)K_2(\lambda/\xi)\rho_{\alpha,\beta}(\lambda)\rho_{\alpha,\beta}^{-1}(\xi) = Q_1(\lambda)Q_1(\xi)K_2(\lambda/\xi)(\lambda/\xi)^{\beta},$$

we can reduce the operator C to the form

$$C = I + P_1 A_1 P_1 + Q_1 A_2 Q_1 + T_0, \tag{16}$$

where

$$(A_1 u)(\lambda) = \rho_{\alpha,\beta}^{-1}(\lambda) \int_0^{\infty} \Psi_1(\lambda/\xi)(\lambda/\xi)^{\alpha}\rho_{\alpha,\beta}(\xi)u(\xi)\,d\xi/\xi,$$

$$(A_2 u)(\lambda) = \rho_{\alpha,\beta}^{-1}(\lambda) \int_0^{\infty} \Psi_2(\lambda/\xi)(\lambda/\xi)^{\beta}\rho_{\alpha,\beta}(\xi)u(\xi)\,d\xi/\xi,$$

$$(T_0 u)(\lambda) = \int_0^{\infty} K_0(\lambda,\xi)\Psi(\lambda,\xi)u(\xi)\,d\xi.$$

The operator T_0 is compact according to Lemma 6. The compactness of the operators $Q_1 A_i P_1$, $P_1 A_i Q_1$ $(i = 1,2)$ follows from similar arguments as in Lemma 4 and Lemma 6. So the operator C is represented of the form: $C = P_1(I + A_1) + Q_1(I + A_2) + T_*$, where T_* is the compact operator. Changing arguments similarly as in (11) we obtain the required conclusion.

This Theorem makes possible to use the results from [2, 3, 13] to investigate the operator C. Specifically, the symbol [3, 13] of the pair operator $P\mathcal{C}_1 + Q\mathcal{C}_2$ in $L^p(\mathbb{R})$ is identical to the symbol $c(\lambda,\theta)$ of the operator C in $L^{p,\alpha,\beta}(\mathbb{R}_+)$. Hence

$$c(\lambda,\theta) = c_1(\lambda)\frac{1+\theta}{2} + c_2(\lambda)\frac{1-\theta}{2}, \qquad \lambda \in \mathbb{R},\ \theta = \pm 1.$$

Here $c_1(\lambda) = 1 + \widetilde{\Psi}_1(\alpha - i\lambda)$, $c_2(\lambda) = 1 + \widetilde{\Psi}_2(\beta - i\lambda)$, and $\widetilde{u}(s)$ is Mellin transform of a function $u(\lambda)$:

$$\widetilde{u}(s) = \int_0^\infty u(\lambda)\lambda^{s-1}d\lambda.$$

Let us note that from (15) it follows that $c_1(\lambda), c_2(\lambda)$ are analytical functions in layers $0 \le \Im\lambda \le \beta - \alpha - b$, $-\beta + \alpha + b \le \Im\lambda \le 0$ respectively (or more precisely $\alpha_* - \alpha < \Im\lambda < \beta^* - \alpha - b$, $-\beta + \alpha_* + b < \Im\lambda < \beta^* - \beta$.

COROLLARY 1. *Under the conditions of Theorem be satisfied and* $c(\lambda,\theta) \ne 0$ *for all* $\lambda \in \mathbb{R}$, $\theta = \pm 1$, $C : L^{p,\alpha,\beta}(\mathbb{R}_+) \to L^{p,\alpha,\beta}(\mathbb{R}_+)$ *is a* Φ *operator* [14], *and its index can be determined by the formula* (*Theorems* 3.2, 3.3 *section* 7 *from* [2]):

$$\kappa = \operatorname{ind} C = \operatorname{ind} c_2(\lambda) - \operatorname{ind} c_1(\lambda).$$

COROLLARY 2. *Assume the hypotheses of Theorem, and besides:*

1^0. $c(\lambda,\theta) \ne 0$, $\lambda \in \mathbb{R}$, $\theta = \pm 1$,
2^0. $\kappa = \operatorname{ind} C = 0$,
3^0. $\dim\ker C = 0$ *in space* $L^{p,\alpha,\beta}(\mathbb{R}_+)$. *Then:*

(i) *the equation* $Cu = f$ *has a unique solution* $u \in L^{p,\alpha,\beta}(\mathbb{R}_+)$ *for any* $f \in L^{p,\alpha,\beta}(\mathbb{R}_+)$,
(ii) *"truncated" equation:* $P_RQ_{1/R}CP_RQ_{1/R}u_R = P_RQ_{1/R}f$ *has a unique solution* $u_R \in L^p(R^{-1}, R)$, *beginning from a certain* $R > 0$. *After extending the function* u_R *on the entire axis* ($\widetilde{u}_R$) *by zero, it converges for* $R \to \infty$ *to the solution of the equation* $Cu = f$ *with respect to norm of the space* $L^{p,\alpha,\beta}(\mathbb{R}_+)$.

Proof of Corollary 2 follows immediately from Theorem and Theorem 5.1 section 11 [2], using the stability of the index [14] and the projective process with respect to the perturbation on a compact operator (Theorem 3.1 section II [2]). Let us here note that the "truncated" equation is not a singular equation but it is Fredholm's type one. Furthermore, it is easy to show, using the results from [2], that the Galerkin method [6] for the equation $Cu = f$ in Hilbert space $L^{2,\alpha,\beta}(\mathbb{R}_+)$ is valid with respect to the set of functions $\phi_n^{\alpha,\beta} = \phi_n\rho_{\alpha,\beta}^{-1}$:

$$\phi_n(\lambda) = \begin{cases} \sqrt{2}\lambda\Lambda_n(-2\ln\lambda), & 0 < \lambda < 1; \\ 0, & 1 < \lambda < \infty, \end{cases} \qquad n = 0, 1, 2, \dots$$

$$\phi_n(\lambda) = -\phi_{-n-1}(\lambda^{-1}), \qquad n = -1, -2, -3, \dots,$$

where $\Lambda_n(t)$, $(n = 0, 1, 2, \dots)$ are normed Laguerre polynomials.

COROLLARY 3. *In the case when the statements of Corollary 2 and Lemma 2 hold, the equation* $Cu = f$ *has a unique solution* $u \in W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ *for any* $f \in W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$.

COROLLARY 4. *Let* $\beta-\alpha > b$ *and* $w_b K \in \mathcal{Z}^*_\delta$ *for certain* $\delta > 0$ *be (instead of* $\beta-\alpha \geq b$ *and* $w_b K \in \mathcal{Z}^*_0$*) in the hypotheses of Theorem* (see Lemma 3). *Besides, presuppose that the conditions 1–3 of Corollary 2 hold not only for the values* α, β *but for* $\alpha+h$, $\beta-h$ *with certain* h $(0 < 2h < \min\{2\delta, \beta-\alpha-b\})$ *also. Then, as it follows from Corollary 2, the equation* $Cu = f$ *has the unique common solution* $u \in L^{p,\alpha,\beta}(\mathbb{R}_+)$ *in each of the spaces* $L^{p,\alpha,\beta}(\mathbb{R}_+)$, $L^{p,\alpha+h,\beta-h}(\mathbb{R}_+)$ $(L^{p,\alpha,\beta}(\mathbb{R}_+) \subset L^{p,\alpha+h,\beta-h}(\mathbb{R}_+))$ *and for the convergence of the projective process the following estimate holds*

$$\|u - \widetilde{u}_R\|^{p,\alpha+h,\beta-h} = o(R^{-h}), \qquad R \to \infty.$$

In fact, from Theorem 2.1 section 11.2 [13] it follows that $\|u-\widetilde{u}_R\|^{p,\alpha,\beta} \leq D\|f - P_R Q_{1/R} f\|^{p,\alpha,\beta}$, $\|u - \widetilde{u}_{R,h}\|^{p,\alpha+h,\beta-h} \leq D_h \|f - P_R Q_{1/R} f\|^{p,\alpha+h,\beta-h}$, where the constants D, D_h do not depend on the value of R and the function f. It remains to be noted that

$$\|u - \widetilde{u}_{R,h}\|^{p,\alpha+h,\beta-h} \leq D_h\, R^{-h} \|f - P_R Q_{1/R} f\|^{p,\alpha,\beta}.$$

Of course, in the space $L^{p,\alpha,\beta}(\mathbb{R}_+)$ the estimate for the convergence is weaker.

Let us here note that if the hypothesis of Corollary 4 are satisfied then the functions $c_1(\lambda)$, $c_2(\lambda)$ do not have any zeros in the corresponding domains of analyticity $0 \leq \Im\lambda \leq h$, $-h \leq \Im\lambda \leq 0$.

Now assume that the functions $c_1(\lambda), c_2(\lambda)$ do not satisfy the conditions of Corollary 4 and have numerous zeros in respective layers. By a_{lj} $(l = 1,2;\ j = 1,2,\ldots,m_l)$ we denote all different zeros of these functions in the regions $0 < \Im\lambda < h$, $-h < \Im\lambda < 0$, respectively, p_{lj} — their multiplicity. Then it can be shown the following corollary

COROLLARY 5. *Let* $p \in [1,\infty)$, $\alpha > \alpha_*$, $\beta^* > \beta$, $\delta > 0$, $\beta - \alpha > b$ *and* $w_b K \in \mathcal{Z}^*_\delta$. *Besides, let the following conditions be satisfied for a certain* $0 < 2h < min\{2\delta, \beta-\alpha-b\}$:

1^0. $c_1(\lambda) \neq 0$, $c_2(\lambda) \neq 0$,
2^0. $c_1(\lambda + ih) \neq 0$, $c_2(\lambda - ih) \neq 0$.

Then any nontrivial solution of the homogeneous equation $Cu_0 = 0$ *in the space* $L^{p,\alpha+h,\beta-h}$ *furnishes the asymptotics*

$$u_0(\lambda) = \lambda^{-\alpha}\Big\{\sum_{j=1}^{m_1} \lambda^{ia_{1j}} P_{1j}(\ln\lambda)\Big\} + o(\lambda^{-\alpha}), \qquad \lambda \to 0,$$

$$u_0(\lambda) = \lambda^{-\beta}\Big\{\sum_{j=1}^{m_2} \lambda^{ia_{2j}} P_{2j}(\ln\lambda)\Big\} + o(\lambda^{-\beta}), \qquad \lambda \to \infty.$$

Here P_{lj} denote polynomials of the degree not greater then $p_{lj} - 1$.

Proof. We take advantage of the representation of the operator C as in (16): $C = I + P_1A_1P_1 + Q_1A_2Q_1 + T_\delta P_1 + T_\delta Q_1$. According to Lemma 3 it is easy to see that the operators: $T_\delta P_1 : L^{p,\alpha+h,\beta-h} \to (L^{p,\alpha,\beta}) \to L^{p,\alpha,\beta-h}$, $T_\delta Q_1 : L^{p,\alpha+h,\beta-h} \to (L^{p,\alpha,\beta}) \to L^{p,\alpha+h,\beta}$ are bounded. After exchanging the arguments similarly as in (11) and denoting $\phi(t) = \rho_{\alpha+h_1,\beta-h_1}(e^{-t})u(e^{-t})$ with $h_1 = h/2$, we arrive to the pair operator C^* in space $e^{h_1|t|}L^p(\mathbb{R})$

$$(C^*\phi)(t) = \begin{cases} \phi(t) + \int\limits_0^\infty \Phi_1^*(t-s)\phi(s)\,ds + (T_1^*\phi)(t), & 0 < t < \infty; \\ \phi(t) + \int\limits_0^\infty \Phi_2^*(t-s)\phi(s)\,ds + (T_2^*\phi)(t), & -\infty < t < 0. \end{cases}$$

Here $\Phi_1^*(t) = \Psi_1(e^{-t})e^{-(\alpha+h_1)t}$, $\Phi_2^*(t) = \Psi_2(e^{-t})e^{(h_1-\beta)t}$. Let us note that $T_1^* : e^{h_1|t|}L^p(\mathbb{R}) \to e^{-h_1t}L^p(\mathbb{R})$, $T_2^* : e^{h_1|t|}L^p(\mathbb{R}) \to e^{h_1t}L^p(\mathbb{R})$ are bounded operators, and from (8) one can conclude that $\Phi_i^* \in e^{-h_1|t|}L^1(\mathbb{R})$. At this point the statement of Corollary 5 follows from the results of Appendix [2].

COROLLARY 6. *It is evident that any nontrivial solution of the homogeneous equation from Corollary 5 can be represented in the form: $u_0 = v + w$, where $v \in W^{p,\alpha+h,\beta-h}_{(n)}(\mathbb{R}_+)$ for any $n \in N$, $w \in L^{p,\alpha,\beta}(\mathbb{R}_+)$. Besides, if we assume that the conditions of Lemma 2 are also satisfied, then $w \in W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ with the respective value of l.*

Remark 4. Corollary 3 and Corollary 6 make possible to obtain some simple result on a regularity of solutions of the equations. In fact it is evident that $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+) \subset W^l_p(\Omega_\varepsilon)$ ($\Omega_\varepsilon = (\varepsilon, 1/\varepsilon)$) for any $\varepsilon \in (0,1)$. Here $W^l_p(\Omega_\varepsilon)$ is the usual Sobolew space [6]. Then from the embedding theorem [6] it follows that $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+) \subset C^r(\mathbb{R}_+)$ if $r+1 \in N$ and $r < l - 1/p$ ($r \le l-1$ for $p = 1$). Moreover, the behaviour of these functions and its classical derivatives near zero and infinity points can be investigated by their asymptotics.

In conclusion, let us note that results of the paper can easily be extended on systems of the singular equations with analogous kernels. Those systems present some boundary value problems (see [8]).

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