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A. M. Linkov

REAL AND COMPLEX HYPERSINGULAR INTEGRALS AND INTEGRAL EQUATIONS IN COMPUTATIONAL MECHANICS*

Dedicated to Professor Janina Wolska-Bochenek

Hypersingular integrals and integral equations became very popular last decade in computational mechanics. The reason is quite clear: they provide a natural and effective means to solve problems involving discontinuities. These are problems of cracks and interacting blocks in elasticity; thin wings in fluid dynamics; shields in electroplating; low permeability walls and geotextile layers in groundwater studies, etc.

But these integrals and their direct values have a rather long history. Direct values of hypersingular integrals became generally known as a result of publication in 1923 of the famous Hadamard lectures on Cauchy problem for hyperbolic equations (supplemented French edition was published in 1932 [1]). J. Hadamard termed the direct values as "finite part integrals". They are now also widely known as Hadamard's integrals. Many years later, in his book "Psychology of invention in mathematical field" edited in 1954 [2], Hadamard wrote (my back translation from Russian): "I could no more avoid this method than the prisoner in Edgar Allen Poe's poem 'The pit and the pendulum' could avoid the pit in the center of his dungeon."

These integrals were a great invention and became an important stimulus to the development of distribution theory. R. Courant in his course "Partial differential equations" [3] wrote (again my back translation): "Actually,

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introduction of the finite part integrals by Hadamard became an essential motive for creation of the modern distribution theory".

But this development, in fact, provided such a powerful mathematical means of investigations that, it seemed, there was no need for Hadamard's integrals. Indeed, an Hadamard integral is simply an explicit form of the corresponding functional.

Interest in hypersingular integrals was revived with the development of computers, when numerical calculations based on integral equations became very popular, providing good reasons for using hypersingular equations. As mentioned above, these equations provide a natural means for solving problems with discontinuities on open or closed surfaces, such as cracks or the interacting surfaces between blocks or cracks.

Over the last decade there have been many publications on the application of hypersingular integrals in numerical calculations (see, e.g. [4]) and, what originally appeared to be new and complicated, now seems simple and even naive.

Unfortunately, there is no single book or single reference that provides a good overview of hypersingular integrals. So, it seems reasonable to present new results, obtained recently by the author and S. G. Mogilevskaya, in frames of some simple methodology.

1. Methodological concepts

1.1. On the term "hypersingular integral"

The term "hypersingular integral" is used in three different ways:

(i) as a proper integral, when a field point is outside of the surface of integration;

(ii) as a limit which results from a normal limit process, when a field point tends to be a point of the integration surface;

(iii) as a direct value, i. e., in the sense of a finite part (Hadamard) integral; in this case a field point is situated on the surface of integration and we need to define how to interpret the integral.

Let us discuss in brief these three meanings and start with their source that is singular solutions.

1.2. Singular (fundamental) solutions and potentials

Singular solutions play a key role in problems described by partial differential equations. Consider for example the two-dimensional Laplace equation. In all applications in acoustics, hydrodynamics and elasticity the main terms entering into the differential equations are generated by the singular solution U of the Laplace equation. Additional terms occur in elasticity problems, but the discussion of these terms can be reduced to that of Laplacian terms. In two-dimensional case $U = -\ln r/(2\pi)$ while in three dimensions $U = r/(4\pi)$ where r is a distance between a field point $\mathbf{x}$ and the point ξ at which a unit source is acting.

Clearly, all the partial derivatives of the singular solution also represent solutions of Laplace equation for $\mathbf{x} \neq \xi$. This can be seen by differentiating the Laplace equation — this simply changes the order of the derivatives. Integrals over any surface, open or closed, are also solutions. Thus, we have a large variety of potentials. Some of these functions have specific names, e. g. single layer potential, double layer potential, hypersingular potential. The latter for $\mathbf{x} \neq \xi$ is an usual proper integral.

For a closed surface we can also use a following consequence of Green's (or in elasticity Betti's) formula for a solution u of the Laplace equation:

$$c(\mathbf{x})u(\mathbf{x}) = \int\limits_{\Sigma} U \frac{\partial u}{\partial n_\xi} \, d\Sigma - \int\limits_{\Sigma} \frac{\partial U}{\partial n_\xi} u \, d\Sigma, \quad \mathbf{x} \notin \Sigma,$$

where $c(\mathbf{x}) = 1$ for $\mathbf{x}$ inside of Σ, $c(\mathbf{x}) = 0$ for $\mathbf{x}$ outside of Σ; the right-hand side and all its partial derivatives are also solutions of the Laplace equation for $\mathbf{x} \neq \xi$.

Let us assume that Σ is sufficiently smooth and the density $\phi(\xi)$ has continuous derivatives up to the k-th order. Then *all the potentials* up to the k-th order *have limit values*, when $\mathbf{x}$ tends to $\mathbf{x}_0 \in \Sigma$ from any side of Σ. Indeed, potentials involving only tangent derivatives are continuous, as are potentials involving normal derivatives of even orders; potentials involving normal derivatives of an odd order also have limits, although these limits in general are different, when approaching from different sides of the surface. Thus, we have included limit values of these integrals, in particular hypersingular integrals.

Since the potentials satisfy the Laplace equation, we can use their limit values to satisfy prescribed boundary conditions. Hence, they can serve to solve boundary value problems. Using them in numerical calculations, we can approximate functions in Σ and find quadrature rules for their limit values. This procedure can be carried out in various ways (see e. g. [5], [8]).

Note however that not all the potentials are of real use. For example, some of them attenuate too rapidly towards infinity. From the uniqueness theorem it can be seen that such potentials can not serve to satisfy arbitrary prescribed values on Σ in external problems.

Among these potentials, the hypersingular potential is especially attractive, when considering surfaces of discontinuity of the solution. To see this, let us consider such a surface, where the harmonic function u has discontinuity and the normal derivative $\partial u/\partial n_x = t_0(\mathbf{x}_0)$ is prescribed on Σ. Writing the solution u in the form of double layer potential

$$u(\mathbf{x}) = \int_\Sigma \frac{\partial U}{\partial n_\xi}\phi(\xi)\, d\Sigma$$

with unknown density $\phi(\xi)$, we have $u^+ - u^- = -\phi(\xi)$, i. e., $\phi(\xi) = -\Delta u$, where $\Delta u = u^+ - u^-$ is the solution discontinuity. Hence, the density has a simple physical meaning: it represents solution discontinuities. So we can write

$$u(\mathbf{x}) = \int_\Sigma \frac{\partial U}{\partial n_\xi}(-\Delta u)\, d\Sigma.$$

By the boundary condition $\lim_{\mathbf{x}\to\mathbf{x}_0^\pm} \partial u/\partial n_x = t_0(\mathbf{x}_0)$ for each $\mathbf{x}_0 \in \Sigma$, we obtain

$$\lim_{\mathbf{x}\to\mathbf{x}_0^\pm} \int_\Sigma \frac{\partial^2 U}{\partial n_x \partial n_\xi}(-\Delta u)\, d\Sigma = t_0(\mathbf{x}_0), \qquad \forall \mathbf{x}_0 \in \Sigma.$$

In elasticity: U is a matrix, a normal derivative is changed into a traction operator, Δu is a displacement discontinuity vector, $t_0(\mathbf{x}_0)$ is a traction vector. Thus, we obtain the equation which solves the problem in terms of discontinuities and which provides a clear physical interpretation of the density: it represents a displacement discontinuity useful in considering block interactions and cracks.

So far we have used only the *limit values* of potentials in order *to satisfy boundary conditions.* Such a treatment is quite sufficient in many applications. Since it allows quadrature rules to be established and to solve discretized problems, using the Boundary Element Method, *it would appear then that there is no need for direct values.*

Meanwhile, in some cases which I shall mention below, it is useful also to discuss direct values of potentials. Then, dealing with hypersingular potential, we have integrals which do not exist in the usual improper integral form and we need to develop special definitions in order to use direct values of the hypersingular integrals.

1.3. Direct values of integrals

Consider the one-dimensional case of an integral

$$\int_0^b \frac{\phi(x)}{x^k}\, dx$$

with kernel $\frac{1}{x^k}$ and density $\phi(x)$. For $k \leq 0$ this is a proper integral. For $0 < k < 1$ we have the usual improper integral. If $k = 1$ the integral can be termed a Cauchy integral. If $k > 1$ the integral is called a Hadamard integral. Consider, for simplicity, k being natural number (the general case does not introduce any new features). We have

$$\frac{1}{x^k} = (-1)^{k-1} \frac{1}{(k-1)!} \frac{d^k}{dx^k} \ln|x|,$$

i. e., the kernel is expressed by the k-th derivative of $\ln|x|$.

We will assume that the function $\phi(x)$ is sufficiently smooth. Then, excluding some small ε-vicinity of the origin $x = 0$, we can integrate by parts. This can be done for any exponent k however large. Gathering all the "good" terms (i. e., those which have a finite value for $\varepsilon = 0$) on the right-hand side and all the "bad" terms (i. e., each term tending to infinity as $\varepsilon \to 0$) on the left-hand side, we get

$$\int_\varepsilon^b \tfrac{\phi(x)}{x}\, dx + \phi(x)\ln|\varepsilon| = \phi(b)\ln|b| - \int_\varepsilon^b \phi'(x)\ln|x|\, dx,$$
$$\int_\varepsilon^b \tfrac{\phi(x)}{x^2}\, dx + \phi'(x)\ln|\varepsilon| - \tfrac{\phi(\varepsilon)}{\varepsilon} = \phi'(b)\ln|b| - \tfrac{\phi(b)}{b} - \int_\varepsilon^b \phi''(x)\ln|x|\, dx,$$
$$\int_\varepsilon^b \tfrac{\phi(x)}{x^3}\, dx + \ldots$$

....

All the right-hand sides have limits; hence,the sums of the left-hand side terms must also have limits, and direct values can be determined as these limits

$$\int_0^b \tfrac{\phi(x)}{x}\, dx = \lim_{\varepsilon\to 0}[\int_\varepsilon^b \tfrac{\phi(x)}{x}\, dx + \phi(x)\ln|\varepsilon|],$$
$$\int_0^b \tfrac{\phi(x)}{x^2}\, dx = \lim_{\varepsilon\to 0}[\int_\varepsilon^b \tfrac{\phi(x)}{x^2}\, dx + \phi'(x)\ln|\varepsilon| - \tfrac{\phi(\varepsilon)}{\varepsilon}],$$
$$\int_0^b \tfrac{\phi(x)}{x^3}\, dx = \ldots$$

....

Clearly, there are analogous formulae for integrals over $[a,0]$. Now, we can use small ε_1.Then for an interval $[a,b]$, taken (only for simplicity) $\varepsilon_1 = \varepsilon$, we obtain the following definitions for the direct value integrals on $[a,b]$:

$$\int_a^b \tfrac{\phi(x)}{x}\, dx = \lim_{\varepsilon\to 0}[\int_a^{-\varepsilon} \tfrac{\phi(x)}{x}\, dx + \int_\varepsilon^b \tfrac{\phi(x)}{x}\, dx],$$
$$\int_a^b \tfrac{\phi(x)}{x^2}\, dx = \lim_{\varepsilon\to 0}[\int_a^{-\varepsilon} \tfrac{\phi(x)}{x^2}\, dx + \int_\varepsilon^b \tfrac{\phi(x)}{x^2}\, dx - 2\tfrac{\phi(\varepsilon)}{\varepsilon}],$$
$$\int_a^b \tfrac{\phi(x)}{x^3}\, dx = \ldots$$

....

Note that the integral, corresponding to $k = 1$, is the well-known Cauchy principle value integral. Thus, we can see that there is no principal difference (except historical) between singular Cauchy integrals and hypersingular Hadamard integrals. This is also quite obvious from the clear connection of such integrals with distribution theory. Indeed, suppose that $\phi(x)$ is a probe function within $[a,b]$. Then, the right-hand sides in the initial formulae contain only integrals; hence, the definitions turn into the usual formulae of distribution theory. It is quite obvious also that the definitions are such that the Newton-Leibnitz formula remains true. This means that if a direct value integral can be formally evaluated, then simple substitution of the end points of an interval satisfies the definition.

The same is true for two-dimensional case. The only difference is that, instead of integrating by parts, we use its extension, Green's formula, to transform a surface integral into a contour integral. Note also that, just as in the one-dimensional case, if a direct value integral can be formally evaluated, then simple substitution of points of the contour satisfies the definition. In the same way, we may consider, if needed, direct values of n-dimensional integrals.

But *the above is still* no more than an elegant construction, *a pure mental exercise.* In applications, limit values of potentials are needed to satisfy prescribed boundary conditions. Hence, in order to use direct values we must define their connection with limit values.

1.4. Connection between limit and direct values

This connection becomes clear, if we consider, for simplicity, a straight (planar) element of the surface. Then for $\mathbf{x} \neq \mathbf{x}_0$ we can transfer the *tangential* derivation from the fundamental solution to the density function $\phi(\mathbf{x})$. In this case we perform all *the same operations* as those used in defining the direct values. Naturally, in limit $\mathbf{x} \to \mathbf{x}_0$ we arrive at expressions which coincide with the direct values. Thus, we see that the limit values of tangent derivatives coincide with the direct values. For potentials containing *normal* derivatives, the only difference is that the odd order normal derivatives generate discontinuities. Indeed, for the first normal derivative, we immediately arrive at the well-known elementary result of potential theory. For the second normal derivative, we can allow the fact that $\ln r$ satisfies the Laplace equation. Hence, we reduce the case to that of the tangent derivatives which we have already discussed. The potential remains continuous. For the third normal derivative, we use the same procedure and obtain a discontinuity, involving the second tangent derivative of the density...etc. The same arguments apply to the three-dimensional case or to the n-dimensional case.

In elasticity we have additional terms, but these present no changes in principle.

The above discussion indicates that there are simple formulae by which limit values may be connected to direct values. Thus, when satisfying boundary conditions, we can always express limit values via direct values. In this way, we arrive at integral equations, involving only points on the boundary.

The analytical simplification is obvious, i. e., we deal only with functions and points of the surface itself. But are there any other virtues to the use of direct values?

1.5. On computational applications of direct values

I see at least four applications in computational mechanics:

(i) direct values allow us to use results of well established mathematical theories (in particular, the theory of complex variables in two-dimensional problems or distribution theory);

(ii) they serve to simplify the derivation of new useful integral equations, e. g., for blocky systems or for layered systems with cracks and cavities;

(iii) sometimes, but not always, they simplify the derivation of quadrature rules (e. g., for complex hypersingular integrals);

(iv) they are useful in studying the accuracy of quadrature formulae and numerical methods.

This concludes the first part of this paper. It does not contain new analytical or computational results and is mostly of methodological nature. In the next part it will be illustrated by some new results obtained recently.

2. New complex and real hypersingular equations in elasticity theory

2.1. Complex hypersingular equation for plane elasticity problems

We know that complex variables are very attractive in plane problems. Using them, one can apply well-established classical theories of analytical functions, singular integrals and equations (e.g., [7], [8]). Then conclusions concerning solutions are relevant.

Computational advantages are also clear. Since modern computers handle complex arithmetic, we can gain, when using in plane problems one (complex) function of one (complex) variable. And what is much more important: the most crucial stage, that is computation of singular integrals does not present difficulties, when applying *complex* variables. There are also some additional advantages concerning accuracy control as well as time and memory consumption for calculations.

From the other hand, as it has been mentioned, there are good reasons to use *hypersingular* forms of equations, when dealing with discontinuities. Thus, it seemed very promising to combine virtues of complex variables with those of hypersingular equations for plane problems. The way to do this is quite clear from the methodology described above. It was used in [9], where presented all the subsequent details.

First of all a complex hypersingular potential is introduced as a derivative of the Cauchy type integral. It can be used to satisfy the Laplace or Navier plane equations. Limit values of the hypersingular potential exist. Hence, they can serve to satisfy prescribed boundary conditions. This can be done without defining direct values of hypersingular integral.

Meanwhile, as mentioned, direct values are also of interest for computational purposes. So, their definitions should be introduced. The way to introduce them is quite similar to that described above for real finite-part integrals. We exclude small ε-vicinity of the point, use integration by parts, collect all the "good" terms in the right-hand side, all the "bad" terms in the left-hand side, pass to the limit, when ε tends to zero, and define this limit to be "direct value" (Hadamard, finite-part) integral. Thus, we obtain a definition consistent with further applications of these integrals to satisfy boundary conditions. The next step is to study connection between limit values and direct values of hypersingular integrals. This connection is expressed by formulae similar to those of Sokhotsky-Plemelj. Hence, we can formulate boundary value problems in terms of direct values, if we like. In complex variables these direct values have significant virtues both in analytical and computational sense. Indeed, we can state the theorem of holomorphity. It provides an equality necessary and sufficient for functions ϕ^+ and ϕ^- to be limit values of holomorphic function $\phi(z)$. This theorem serves to obtain hypersingular equations not referring to their singular counterparts.

A new hypersingular equation follows from these results [9]. It serves for cracks and blocky inhomogeneous systems; for internal and external problems. For blocky systems, its advantage, as compared with real equations, is that it contains only tractions and displacement discontinuities, i. e., the values which enter constitutive equations for contact interaction. In particular case, the equation refers to systems of curvilinear cracks. In this case, the index of the equation is zero. This provides a significant simplification in numerical calculations: we do not need satisfy additional conditions as it is the case for complex *singular* equations (the index of the latter is not zero).

To use complex hypersingular equations for numerical calculations, we need quadrature rules for direct values of integrals. There are no problems in computing such integrals over arbitrary *curvilinear* segment for a variety of approximations of the density. This provides a drastic simplification of the crucial step as compared with real equations. For instance, we have simple quadrature formulae, if the density is approximated by a complex Lagrange polynomial. All the coefficients are expressed by analytical formulae. For tip elements we also have analytical quadrature formulae for any rational exponent in asymptotic [9].

Numerical results for cracks [9], and recently produced calculations for two and four interacting blocks, confirm efficiency and accuracy of the method employing complex hypersingular equations.

2.2. Hypersingular equation for three-dimensional blocky systems

Another example refers to elasticity problems for three-dimensional blocky systems. Such systems can present rock, grains or composite construction. Consider the case of p isotropic elastic blocks. For the i-th block with the boundary B^i we have a hypersingular equation (see, e.g., [10], [11])

$$c^i(\mathbf{x})\sigma^i(\mathbf{x}) = \int_{B^i} [\mathbf{T}^i(\xi,\mathbf{x})]^t \sigma^i(\xi)\, d\Sigma - \int_{B^i} \mathbf{Q}^i(\mathbf{x},\xi)\mathbf{u}^i(\xi)\, d\Sigma,$$

where $c^i(\mathbf{x}) = 1$, if $\mathbf{x}$ is inside of the i-th block, $c^i(\mathbf{x}) = 0$, if $\mathbf{x}$ is outside of it, and $c^i(\mathbf{x}) = 1/2$, if x belongs to its boundary B^i, σ^i is a traction vector, $\mathbf{u}^i$ is a displacement vector,

$\mathbf{T}^i(\mathbf{x},\xi) = [\mathbf{T}_{n(\xi)}\mathbf{U}^i(\xi,\mathbf{x})]^t$, $\mathbf{Q}^i(\mathbf{x},\xi) = \mathbf{T}_{n(x)}\mathbf{T}^i(\mathbf{x},\xi)$,

$\mathbf{U}^i(\mathbf{x},\xi)$ is Kelvin's matrix, $\mathbf{T}_{n(x)}$ is the traction operator.

It is essential that matrices $\mathbf{T}^i$ and $\mathbf{Q}^i$ can be represented in a form

$$\mathbf{T}^i = \frac{1}{1-\nu^i}\mathbf{T}^i_\nu, \qquad \mathbf{Q}^i = \frac{2\mu^i}{1-\nu^i}\mathbf{Q}^i_\nu,$$

where μ^i is a shear modulus of the i-th block, ν^i is its Poisson's ratio. Matrices $\mathbf{T}^i_\nu$, $\mathbf{Q}^i_\nu$ depend only on Poisson's ratio (and do not depend on shear modulus).

Suppose that Poisson's ratios are the same for all the blocks $\nu^i = \nu$, $i = 1,\dots,p$. Hence, $\mathbf{T}^i_\nu = \mathbf{T}^i_\nu$, $\mathbf{Q}^i_\nu = \mathbf{Q}_\nu$. Then dividing a hypersingular equation for each i-th block by $2\mu/(1-\nu)$ and summing over i, one gets an hypersingular equation

$$\frac{1}{2}a_2(\mathbf{x})\sigma(\mathbf{x}) - \int_B a_1(\xi)[\mathbf{T}_\nu(\xi,\mathbf{x})]^t\sigma(\xi)\, d\Sigma + \int_B \mathbf{Q}_\nu(\mathbf{x},\xi)\Delta\mathbf{u}(\xi)\, d\Sigma = \mathbf{0}, \quad \mathbf{x}\in B,$$

where $a_1 = 0.5(1-\nu)(1/\mu^+ - 1/\mu^-)$, $a_2 = 0.5(1-\nu)(1/\mu^+ + 1/\mu^-)$, $\sigma(\mathbf{x})$ is a traction in the point $\mathbf{x}$, $\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ is a displacement discontinuity, the normal $\mathbf{n}$ is fixed on the contacting surfaces of the blocks, the sign "plus" ("minus") corresponds to the block for which the normal $\mathbf{n}$ is outward (inward), for external boundaries the normal $\mathbf{n}$ is assumed to be outward and $1/\mu^- = 0$, $\mathbf{u}^- = \mathbf{0}$.

The derived hypersingular equation contains only tractions and displacement discontinuities on contacts of blocks. These are just the values entering constitutive equations for contact interaction. Hence, we do not need to find limit values $\mathbf{u}^+$ and $\mathbf{u}^-$, but only their difference $\mathbf{u}^+ - \mathbf{u}^-$. The number of unknowns on contacts becomes twice less as compared with other methods.

In a case when Poisson's ratios of the blocks are not equal, we can use Taylor's expansion over deviations of ν^i from some fixed value ν. Then we arrive at a chain of such equations; only their right-hand sides change at each successive step.

3. Conclusions

The conclusions can be summarized as follows:

(i) The hypersingular potentials provide a natural and effective means for solving problems involving discontinuities on surfaces of cracks, interacting blocks, shields used in electroplating or low permeability walls and geotextile layers in groundwater flow studies.

(ii) The *limit* values of hypersingular integrals have a clear physical meaning and can be applied directly in numerical computations. Boundary equations obtained in this form can be solved by using approximations and quadrature rules derived for limit values. The *direct* values of hypersingular integrals, although introduced formally, are also of practical use, due to their simple connection with limit values, and the ability they provide for dealing with points and functions on the integration surface only.

(iii) *Complex* hypersingular equations can serve to use virtues of complex variables and to overcome the main difficulty: computation of hypersingular integrals over curvilinear contours. In important applications they have zero index what simplifies numerical solution. Numerical results confirm high efficiency of employing complex hypersingular equations.

(iv) New *real* hypersingular equations for three-dimensional blocky systems may serve to diminish the number of unknowns when accounting for contact interaction.

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INSTITUTE FOR PROBLEMS OF MECHANICAL ENGINEERING
RUSSIAN ACADEMY OF SCIENCES
St.-PETERSBURG, RUSSIA

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