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INVERSE PROBLEM FOR THE SYSTEM OF DIFFUSION EQUATIONS WITH A CHOSEN CONTROL POINT

Dedicated to Professor Janina Wolska-Bochenek

1. Introduction

The inverse problem (P) for the parabolic in Petrovskii's sense system in $\mathbb{R}^k$

$$\frac{\partial}{\partial t}u(t,x) = \mathbf{A}\Delta u(t,x) + \mathbb{F}(t,x) \tag{1.1}$$

with the function $\mathbb{F}$ of the form

$$\mathbb{F}(t,x) = \mathbb{H}(t,x)f(t) + w(t,x), \quad (t,x) \in]0,T[\times \Omega, \tag{1.2}$$

where

$$\Omega = \{(x_1,x_2): \ x_1^2 + x_2^2 \le a^2\}, \tag{1.3}$$

$\mathbf{A}$ is a given real $k \times k$ matrix, $\mathbb{H}$ is a given $k \times k$ matrix-valued function

$$\mathbb{H}: \ [0,T] \times \overline{\Omega} \ni (t,x) \mapsto \mathbb{H}(t,x) \in \mathbb{R}^{k^2},$$

w is a given $\mathbb{R}^k$-valued function

$$w: [0,T] \times \overline{\Omega} \ni (t,x) \mapsto w(t,x) \in \mathbb{R}^k,$$

consists in determining the unknown $\mathbb{R}^k$-valued functions u and f:

$$u: [0,T] \times \overline{\Omega} \ni (t,x) \mapsto u(t,x) \in \mathbb{R}^k,$$

$$f: [0,T] \ni t \mapsto f(t) \in \mathbb{R}^k,$$

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satisfying the system (1.1) and the usual initial and boundary Fourier conditions (first Fourier problem) which are overspecified by the additional measurement of the function $u(t,x)$ at the point $\mathring{x} \in \mathrm{INT}\,\Omega$, i.e.

$$u(t,\mathring{x}) = h(t) \quad \text{for } t \in [0,T], \tag{1.4}$$

where h is a given $\mathbb{R}^k$-valued function

$$h : [0,T] \ni t \mapsto h(t) \in \mathbb{R}^k.$$

A similar problem was investigated by M. Majchrowski, J. Rogulski in [6] but for $\Omega =]0,1[$ only.

The method of solving our problem is by resolving it into the system of Volterra integral equations of the second kind. At first we have to find the solution of an auxiliary problem (F1) for the system (1.1) with the initial and boundary conditions:

$$u(0,x) = g(x) \quad \text{for } x \in \Omega, \tag{1.5.1}$$

$$u(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \partial\Omega, \tag{1.5.2}$$

where g is a given $\mathbb{R}^k$-valued function

$$g : \overline{\Omega} \ni x \to g(x) \in \mathbb{R}^k.$$

Such first Fourier problem (F1) was investigated in detail in [1]. Solution of the problem (F1) is represented in a sum of two integrals being counterparts of the Poisson-Weierstrass and the potential of a plane domain. Kernels of these potentials are represented by the matrix-function G. The form of G and the properties of this matrix-function are discussed in [1]. The representation of solution of the problem (F1) with G being used allows us to solve an inverse problem by an application of the theory of systems of Volterra integral equations.

Inverse problems formulated for the above systems are in a sense some kind of problems discussed in Control Theory. In our paper the source $\mathbb{F}$ in a system of diffusion equations becomes an unknown function as well as u itself. This sort of problems is discussed in Cannon's papers [2], [3], [4], [5], [7], and others papers (see [8]).

2. Assumptions

We make the following assumptions for the functions $\mathbb{H}, w, g, h$ and for the matrix $\mathbb{A}$:

(2.1) $\mathbb{H}(t,x) = 0$ for $(t,x) \in [0,T] \times \partial\Omega$, $\mathbb{H}(t,\mathring{x})$ is invertible for every

$t \in [0,T]$, and the function

$$\widetilde{\mathbb{H}} : [0,T] \times [0,a] \times [0,2\Pi] \to \mathbb{R}^{k^2}$$

which we define by

$$\widetilde{H}(t,\rho,\gamma) := \mathbb{H}(t,\rho\cos\gamma,\rho\sin\gamma)$$

is of the class C^1 and of the class C^2 relative to the second variable and of the class C^4 relative to the third variable and

$$\frac{\partial^S}{\partial\gamma^S}\widetilde{\mathbb{H}}(t,\rho,0) = \frac{\partial^S}{\partial\gamma^S}\widetilde{\mathbb{H}}(t,\rho,2\Pi) \quad \text{for } s \in \{0,1,2,3\},$$

$$\frac{\partial^{l+2}}{\partial\gamma^l\partial\rho^2}\widetilde{\mathbb{H}} \quad \text{exist, are continuous, bounded and equal zero}$$

on $\partial\Omega$ for $l \in \{1,2\}$,

(2.2) $w(t,x) = 0$ for $(t,x) \in [0,T] \times \partial\Omega$, the function

$$\widetilde{w} : [0,T] \times [0,a] \times [0,2\Pi] \to \mathbb{R}^k$$

which we define by

$$\widetilde{w}(t,\rho,\gamma) := w(t,\rho\cos\gamma,\rho\sin\gamma)$$

is of the class C^1 and of the class C^2 relative to the second variable and of the class C^4 relative to the third variable and

$$\frac{\partial^S}{\partial\gamma^S}\widetilde{w}(t,\rho,0) = \frac{\partial^S}{\partial\gamma^S}\widetilde{w}(t,\rho,2\Pi) \quad \text{for } s \in \{0,1,2,3\},$$

$$\frac{\partial^{l+2}}{\partial\gamma^l\partial\rho^2}\widetilde{w} \quad \text{exist, are continuous, bounded and equal zero}$$

on $\partial\Omega$ for $l \in \{1,2\}$,

(2.3) $h(0) = g(\mathring{x})$ and h is of class C^1,

(2.4) $g(x) = 0$ for $x \in \partial\Omega$, the function

$$\widetilde{g} : [0,a] \times [0,2\Pi] \to \mathbb{R}^k$$

which we define by

$$\widetilde{g}(\rho,\gamma) := g(\rho\cos\gamma,\rho\sin\gamma)$$

is of the class C^2 and of the class C^4 relative to the second variable and

$$\frac{\partial^S}{\partial\gamma^S}\widetilde{g}(\rho,0) = \frac{\partial^S}{\partial\gamma^S}\widetilde{g}(\rho,2\Pi) \quad \text{for } s \in \{0,1,2,3\},$$

$$\frac{\partial^{l+2}}{\partial\gamma^l\partial\rho^2}\widetilde{g} \quad \text{exist, are continuous, bounded and equal zero}$$

on $\partial\Omega$ for $l \in \{1,2\}$,

(2.5) all eigenvalues of $\mathbb{A}$ have positive real parts.

We impose an additional condition in the form: we require that there exists $\lim_{t\to 0}\frac{\partial}{\partial t}u(t,\mathring{x})$.

3. Existence of a solution of the problem (P)

From [1], Theorem 5.1, it follows the existence of a function $v : [0,T] \times \overline{\Omega} \to \mathbb{R}^k$, which is a solution of the problem (F1) with $\mathbb{F} = 0$.

Put $U(t,x) = u(t,x) - v(t,x)$ and observe that problem (P) becomes

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial t}v = \mathbb{A}\Delta U + \mathbb{A}\Delta v + \mathbb{H}(t,x)f(t) + w(t,x), \quad \text{for } (t,x) \in]0,T[\times\Omega,$$

$$U(0,x) + v(0,x) = g(x), \quad \text{for } x \in \Omega,$$

$$U(t,x) + v(t,x) = 0, \quad \text{for } (t,x) \in [0,T] \times \partial\Omega,$$

$$U(t,\mathring{x}) + v(t,\mathring{x}) = h(t), \quad \text{for } t \in [0,T].$$

Hence we obtain the following problem (P0) with unknown functions U and f

(3.1) $$\frac{\partial}{\partial t}U(t,x) = \mathbb{A}\Delta U(t,x) + \mathbb{H}(t,x)f(t) + w(t,x), \quad \text{for } (t,x) \in]0,T[\times\Omega,$$

(3.2) $U(0,x) = 0$, for $x \in \Omega$,

(3.3) $U(t,x) = 0$, for $(t,x) \in [0,T] \times \partial\Omega$,

(3.4) $U(t,\mathring{x}) = h(t) - v(t,\mathring{x})$, for $t \in [0,T]$, $\mathring{x} \in \mathrm{I\mathring{N}T}\,\Omega$.

It follows from [1], Theorem 5.2, that for a given f the problem (3.1-3) has a solution.

THEOREM 3.1 *If the functions $\mathbb{H}, w, h$, satisfy assumptions* (2.1-3) *then there exist continuous functions*

$$U : [0,T] \times \overline{\Omega} \to \mathbb{R}^k,$$

$$f : [0,T] \to \mathbb{R}^k,$$

such that the derivatives $\frac{\partial}{\partial t}U$, $\frac{\partial}{\partial x_i}U$, $\frac{\partial^2}{\partial x_i^2}U$, where $i \in \{1,2\}$ are defined and continuous on $]0,T[\times\Omega$, and the pair (U,f) satisfies (3.1-4).

Proof. From Theorem 5.2 in [1] it follows that the problem (3.1-3) has the solution v_2 given by the formula (for $\mathbb{F} = \mathbb{H}f + w$)

$$(3.5)\quad \widetilde{v}_2(t,r,\beta) = \frac{2}{a^2\pi}\int_0^t\Big[\int_0^{2\pi}\Big[\int_0^a \rho G(t-\eta,r,\rho,\beta-\gamma)\widetilde{\mathbb{F}}(\eta,\rho,\gamma)d\rho\Big]d\gamma\Big]d\eta$$

where $v_2(t,r,\beta) := v_2(t, r\cos\beta, r\sin\beta)$, $\widetilde{\mathbb{F}}(t,r,\beta) := \mathbb{F}(t, r\cos\beta, r\sin\beta)$ and the matrix-function G is given in [1] by the formula (3.3). Then applying condition (3.4), we obtain the following system of equations for the function f.

$$(3.6)\quad \frac{2}{a^2\pi}\int_0^t\Big[\int_0^{2\pi}\Big[\int_0^a \rho G(t-\eta,r_0,\rho,\beta_0-\gamma)\widetilde{\mathbb{F}}(\eta,\rho,\gamma)d\rho\Big]d\gamma\Big]d\eta = h(t) - \widetilde{v}_1(t,r_0,\beta_0),$$

where $\widetilde{v}_1(t,r,\beta) := v_1(t, r\cos\beta, r\sin\beta)$, v_1 is the mentioned above solution of the problem F1 with $\mathbb{F} = 0$ (see [1] theorem 5.1) and has the form

$$\widetilde{v}_1(t,r,\beta) = \frac{2}{a^2\pi}\int_0^{2\pi}\Big[\int_0^a \rho G(t,r,\rho,\beta-\gamma)\widetilde{g}(\rho,\gamma)d\rho\Big]d\gamma,$$

$$(r_0\cos\beta_0, r_0\sin\beta_0) = \mathring{x} \in \mathrm{INT}\,\Omega,$$

$$\widetilde{\mathbb{F}}(\eta,\rho,\gamma) = \widetilde{\mathbb{H}}(\eta,\rho,\gamma)f(\eta) + \widetilde{w}(\eta,\rho,\gamma),$$

$$\widetilde{\mathbb{H}}(\eta,\rho,\gamma) := \mathbb{H}(\eta,\rho\cos\gamma,\rho\sin\gamma),$$

$$\widetilde{w}(\eta,\rho,\gamma) := w(\eta,\rho\cos\gamma,\rho\sin\gamma).$$

The equation (3.6) is a system of Volterra integral equations of the first kind for the function f of the form

$$(3.7)\quad \int_0^t \mathbb{N}(t,\eta)f(\eta)d\eta = \Psi(t), \quad t \in [0,T],$$

where the matrix-valued function $\mathbb{N}(t,\eta)$ and the $\mathbb{R}^k$-valued function Ψ are defined as follows:

$$\mathbb{N}(t,\eta) := \frac{2}{a^2\pi}\int_0^{2\pi}\Big[\int_0^a \rho G(t-\eta,r_0,\rho,\beta_0-\gamma)\widetilde{\mathbb{H}}(\eta,\rho,\gamma)d\rho\Big]d\gamma, \quad \text{for } \eta \le t,$$

$$\Psi(t) := h(t) - \widetilde{v}_1(t, r_0, \beta_0) + \frac{2}{a^2\pi} \int_0^t \Big[\int_0^{2\pi} \Big[\int_0^a \rho G(t-\eta, r_0, \rho, \beta_0 - \gamma)\widetilde{w}(\eta, \rho, \gamma) d\rho \Big] d\gamma \Big] d\eta, \quad \text{for } t \in [0, T].$$

By definition of the matrix-function G (see [1]), we obtain another form of the kernel of the equation (3.7)

$$\mathbb{N}(t, \eta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \exp\Big(- \Big(\mu_{n,m} \frac{1}{a}\Big)^2 (t-\eta)\mathbb{A}\Big) \mathbb{H}_{n,m}(\eta),$$

where

$$\mathbb{H}_{n,m}(\eta) := J_n\Big(\mu_{n,m}\frac{r_0}{a}\Big)[\cos n\beta_0 \overset{1}{\underset{n,m}{\mathbb{H}}}(\eta) + \sin n\beta_0 \overset{2}{\underset{n,m}{\mathbb{H}}}(\eta)],$$

$$\overset{1}{\mathbb{H}}_{n,m}(\eta) := \frac{2}{\varepsilon_n a^2 \pi [J_{n+1}(\mu_{n,m})]^2} \times \int_0^{2\pi} \Big[\int_0^a \rho \cos n\gamma J_n\Big(\mu_{n,m}\frac{\rho}{a}\Big) \times \widetilde{\mathbb{H}}(\eta, \rho, \gamma) d\rho \Big] d\gamma, \quad (n, m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N},$$

$$\overset{2}{\mathbb{H}}_{n,m}(\eta) := \frac{2}{\varepsilon_n a^2 \pi [J_{n+1}(\mu_{n,m})]^2} \times \int_0^{2\pi} \Big[\int_0^a \rho \sin n\gamma J_n\Big(\mu_{n,m}\frac{\rho}{a}\Big) \times \widetilde{\mathbb{H}}(\eta, \rho, \gamma) d\rho \Big] d\gamma, \quad (n, m) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}.$$

For fixed $\eta \in [0, T]$ and $\eta < t$ we analogously observe as in the proof of Theorem 5.1 in [1] that

$$\mathbb{H}(\eta, \mathring{x}) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \mathbb{H}_{n,m}(\eta), \quad \text{where } \mathring{x} = (r_0 \cos \beta_0, r_0 \sin \beta_0).$$

Hence for every $t \in [0, T]$ we have

$$\mathbb{N}(t, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} I\mathbb{H}_{n,m}(t) = \mathbb{H}(t, \mathring{x}), \tag{3.8}$$

where I is the unit matrix.

Since the matrix $\mathbb{H}(t,\mathring{x})$ is invertible for every $t \in [0,T]$, the matrix $\mathbb{N}(t,t)$ is also invertible for every $t \in [0,T]$. From [1] see Theorem 4.1) it follows that the kernel $\mathbb{N}(t,\eta)$ is continuous function on a closed triangle $0 \le \eta \le t \le T$.

Also from the same theorem it follows that the function $\mathbb{N}_t(t,\eta)$ given by the formula

$$\mathbb{N}_t(t,\eta) = \frac{1}{a^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\mu_{n,m}^2 \exp\left(-\left(\mu_{n,m}\frac{1}{a}\right)^2(t-\eta)\mathbb{A}\right)\mathbb{H}_{n,m}(\eta),$$

is continuous on the closed triangle $0 \le \eta \le t \le T$.

Differentiating both sides of (3.7) we obtain a system of Volterra integral equations of the second kind for the function f

$$\mathbb{N}(t,t)f(t) + \int_0^t \mathbb{N}_t(t,\eta)f(\eta)d\eta = \Psi'(t). \tag{3.9}$$

From (3.8) and (3.9) we obtain

$$f(t) + \int_0^t \mathbb{K}(t,\eta)f(\eta)d\eta = \mathbb{H}(t,\mathring{x})^{-1}\Psi'(t), \tag{3.10}$$

where the kernel $\mathbb{K}(t,\eta) = \mathbb{H}(t,\mathring{x})^{-1}\mathbb{N}_t(t,\eta)$ is continuous on the closed triangle $0 \le \eta \le t \le T$. Hence the equation (3.10) has a unique continuous solution $f : [0,T] \to \mathbb{R}^k$. Taking $\mathbb{F} = \mathbb{H}f + w$, the $\mathbb{R}^k$-valued function $U = v_2$ given by the formula (3.5) is the solution of the problem (F1). Hence we obtain the function U which together with f has all properties listed in Theorem 3.1.

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