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ON LOCALLY SYMMETRIC SETS

Introduction

Locally symmetric sets were considered by S. Marcus, among others, in papers [3] and [4]. This notion was useful in investigating the problems connected with continuity and symmetric differentiability. In the paper we give various theorems on the structure of locally symmetric sets. These results will be used to examine a symmetric generalization of the derivative of a set and a function as well as to study symmetric continuity.

1. Points of local symmetry of sets

DEFINITION. A point x will be called a point of local symmetry of a set A if there exists a positive number d such that, for each number h satisfying the condition $|h| < d$, the implication

$$x + h \in A \Rightarrow x - h \in A \tag{1}$$

holds.

The set of all points of local symmetry of the set A will be denoted by the symbol $S(A)$.

1.1. *If $B = R \setminus A$, then $S(B) = S(A)$.*
(R will always denote the set of all real numbers). In particular, $S(\emptyset) = S(R) = R$.

1.2. $S(A \cap B) \supset S(A) \cap S(B)$,
$S(A \cup B) \supset S(A) \cap S(B)$,
$S(A \setminus B) \supset S(A) \cap S(B)$.

More generally, if $A_j \subset S(B_j)$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subset S\Big(\bigcap_{j=1}^{n} B_j\Big) \quad \textit{and} \quad \bigcap_{j=1}^{n} A_j \subset S\Big(\bigcup_{j=1}^{n} B_j\Big).$$

In particular, if $A \subset S(B_j)$ *for* $j = 1, 2, \dots, n$, *then*

$$A \subset S\Big(\bigcap_{j=1}^{n} B_j\Big) \quad \textit{and} \quad A \subset S\Big(\bigcup_{j=1}^{n} B_j\Big).$$

Equalities in the above inclusions may not hold, for example, for $A = Q$ (Q will always denote the set of all rational numbers) and $B = R\backslash Q$ because, then, $S(A) = S(B) = Q$ and $S(A \cap B) = S(A \cup B) = R$. Whereas, for $A = Q$ and $B = Q$, we have $S(A\backslash B) = R \neq Q = S(A \cap B)$.

1.3. $S(A) \subset S[S(A)]$.

Proof. Let $x \in S(A)$. Then there exists a positive number d such that

$$\forall_{|h|<d}(x + h \in A \Longleftrightarrow x - h \in A). \tag{2}$$

Consider an arbitrary fixed number h where $|h| < d$. If $x + h \in S(A)$, then there exists a number $d_h > 0$ such that

$$\forall_{|k|<d_h}(x + h + k \in A \Longleftrightarrow x + h - k \in A). \tag{3}$$

We may, of course, assume that the number d_h is so small that $[x + h - d_h, x + h + d_h] \subset (x - d, x + d)$, that is, $|h| + d_h < d$. Putting successively in (2) $h + k$ and $h - k$ instead of h, we obtain

$$x + h + k \in A \Longleftrightarrow x - h - k \in A \quad \text{and} \quad x + h - k \in A \Longleftrightarrow x - h + k \in A.$$

Hence and from (3) we have

$$\forall_{|k|<d_h}(x - h - k \in A \Longleftrightarrow x - h + k \in A),$$

which yields $x - h \in S(A)$. We have demonstrated that

$$\forall_{|h|<d}[x + h \in S(A) \Rightarrow x - h \in S(A)],$$

i.e. that $x \in S[S(A)]$. ■

Inclusion 1.3 cannot be replaced by the equality since, for instance, for the set $A = \{0, 1, 1/2, 1/3, \dots\}$, we have $S(A) = R\backslash\{0\}$ and $S[S(A)] = R$.

1.4. $S(A) \cap A \subset S[S(A) \cap A]$ *and* $S(A)\backslash A \subset S[S(A)\backslash A]$.

Proof. Let, for example, $x \in S(A)\backslash A$. Then $x \notin A$ and there exists a number $d > 0$ such that condition (2) is satisfied.

Consider an arbitrarily fixed number h where $0 < |h| < d$. If $x + h \in S(A)\backslash A$, then $x + h \notin A$ and, in virtue of (2), $x - h \notin A$. Moreover, there exists $d_h \in (0, d - |h|)$ such that

$$\forall_{|k|<d_h}(x + h + k \in A \Longleftrightarrow x + h - k \in A).$$

Replacing successively in (2) h by $h + k$ and $h - k$, we have the equivalences $x + h + k \in A \Longleftrightarrow x - h - k \in A$ and $x + h - k \in A \Longleftrightarrow x - h + k \in A$ from which we now infer that, for $|k| < d_h$, we have $x - h - k \in A \Longleftrightarrow x - h + k \in A$, i.e. that $x - h \in S(A)$. Since $x - h \notin A$, therefore $x - h \in S(A)\backslash A$ provided $|h| < d$. In consequence, $x \in S[S(A)\backslash A]$.

1.5. $S(A) \subset S(\operatorname{Int} A)$.

Proof. Let $x \in S(A)$. Then there exists a number $d > 0$ such that, for $|h| < d$, the equivalence

$$x + h \in A \Longleftrightarrow x - h \in A \tag{4}$$

holds. If now $x + h \in \operatorname{Int} A$, then there exists an interval (a, b) such that $x + h \in (a, b) \subset A$. From condition (4) we deduce that $x - h \in (2x - b,\ 2x - a) \subset A$. For $|h| < d$, we have shown the implication $k + h \in \operatorname{Int} A \Rightarrow x - h \in \operatorname{Int} A$ which gives the relation $x \in S(\operatorname{Int} A)$.

1.6. *If a set A is nowhere dense, then the set $\overline{A} \cap S(A)$ (the more so, the set $A \cap S(A)$) is countable.*

Proof. Let $B = \overline{A}$. The set B is closed and nowhere dense. Without loss of generality we may assume that the set B is contained in some interval (a, b) where $b - a < \infty$.

Denote by $(a_1, b_1), (a_2, b_2) \ldots$ components of the set $(a, b)\backslash B$, and by D the set of all numbers of the form $(a_m + a_n)/2$, $(b_m + b_n)/2$, $(a_m + b_n)/2$ where $m, n = 1, 2, \ldots$.

Since $B \cap S(A) \subset [(B\backslash D) \cap S(A)] \cup D$, and the set D is at most countable, the assertion of 1.6 will be proved when we show the equality

$$(B\backslash D) \cap S(A) = \emptyset. \tag{5}$$

Let $x \in (B\backslash D) \cap S(A)$. The point x is not isolated in the set A since, in the contrary case, $x = b_m$, $x = a_n$ and $x = (b_m + a_n)/2 \in D$. Being an accumulation point of the set A, thus of the set B, x is a limit from at least one side, say from the right, of some subsequence (a_{k_n}) of the sequence (a_n). Since $x \in S(A)$, therefore, for n sufficiently large, we have $(2x - b_{k_n}, 2x - a_{k_n}) \cap A = \emptyset$ because $(a_{k_n}, b_{k_n}) \cap A = \emptyset$.

It is easily seen that the interval $(2x - b_{k_n}, 2x - a_{k_n})$ is, for a sufficiently large n, some interval (a_m, b_m). Hence $a_m = 2x - b_{k_n}$ and $x = (a_m + b_{k_n})/2 \in D$, which contradicts the relation $x \notin D$. We have justified relation (5) and, thereby, the assertion is proved.

1.7. *If a set A is measurable (in the sense of Lebesgue), then the set $S(A)$ is measurable.*

Proof. It can easily be seen that $S(A)$ is the set of all those points at which the symmetric derivative $D1_A(x) = 0$ and, at the same time, $S(A) = \{x :$ the finite $D1_A(x)$ exists$\}$, where 1_A is the characteristic function of the set A. In virtue of a theorem from paper [1] (see e.g. Theorem 2.3), the set $S(A)$ is therefore measurable.

We shall now give two examples in which the set $S(A)$ is non-measurable or of cardinality continuum and measure zero.

1.8. *There exists a set $A \subset R$ for which the set $S(A) = A$ is non-measurable.*

Proof. Denote by H a (Hamel) basis of a linear space R over the field Q of rational numbers, such that $1 \in H$. Every real number a has exactly one representation of the form

$$a = \sum_{h \in H} q_h \cdot h$$

where the number of coefficients $q_h \in Q$ different from zero is finite. Let us adopt

$$A = \Big\{a = \sum_{h \in H} q_h \cdot h : q_1 = 0\Big\}.$$

It can easily be proved that the sets A and $R \setminus A$ are dense and non-measurable, A being a linear subspace of the space R. Hence we infer that $S(A) = A$.

1.9. *There exists a Borel set A of cardinality continuum and measure zero, such that $S(A) = A$.*

Proof. In paper [2] (see also [6]) the author built (Lemma 3) an example of a linear space $A \subset R$ over the field of rational numbers which is a Borel set of cardinality continuum and measure zero. Since $S(A) = A$, therefore $S(A)$ is a Borel set of cardinality continuum and measure zero.

2. Locally symmetric sets

DEFINITION. A set $A \subset R$ is called locally symmetric if each point of the set A is its point of local symmetry, i.e. if $A \subset S(A)$.

2.1. *The intersection of a finite number of locally symmetric sets is a locally symmetric set.*

Indeed, if the sets $A_1, A_2, \ldots, A_n$ are locally symmetric, then $A_j \subset S(A_j)$ for $j = 1, 2, \ldots, n$ and, in virtue of Theorem 1.2, we have $\bigcap_{j=1}^{n} A_j \subset S(\bigcap_{j=1}^{n} A_j)$, that is, the set $\bigcap_{j=1}^{n} A_j$ is locally symmetric.

2.2. *The intersection of infinitely many locally symmetric sets need not be a locally symmetric set. There also exist locally symmetric sets whose complements are not locally symmetric sets. Similarly, the union and the difference of two locally symmetric sets need not be locally symmetric sets.*

Indeed, putting $A_n = \bigcup_{k=1}^{\infty}(k^{-1} - n^{-3}, k^{-1} + n^{-3}) \cup (-n^{-3}, n^{-3})$, we obtain open, thus locally symmetric, sets for which $\bigcap_{n=1}^{\infty} A_n = \{0, 1, 1/2, 1/3, \ldots\}$ is not a locally symmetric set.

The sets Q and $A = \{q\sqrt{2} : q \in Q\}$ are locally symmetric, whereas the set $Q \cup A$ has only one point of local symmetry - zero. The set $R \backslash Q$ is not locally symmetric, either.

2.3. *For any set $A \subset R$, the sets $S(A), S(A) \cap A$ and $S(A) \backslash A$ are locally symmetric.*

The above assertion follows from Theorems 1.3 and 1.4.

2.4. *For each set $A \subset R$, the locally symmetric set $S(A)$ is the union of two locally symmetric sets $S(A) \cap A$ and $S(A) \backslash A$.*

2.5. *If a set A is locally symmetric,then $A \cap A^d$ where A^d stands for the derivative of the set A is a locally symmetric set.*

For the set $A = (0, 1)$, the derivative $A^d = [0, 1]$ is not locally symmetric set.

Theorem 2.5 is a corollary from more general Theorem 2.8.

2.6. *If $A \subset S(B) \cap S(C)$, then $A \subset S(B \backslash C)$. In particular, for $A = B \backslash C$ or $A = B$, we have $B \backslash C \subset S(B \backslash C)$, that is, the set $B \backslash C$ is locally symmetric.*

2.7. *Every locally symmetric set is the union of two locally symmetric sets: of a dense-in-itself set and a dispersed (thus countable) one.*

Proof. The locally symmetric set $A = B \cup C$ where B is the largest dense-in-itself subset of A, and $C = A \backslash B$. Suppose that, for some point x of the set A and each number $a > 0$, there exists a number $h_a \in (-a, a)$ such that $x + h_a \in B$ and $x - h_a \notin B$. Since, for some $d > 0$, the set $A \cap (x - d, x + d)$ has its symmetry centre at x, therefore the symmetric reflection B' of the set $B \cap (x - d, x + d)$ with respect to x is a dense-in-itself subset of A. From this we obtain the relation $x - h_d \in B' \subset B$ contradictory with the condition $x - h_d \notin B$. Consequently, the set B is locally symmetric and, what is more, $A \subset S(B)$. Since $A \subset S(A)$ and $A \subset S(B)$, therefore, in virtue of Theorem 2.6, we have $A \backslash B \subset A \subset S(A \backslash B)$, which means that the set C is locally symmetric.

Let W be a condition (predicate function) defined for certain real numbers. The symbol $W(x)$ will mean that a number x satisfies the condition W.

DEFINITION. We say that a set $A \subset R$ satisfies the condition W locally symmetrically if and only if

$$\forall_{x \in A} \exists_{d > 0} \forall_{|h| < d} [W(x + h) \Rightarrow W(x - h)].$$

Every locally symmetric set satisfies locally symmetrically, among other things, the following conditions:

(1) A point is an accumulation point of a set.
(2) A point is an isolated point of a set.
(3) A point is a condensation point of a set.
(4) A point is an (outer) density point of a set.
(5) A point is a boundary point of a set.

2.8. *If a set $A \subset S(A)$ satisfies the condition W locally symmetrically, then the sets*

$$\{x \in A : W(x)\} \quad \textit{and} \quad \{x \in A :\sim W(x)\}$$

are locally symmetric.

Proof. Let us adopt $B = \{x \in A : W(x)\}$. For each $x \in A$, some $d_x > 0$ and each h where $|h| < d_x$, we have the implications

$$x + h \in A \Rightarrow x - h \in A \quad \text{and} \quad W(x+h) \Rightarrow W(x-h)$$

from which

$$x + h \in B \Rightarrow x - h \in B.$$

We have shown that $A \subset S(B)$, whence we get the inclusion $B \subset S(B)$. Consequently, the set B is locally symmetric. Since $A \subset S(A)$ and $A \subset S(B)$, therefore, in virtue of Theorem 2.6, $A \backslash B \subset A \subset S(A \backslash B)$, which means that the set $\{x \in A :\sim W(x)\}$ is locally symmetric.

2.9. *If a set A is closed or open, then the set $S(A)$ is the union of two locally symmetric sets: of an open set and a countable one.*

Proof. Let A be a closed set, $B = (R \backslash A) \cup \operatorname{Int} A$ and $C = S(A) \backslash B$. Then $S(A) = B \cup C$, and B is a open subset of $S(A)$. Moreover, $C = S(A) \cap A \backslash \operatorname{Int} A$. By Theorem 1.5, $C = [S(A) \cap A] \cap [S(\operatorname{Int} A) \backslash \operatorname{Int} A]$. Hence, according to 2.3 and 2.1, the set C is locally symmetric. The inclusion $C \subset A \backslash \operatorname{Int} A = \operatorname{Fr} A$ implies that C is nowhere dense, and, by the inclusion $C \subset C \cap S(C)$, on the ground of Theorem 1.6, the set C is countable.

The case when A is an open set reduces to the above one by help of Theorem 1.1.

2.10. *Every locally symmetric and measurable set A is the union of an open set $B = \operatorname{Int} A$ and a locally symmetric set C of measure zero, disjoint from it.*

Proof. The characteristic function 1_A of the set A is measurable. It is easily seen that $S(A) = \{x : D1_A(x) = 0\} = \{x : D1_A(x)$ exists and is finite$\}$ where $Df(x)$ denotes the symmetric derivative of the function f at the point x.

Put

$$E = \{x : 1'_A(x) \quad \text{exists and is finite}\}.$$

We have

$$E = \{x : 1'_A(x) = 0\} \subset S(A).$$

In virtue of Theorem 2.3 of paper [1], the set $S(A)\backslash E$ has measure zero:

$$m[S(A)\backslash E] = 0.$$

Put $B = A \cap E$ and $C = A\backslash E$. Then $A = B \cup C$ and $B \cap C = \emptyset$, the set B being open. Indeed, if $x \in B = A \cap E$, then $1_A(x) = 1$ and $1'_A(x) = 0$. Hence it appears that $1_A(t) = 1$ in some neighbourhood U of the point x. Thus the set $U \subset A \cap E = B$, and the set B is open. Of course, $B = \operatorname{Int} A$.

Since $A \subset S(A)$, therefore $0 \leq m[A\backslash(A \cap E)] \leq m[S(A)\backslash E] = 0$, which means that $m(C) = 0$.

Now, we shall demonstrate that the set $C = A\backslash \operatorname{Int} A$ is locally symmetric. Let $x \in C$. Then $x \in A \subset S(A)$ and $x \notin \operatorname{Int} A$. The condition $x \in S(A)$ implies the existence of a number $d > 0$ such that, for each h satisfying the condition $|h| < d$, the equivalence $x + h \in A \iff x - h \in A$ holds. Hence we deduce that if $x + h \in A\backslash \operatorname{Int} A$, then $x - h \in A\backslash \operatorname{Int} A = C$. In consequence, $x \in S(C)$, which yields $C \subset S(C)$.

As a corollary we obtain the following theorem:

2.11. *Every measurable and boundary locally symmetric set is of measure zero.*

Theorem 1.9 implies that there exist Borel locally symmetric sets of cardinality continuum and measure zero. We shall now show

2.12. *There exists a non-Borel locally symmetric set of cardinality continuum and measure zero.*

Proof. Let A be the set from the proof of 1.9. Then A is a linear space over the field Q. Moreover, A is a set of cardinality continuum and measure zero. Denoting by B a basis of the space A, we ascertain that it is of cardinality c (continuum). So, it contains 2^c subsets of cardinality c. Any two distinct subsets of those ones determine distinct linear subspaces of A. Hence it appears that A contains 2^c linear subspaces of cardinality c. Among them there are, of course, non-Borel linear subspaces. Let C be such a subspace. Then the set C is of measure zero, of cardinality continuum and non-Borel, with that $C = S(C)$, whence it appears that C is a locally symmetric set.

2.13. *A set A is locally symmetric if and only if the set $A\backslash \operatorname{Int} A$ is locally symmetric and contained in the set $S(\operatorname{Int} A)$.*

Proof. The necessity of the condition follows from Theorems 2.6 and 1.5. To prove its sufficiency, let us assume that the set $A\setminus \operatorname{Int} A \subset S(A\setminus \operatorname{Int} A)$, and $A\setminus \operatorname{Int} A \subset S(\operatorname{Int} A)$. Then, for any point x from the set $A\setminus \operatorname{Int} A$, there exists a neighbourhood U such that x is the centre of symmetry of the sets $U \cap (A\setminus \operatorname{Int} A)$ and $U \cap \operatorname{Int} A$. Thus x is the centre of symmetry of the set $U \cap A$. We have demonstrated that $A\setminus \operatorname{Int} A \subset S(A)$. Since $\operatorname{Int} A \subset S(A)$, therefore $A \subset S(A)$, q.e.d.

References

[1] F. M. Filipczak, *Sur la structure de l'ensemble des points où une fonction continue n'admet pas de dérivée symétrique*, Dissertationes Math. 130, 1–49.

[2] F. M. Filipczak, *Sur les dérivées symétriques des fonctions continues*, Colloq. Math. 34 (2) (1976), 249–256.

[3] S. Marcus, *Les ensembles F_σ et la continuité symétrique*, Bull. St. Acad. R.P.R., série Mathématique (1955).

[4] S. Marcus, *Sur un problème de F. Hausdorff concernant les fonctions symétriques continues*, Bull. Acad. Polon. Sc. C1.III, IV.4 (1956), 201–205.

[5] S. Marcus, *Sur un problème de la théorie de la mesure de H. Steinhaus et S. Ruziewicz*, ibid. (1956), 197–199.

[6] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Mathematische Ann. 99 (1928), 134–141.

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