DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 3 1995

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ON FUNCTIONS WITH QUASI-COMPANION

1. Introduction

P. M. Kogge [2] considered recurrences of the form $x_i = f(a_i, x_{i-1})$ where the $x_i (i = 0, \ldots, n)$ belong to a set X and the a_i belong to a set A of parameters. He showed that such recurrences may be evaluated in parallel if there exists a companion function $g(a, b)$ satisfying the transformation equation

$$f(a, f(b, x)) = f(g(a, b), x)$$

In [5] a function $f : A \times X \to X$ is called a recurrence function and it is shown that parallel evaluation of the corresponding recurrence is also possible if there exist mappings $\varphi : A \to A$ and $\psi : X \to X$, and a quasi-companion $g : A \times A \to A$ with respect to these mappings defined by the functional equations

$$\text{(q1)} \qquad f(a, f(b, x)) = f(g(a, \varphi(b)), \psi(x))$$

$$\text{(q2)} \qquad \varphi(g(a, b)) = g(\varphi(a), \varphi(b))$$

$$\text{(q3)} \qquad \psi(f(a, x)) = f(\varphi(a), \psi(x)).$$

For real valued functions defined on real intervals the general solution of these equations is given under the following additional assumptions (see [5] theorem 8.6, page 97):

(a1) $f(a, x)$ is strictly monotonic in both variables and continuous in x,

(a2) $g(a, b)$ is strictly monotonic and continuous in both variables,

(a3) ψ is surjective.

In this note we shall determine the solution without assuming the surjectivity of ψ (cf. section 2, theorem 1).

Once the equations are solved the question arises naturally as to what extent they are independent. From (q1) and (q3) alone it is easy to derive the following pair of functional equations for g and φ if we still assume condition (a1):

$$\text{(g1)} \qquad \varphi[g(a,\varphi(b))] = g[\varphi(a),\varphi^2(b)]$$

$$\text{(g2)} \qquad g[a,g(\varphi(b),\varphi^2(c))] = g[g(a,\varphi(b)),\varphi^2(c)].$$

To prove (g1) apply ψ to the left and right-hand side of (q1) and use (q3) and the monotonicity of f. To prove (g2) compute $f(a,f(b,f(c,x)))$ in two ways by first working from inside outwards and then from outside inwards. It follows that $g[a,\varphi(g(b,\varphi(c)))] = g[g(a,\varphi(b)),\varphi^2(c)]$ and applying (g1) we obtain (g2).

We could not solve this pair of equations, not even under the assumptions that g satisfies (a2) and φ is strictly monotonic. Only when φ is assumed continuous can we show that g is a quasi-addition. Nevertheless, using this restricted result it is possible to circumvent the problem by a change of equation (q1) suggested by J. Aczél :

$$\text{(q1*)} \qquad f[a,f(b,x)] = f[G(a,b),\psi(x)].$$

If we assume continuity for $G(a,b)$ it is possible to apply our restricted result on the pair of equations (g1) and (g2) and to solve in this way the simultaneous equations (q1*) and (q3) (cf. section 4, theorem 2).

It is equally natural to ask whether equation (q3) was really needed in theorem 1 or theorem 2. Equation (q1*) may be studied as a special case of the generalized associativity equation. In view of the considerable literature on the latter (see e.g. [3], [4]) it might be possible to find its solution without using (q3). This is however, an open question.

In section 2 we give the proof of theorem 1. Section 3 is devoted to the solution of the pair of equations (g1) and (g2) which will be the basis of the proof of theorem 2 given finally in section 4.

2. Equations (q1), (q2), and (q3)

The following theorem gives the complete solution of the simultaneous equations (q1), (q2), and (q3) without assuming the surjectivity of ψ. It should be noted that in cases i), iia), and iiia) of the conclusion of theorem 1 below the mapping ψ turns out to be surjective anyway. In cases iib) or iiib) however, ψ is surjective only when $\mathbf{d} = 0$. Thus our theorem differs from the corresponding theorem 8.6 of [5] mainly in two points: firstly the hypothesis that ψ be surjective is dropped and secondly in the conclusion the possibility that $\mathbf{d} > 0$ is admitted in cases iib) and iiib).

THEOREM 1. *Let A and X be real intervals. Assume that f satisfies condition* (a1) *and that g is continuous in each variable. Under these conditions* *the complete solution of the functional equations* (q1), (q2), *and* (q3) *is as follows*:

$$\begin{aligned} f(a,x) &= \alpha(\beta^{-1}(a)+\alpha^{-1}(\psi(x))) = \alpha(\beta^{-1}(a)+\mathbf{c}\alpha^{-1}(x)+\mathbf{d}) \\ g(a,b) &= \beta(\beta^{-1}(a)+\beta^{-1}(b)) \\ \psi(x) &= \alpha(\mathbf{c}\alpha^{-1}(x)+\mathbf{d}) \\ \varphi(a) &= \beta(\mathbf{c}\beta^{-1}(a)). \end{aligned}$$

Here $\alpha : I_1 \to X$ *and* $\beta : I_0 \to A$ *are strictly monotonic and continuous mappings onto* X *and* A *respectively*, $\mathbf{c}$ *and* $\mathbf{d}$ *are constants*, *and* I_0, I_1, $\mathbf{c}$, *and* $\mathbf{d}$ *are of one of the following forms* (*we denote by* $< l, u)$ *an interval which may be open or closed at its lower end*):

i)	$I_0 = I_1 = \Re$,	$\mathbf{c} \neq 0$, *otherwise* $\mathbf{c}$	*and* $\mathbf{d}$ *are arbitrary*,
iia)	$I_0 = \; < 0,\infty)$,	$I_1 = \Re, \mathbf{c} > 0$,	$\mathbf{d}$ *is arbitrary*,
iib)	$I_0 = \; < 0,\infty)$,	$I_1 = \; < 0,\infty)$,	$\mathbf{c} > 0,\ \mathbf{d} \geq 0$,
iiia)	$I_0 = \; < L,\infty)$,	$I_1 = \Re, L > 0$,	$\mathbf{c} \geq 1, \mathbf{d}$ *is arbitrary*,
iiib)	$I_0 = \; < L,\infty)$,	$I_1 = \; < 0,\infty)$,	$L > 0, \mathbf{c} \geq 1, \mathbf{d} \geq 0$.

Proof of Theorem 1. Before going into the details of the proof it is worth noting that the functions satisfying the hypothesis of theorem 1 are invariant under a large number of transformations (denoted by $f \to \tilde{f}, g \to \tilde{g}$, etc.).

(i) Conjugation with respect to X. Let $\sigma : X \to X_1$ be a continuous bijection and define $\tilde{f}(a,x) = \sigma(f(a,\sigma^{-1}(x)))$, $\tilde{g} = g$, $\tilde{\psi} = \sigma\psi\sigma^{-1}$, $\tilde{\varphi} = \varphi$.

(ii) Conjugation with respect to A. Let $\sigma : A \to A_1$ be a continuous bijection and define $\tilde{f}(a,x) = f(\sigma^{-1}(a),x)$, $\tilde{g}(a,b) = \sigma(g(\sigma^{-1}(a),\sigma^{-1}(b)))$, $\tilde{\varphi} = \sigma\varphi\sigma^{-1}$, $\tilde{\psi} = \psi$.

It is easily checked that the functions occurring in the conclusion of theorem 1 are also invariant under the same transformations. This observation will occasionally be useful in the proof of the theorem: in trying to show that if functions f, g, φ, ψ satisfy the hypothesis they will be of the form as stated in the conclusion, it will be possible to transform them suitably so as to establish whatever property of them may seem desirable. In this way it will be possible to avoid some tedious case distinctions.

It is easy to check that the functions described in the theorem provide solutions of equations (q1), (q2), and (q3). Conversely, we have to show that any solution must have the form described, provided f and g satisfy the assumptions of the theorem.

This part of the proof will be achieved by a series of simple steps. The basic ideas are the same as in the proof of theorem 8.6 of [5] but the lacking surjectivity of ψ causes certain complications. Some of the steps especially up to (2.5) have identical or very similar counterparts in [5].

From now on throughout this section let f, g, φ, and ψ denote functions satisfying the conditions of the theorem.

(2.1) *The functions φ and ψ are strictly monotonic* (cf. [5], *p.* 98).

There exist inverse functions f_1^{-1} and f_2^{-1} defined by $a = f_1^{-1}(y, x)$ and $x = f_2^{-1}(a, y)$ if, and only if, $y = f(a, x)$. Note that f_1^{-1} and f_2^{-1} are strictly monotonic in the first, respectively the second argument.
From (q1) it follows that $\psi(x) = f_2^{-1}(g(a, \varphi(b)), f(a, f(b, x)))$. Hence ψ is strictly monotonic.
From (q3) it follows that $\varphi(a) = f_1^{-1}(\psi(f(a, x)), \psi(x))$. Hence φ is strictly monotonic.

(2.2) *The function ψ is continuous* (*cf.* [5] *p.* 100).

Since $f(a, x)$ is continuous in x the composed function $f(a, f(b, x))$ is continuous in x. The function f_2^{-1} is continuous in the second argument. Since $\psi(x) = f_2^{-1}(g(a, \varphi(b)), f(a, f(b, x)))$ it follows that ψ is continuous.

(2.3) (Definition). *Let $B = \varphi(A)$ and $C = \varphi(B)$.*

(2.4) *g is strictly monotonic in both arguments and associative, i.e. $g[a, g(b, c)] = g[g(a, b), c]$ for $a, b, c \in A$.* (*Compare* [5], *p.* 100).

We show first that g is strictly monotonic. From (q1) it follows that

$$g(a, b_1) = f_1^{-1}(f[a, f(\varphi^{-1}(b_1), x)], \psi(x))$$

where $b_1 = \varphi(b)$. Since φ^{-1}, f, and f_1^{-1} are strictly monotonic functions it follows that $g(a, b_1)$ is strictly monotonic in both variables on $A \times B$ and hence a *fortiori* on $B \times B$. From (q2) and (2.1) it follows that $g(a, b) = g(\varphi^{-1}(a_1), \varphi^{-1}(b_1)) = \varphi^{-1}(g(a_1, b_1))$ where $a_1 = \varphi(a)$ and as before $b_1 = \varphi(b)$. This implies that g is strictly monotonic in both variables on $A \times A$.

Let us now deal with the associativity. From (q2) it follows that $g(B, B) \subseteq B$ and $g(C, C) \subseteq C$. Hence from (g2) it follows that g is an associative operation on C. But now (2.1) implies that φ is an isomorphism of B onto C with respect to the operation g on both sets. Therefore g is associative on B and in a similar way it may be concluded that g is associative on A.

Therefore g is a quasi-addition (see [1], p. 253), i.e. there exists a monotonic and continuous mapping $\beta : I_0 \to A$ such that

$$g(a,b) = \beta(\beta^{-1}(a) + \beta^{-1}(b)).$$

Here we may assume that I_0 is an interval having one of the forms as stated in Theorem 1.

(2.5) $\varphi(a) = \beta(\mathbf{c}\beta^{-1}(a))$ *for some suitable constant* $\mathbf{c} \neq 0$ (*cf.* [5] *p.* 101).

For $a^* \in I_0$ define $\varphi^*(a^*) = \beta^{-1}(\varphi(\beta(a^*)))$. Setting $a = \beta(a^*), b = \beta(b^*)$ we obtain $\varphi^*(a^* + b^*) = \beta^{-1}[\varphi(\beta[\beta^{-1}(a) + \beta^{-1}(b)])] = \beta^{-1}[\varphi(g(a,b))] = \beta^{-1}[g(\varphi(a), \varphi(b))] = \beta^{-1}[\beta[\beta^{-1}(\varphi[\beta(a^*)]) + \beta^{-1}(\varphi[\beta(b^*)])]] = \varphi^*(a^*) + \varphi^*(b^*)$.

From this we may conclude that $\varphi^*(a^*) = \mathbf{c} \cdot a^*$ for a suitable non-zero constant $\mathbf{c}$. This is essentially Cauchy's theorem. Only in place of the continuity of φ^* we have to use the monotonicity which is possible according to an argument going back to G. Darboux (c.f. [1], p. 33). From the definition of φ^* we obtain (2.5).

(2.6) $g(A,B) \subseteq B$.

Let $a \in A, b \in B$, i.e. $b = \varphi(a_1)$ for some $a_1 \in A$. We know that $g(a,b) = \beta(\beta^{-1}(a) + \beta^{-1}(b))$ and $\varphi(a_1) = \beta[\mathbf{c}\beta^{-1}(a_1)] = b$. Thus

$$g(a,b) = \beta[\beta^{-1}(a) + \beta^{-1}(\beta[\mathbf{c}\beta^{-1}(a_1)])] = \beta(\beta^{-1}(a) + \mathbf{c}\beta^{-1}(a_1)).$$

Let $a^* = \beta^{-1}(a), a_1^* = \beta^{-1}(a_1)$. Then $a^* + \mathbf{c}a_1^* = \mathbf{c}a_2^*$ for a suitable element $a_2^* \in I_0$ and so

$$g(a,b) = \beta(\mathbf{c}a_2^*) = \beta(\mathbf{c}\beta^{-1}(a_2)) \quad \text{where } a_2 = \beta(a_2^*).$$

This means that $g(a,b) = \varphi(a_2)$ so that $g(a,b) \in B$ which proves (2.6).

(2.7) (Definition). Set $U = \psi(X)$ and define a new function $F : A \times U \to X$ by $F(a,u) = f(a, \psi^{-1}(u))$.

As ψ is continuous, U is a subinterval of X (i.e. convex). The range of F coincides with the range of f since $\psi^{-1}(u)$ runs through the whole set X as u runs through U.

We omit the easy proofs of the next three statements.

(2.8) *The function* $F(a,u)$ *is strictly monotonic in both arguments and continuous in* u.

(2.9) $\psi^{-1}(f(b,u)) = f(\varphi^{-1}(b), \psi^{-1}(u))$ *for any* $b \in B, u \in U$ (*cf.* [5] *p.* 102).

(2.10) $\psi(F(a,u)) = F(\varphi(a), \psi(u))$ *for all* $a \in A, u \in U$ (*cf.* [5], *p.* 103).

(2.11) $f(A, f(A,X)) \subseteq \psi(f(A,X)) \subseteq U$.

Let $a, b \in A$. We have $f(a, f(b, x)) = f(g(a, \varphi(b)), \psi(x)) =$ (using 2.6) $f(\varphi(a_2), \psi(x)) = \psi(f(a_2, x)) \in U$ which proves (2.11).

(2.12) (Definition). *Set $f(A, f(A, X)) = S$ so that by the previous assertion $S \subseteq U$. Let T denote the convex hull of S, i.e. $t \in T$ whenever $t \in S$ or there exist $s_1, s_2 \in S$ and $s_1 < t < s_2$. Since we know that U is an interval (i.e. convex) we know as well that $T \subseteq U$.*

(2.13) $F(a, t) \in T$ for any $t \in T, a \in A$.

We show first that $F(a, s) \in S$ for all $s \in S$. In fact if $s \in S$ by (2.11) for suitable $a_1 \in A, x \in X$ we have $s = \psi[f(a_1, x)]$ and so

$$F(a, s) = F(a, \psi[f(a_1, x)]) = f(a, \psi^{-1}(\psi[f(a_1, x)])) = f(a, f(a_1, x)) \in S.$$

Now let $t \in T$. If $t \in S$ we have just shown that $F(a, t) \in S \subseteq T$. Hence assume t is not in S. Then there exist $s_1, s_2 \in S$ such that $s_1 < t < s_2$.

As F is monotonic in the second argument we have either

$$F(a, s_1) < F(a, t) < F(a, s_2) \quad \text{or} \quad F(a, s_2) < F(a, t) < F(a, s_1)$$

and furthermore, as we have shown, $F(a, s_1)$ and $F(a, s_2)$ belong to S. This proves that $F(a, t)$ belongs to the convex hull of S which is T.

(2.14) $\psi(T) \subseteq T$.

We show first that $\psi(S) \subseteq S$. For let $s = f(a, f(b, x)) \in S$. Then

$$\psi[f(a, f(b, x))] = f(\varphi(a), \psi[f(b, x)]) = f(\varphi(a), f(\varphi(b), \psi(x))) \in S.$$

Now T was defined to be the convex hull of S and ψ is monotonic. Therefore $\psi(T) \subseteq T$.

(2.15) *$F(a, F(b, u)) = F(g(a, b), u)$ whenever $F(b, u) \in U$. (Compare* [5], *p.* 102).

Assume that $F(b, u) \in U$ so that the expression $F(a, F(b, u))$ makes sense. Let $a_1 = \varphi(a), b_1 = \varphi(b)$, so that $a = \varphi^{-1}(a_1), b = \varphi^{-1}(b_1)$. By (2.7) we get

$$F(b, u) = f(b, \psi^{-1}(u)) = f(\varphi^{-1}(b_1), \psi^{-1}(u)) = \psi^{-1}[f(b_1, u)].$$

By our assumption $\psi^{-1}[f(b_1, u)]$ is an element of U and since ψ maps U into itself, $f(b_1, u)$ also belongs to U. By (2.7) again

$$F(a, F(b, u)) = f(\varphi^{-1}(a_1), \psi^{-1}[\psi^{-1}[f(b_1, u)]]) = \qquad \text{using (2.9)}$$
$$\psi^{-1}[f(a_1, \psi^{-1}[f(b_1, u)])] = \psi^{-1}[f(\varphi^{-1}(a_2), \psi^{-1}[f(b_1, u)])]$$

where $a_2 = \varphi(a_1)$. Using again (2.9) we obtain

$$\psi^{-1}[f(\varphi^{-1}(a_2), \psi^{-1}[f(b_1,u)])] = \psi^{-2}[f(a_2, f(b_1,u))] = \quad \text{by (q1)}$$
$$\psi^{-2}[f(g(a_2,\varphi(b_1)),\psi(u))] =$$
$$\psi^{-2}[f(g(\varphi(a_1),\varphi(b_1)),\psi(u))] = \quad \text{by (q2)}$$
$$\psi^{-2}[f(\varphi[g(a_1,b_1)],\psi(u))] = \quad \text{by (2.9)}$$
$$\psi^{-1}[f(g(a_1,b_1),u)) = \psi^{-1}[f(g(\varphi(a),\varphi(b)),u)] = \quad \text{by (q2)}$$
$$\psi^{-1}[f(\varphi[g(a,b)],u)] = \quad \text{by (2.9)}$$
$$f(g(a,b),\psi^{-1}(u)) = F(g(a,b),u).$$

On T the function F satisfies all requirements needed in Aczél's treatment of the transformation equation. Hence there exist an interval J_1 and a continuous monotonic mapping α of J_1 onto T such that

(2.16) $F(a,t) = \alpha(\beta^{-1}(a) + \alpha^{-1}(t))$ *for* $t \in T$.

J_1 has one of the forms:

a) $J_1 = \Re$ (for I_0 (i), (ii), or (iii) may hold),
b) $J_1 = < 0, \infty)$ (for I_0 only (ii) or (iii) are possible).

(2.17) *For* $t \in T$ *we have* $\psi(t) = \alpha(\mathbf{c}\alpha^{-1}(t) + \mathbf{d})$ *for a suitable constant* $\mathbf{d}$.

Consider the mappings
$\varphi^* = \beta^{-1}\varphi\beta : I_0 \to I_0$ and $\psi^* = \alpha^{-1}\psi\alpha : J_1 \to J_1$.
Let $a^* = \beta^{-1}(a), x^* = \alpha^{-1}(x)$. Then

$$\begin{aligned}\psi^*(a^* + x^*) &= \alpha^{-1}\psi\alpha(\beta^{-1}(a) + \alpha^{-1}(x))\\ &= \alpha^{-1}(\psi[F(a,x)]) = \alpha^{-1}(F[\varphi(a),\psi(x)])\\ &= \alpha^{-1}(\alpha[\beta^{-1}(\varphi(a)) + \alpha^{-1}(\psi(x))])\\ &= \beta^{-1}\varphi\beta(a^*) + \alpha^{-1}\psi\alpha(x^*) = \varphi^*(a^*) + \psi^*(x^*).\end{aligned}$$

This Pexider type equation holds for $a^* \in I_0$ and $x^* \in J_1$. Moreover we know that $\varphi^*(a^*) = \mathbf{c} \cdot a^*$. Therefore $\psi^*(a^* + x^*) = \mathbf{c} \cdot a^* + \psi^*(x^*)$.

Choose x_1^* fixed in J_1. Then $\psi^*(a^* + x_1^*) = \mathbf{c} \cdot a^* + d_1$ for $a^* \in I_0$ where $d_1 = \psi^*(x_1^*)$. For arbitrary x^* choose $a^* \in I_0$ large enough so that $a^* + x^* = b^* + x_1^*$ with $b^* \in I_0$. It follows that $\psi^*(x^*) = \psi^*(a^* + x^*) - \mathbf{c} \cdot a^* = \psi^*(b^* + x_1^*) - \mathbf{c} \cdot a^* = \mathbf{c} \cdot b^* + d_1 - \mathbf{c} \cdot a^* = \mathbf{c} \cdot x^* + d_1 - \mathbf{c} \cdot x_1^* = \mathbf{c} \cdot x^* + \mathbf{d}$ where $\mathbf{d} = d_1 - \mathbf{c} \cdot x_1^*$. From this it follows that

$$\psi(x) = \alpha\psi^*\alpha^{-1}(x) = \alpha(\mathbf{c}\alpha^{-1}(x) + \mathbf{d}).$$

(2.18) (Notation). *Let* $u_\infty = \lim_{x\to\infty} \alpha(x)$ *and, in the case where* $J_1 = (-\infty, \infty)$, *let* $u_{-\infty} = \lim_{x\to-\infty} \alpha(x)$. *Note that* u_∞ *and* $u_{-\infty}$ *are end-points* *of T but do not belong to T and that u_∞ may be the upper or the lower end-point of T and it may also be ∞ or $-\infty$.*

(2.19) *Towards the side of u_∞ (and of $u_{-\infty}$ if it exists) the interval X is not larger than T. In particular u_∞ (and $u_{-\infty}$) are also end-points of X.*

On T we have determined $F(a,t)$ and $\psi(t)$ so we also have determined $f(a,t)$, namely

$$\begin{aligned} f(a,t) = F(a,\psi(t)) &= \alpha(\beta^{-1}(a) + \alpha^{-1}(\alpha(\mathbf{c}\alpha^{-1}(t) + \mathbf{d}))) \\ &= \alpha(\beta^{-1}(a) + \mathbf{c}\alpha^{-1}(t) + \mathbf{d}). \end{aligned}$$

Now as t tends to u_∞ its pre-image $\alpha^{-1}(t)$ tends to ∞ in J_1 and hence $\beta^{-1}(a) + \mathbf{c}\alpha^{-1}(t) + \mathbf{d}$ tends to ∞ or to $-\infty$ according as the constant $\mathbf{c}$ is positive or negative.

Let us consider the case of negative $\mathbf{c}$ first. This implies in particular that $J_1 = (-\infty, \infty)$ and that $f(a,t)$ tends to $u_{-\infty}$ as t tends to u_∞. Similarly it follows that $f(a,t)$ tends to u_∞ as t tends to $u_{-\infty}$. If u_∞ would belong to X then

$$f(a,u_\infty) = \lim_{t\to u_\infty} f(a,t) = u_{-\infty}$$

as we have just seen. This would be true for arbitrary $a \in A$ contradicting the hypothesis that f is strictly monotonic in its first argument. Hence u_∞ does not belong to X. Similarly $u_{-\infty}$ does not belong to X.

If $\mathbf{c}$ is positive then $f(a,t)$ tends to u_∞ as t tends to u_∞ and $f(a,t)$ tends to $u_{-\infty}$ as t tends to $u_{-\infty}$ (if $u_{-\infty}$ exists). If u_∞ (or $u_{-\infty}$) would be contained in X then since f is continuous in its second argument it would follow that $f(a,u_\infty) = u_\infty$ (and that $f(a,u_{-\infty}) = u_{-\infty}$). This contradicts again the hypothesis that f is strictly monotonic in its first argument. This proves (2.19).

By conjugation with respect to X we may henceforth assume without loss of generality that $X = \langle l, u_\infty)$ (which means that α is monotonic increasing).

(2.20) *Assume that $0 \in I_0$ and let a_0 be the element of A corresponding to 0 under β. Then $f(a_0,x) = \psi(x)$.*

For any a in A we have $g(a,a_0) = \beta(\beta^{-1}(a) + 0) = a$. We also have $\varphi(a_0) = \beta(\mathbf{c}\beta^{-1}(a_0)) = \beta(\mathbf{c}\cdot 0) = \beta(0) = a_0$. Using $f(a_0,x)$ as an argument in f we get

$$f(a, f(a_0,x)) = f(g(a,\varphi(a_0)),\psi(x)) = f(g(a,a_0),\psi(x)) = f(a,\psi(x)).$$

It follows that $f(a_0,x) = \psi(x)$.

(2.21) *$F(a,u) \in U$ for all $u \in U$.*

If $T = U$ this is obvious. Hence assume $T \subset U$. Let s be the lower end-point of T so that $T = \langle s, u_\infty)$. As $T \subset U$ we may infer that $s \in U$. Now $F(a,t) \in T$ for all $t \in T$ and since F is continuous in its second argument it follows that $F(a,s) \in T$ or $F(a,s) = s$. Thus on $T_1 = [s, u_\infty)$ the function F again satisfies all the assumptions of Aczél ([1], p. 316) and we may determine F on the interval $[s, u_\infty)$, i.e.

$$[F(a,t) = \alpha(\beta^{-1}(a) + \alpha^{-1}(t))$$

where α^{-1} is now defined on $[s, u_\infty)$. There is an important additional conclusion we may draw from this: since T_1 is closed at its lower end it follows that I_0 may only be $< 0, \infty)$ or $< L, \infty)$ where $L > 0$. Thus the case $I_0 = \Re$ is excluded by the assumption that $T \subset U$.

We shall assume here and in the following that β is monotonic increasing. This assumption means no restriction since it may be achieved by conjugation with respect to the set A. By the assumptions made previously α is increasing and therefore $F(a,t)$ is increasing in its first argument at least as long as $t \in T$. This implies that $F(a,u)$ must be increasing in its first argument regardless of u. Otherwise there would exist u such that $F(a,u) > F(b,u)$ for some $a < b$. We would have $F(a,u) - F(b,u) > 0$ and $F(a,t) - F(b,t) < 0$. Hence $F(a,u_1) - F(b,u_1) = 0$ for some u_1 in between u and t which is impossible.

The case that $I_0 = [0, \infty)$ would cause some special trouble in the main argument that follows. So let us consider this case first. As before let a_0 denote the element of A corresponding to 0. We have

$$F(a_0, u) = f(a_0, \psi^{-1}(u)) = \psi(\psi^{-1}(u)) = u.$$

Now for any a in A we have $a_0 \leq a$ and $F(a_0, u) \leq F(a, u)$ and so $F(a,u) \in U$.

Let us now return to the main argument where we may now assume that $I_0 = (0, \infty)$ or $I_0 = \langle L, \infty)$ for some $L > 0$. If I_0 is not of the form $[L, \infty)$ let us replace I_0 temporarily by a smaller set $I_0^* = [L, \infty)$ (where $L > 0$) and let $A_1 = [a_1, b)$ denote the corresponding subset of A. Thus A_1 definitely has a smallest element.

Let us consider any interval $T_2 \subseteq U$ containing T and such that $F(a,t) \in T_2$ whenever $t \in T_2$, and $a \in A_1$. Note that T and T_1 are such intervals. If T_2 does not contain its lower end-point s_2 but $s_2 \in U$, (i.e. $T_2 \subset U$) we may adjoin the end-point as before, since the continuity of F in its second argument ensures that $F(a, s_2) \in T_2 \cup \{s_2\}$.

Thus as long as $T_2 \subset U$ we may always assume that T_2 is closed at its lower end. We may apply Aczél's theorem to F with domain T_2 and obtain a representation of F as before

$$F(a,t) = \alpha(\beta^{-1}(a) + \alpha^{-1}(t))$$

for a in A_1 and t in T_2. Note that the assumption that α is increasing is still valid for the new α since otherwise $F(a,t)$ would no longer be increasing in its first argument.

From the fact that $I_0^* = [L, \infty)$ we may infer that $F(a_1, s_2) > s_2$. This fact is crucial in our argument. For let us now consider elements $s < s_2$. By the continuity of F in its second argument the inequality $F(a_1, s) > s$ must persist in some interval to the left of s_2. Thus we have $s < F(a,s)$ for some $s < s_2$ and for all $a \in A_1$. Thus $[s, u_\infty)$ again is an interval for which $F(a,t) \in [s, u_\infty)$ whenever $t \in [s, u_\infty)$.

That is to say we have shown that T_2 can always be extended to the left. It is now easy to see that we can reach every point of U by such an extension. For if the set Q that cannot be reached was not empty we could look at $\sup(Q) = q$. Then (q, u_∞) is an interval with the desired property that $F(a,t) \in (q, u_\infty)$ whenever $t \in (q, u_\infty)$ and $a \in A_1$.

Now we can join the end-point q to this interval and obtain $[q, u_\infty)$ and from the above we see that we could extend $[q, u_\infty)$ to the left leading to a contradiction. Hence Q is empty and we can reach all of U. This proves that $F(a,u) \in U$ for all $a \in A_1$.

These considerations can be made with any A_1 corresponding to arbitrary intervals of the form $I_0^* = [L_1, \infty)$ where $L_1 > 0$. Since the case $I_0 = [0, \infty)$ was treated beforehand as a special case the union of these intervals I_0^* is I_0 and the proof of (2.21) is finished.

We now obtain the complete representation of F on U.

(2.22) $F(a,u) = \alpha_1(\beta^{-1}(a) + \alpha_1^{-1}(u))$ *for* $u \in U$.

As before $\alpha_1 : J_1 \to X$ is strictly monotonic and continuous and J_1 has one of the forms:

a) $J_1 = \Re$ (for I_0 (i), (ii), or (iii) may hold)
b) $J_1 = \langle 0, \infty)$ (for I_0 only (ii) or (iii) are possible).

(2.23) *For* $u \in U$ *we have* $\psi(u) = \alpha_1(c\alpha_1^{-1}(u) + \mathrm{d})$.

The proof is similar to the proof of (2.17) and need not be repeated.

It is now easy to finish the proof of Theorem 1. All we have to do is to extend the definition of the mapping α_1 in a suitable way so that (2.23) becomes valid for the whole set X.

Note that when $J_1 = \Re$ we have $\psi(U) = U$ and hence $\psi(X) = X$, i.e. $U = X$. In this case there is nothing to show and we may set $I_1 = J_1$.

Thus we may assume that $J_1 = \langle 0, \infty)$. Let $J_1^* = \langle -\mathbf{c}^{-1}\mathbf{d}, \infty)$ so that $\psi^* : x^* \to \mathbf{c} \cdot x^* + \mathbf{d}$ maps the interval J_1^* onto $J_1 = \langle 0, \infty)$.

For $x^* \in J_1^* \setminus J_1$ we may extend the definition of α_1 by the rule $\alpha_1(x^*) = \psi^{-1}\alpha_1(\psi^*(x^*))$.

It follows that

$$\psi(x) = \psi(\alpha_1[\alpha_1^{-1}(x)]) = \alpha_1(\psi^*[\alpha_1^{-1}(x)])$$
$$= \alpha_1[\mathbf{c}\alpha_1^{-1}(x) + \mathbf{d}] \quad \text{for } x \in X \setminus U.$$

We still need an adjustment to be able to replace the interval J_1^* by $I_1 = \langle 0, \infty)$ as required in Theorem 1. Consider the map $\psi^* : x^* \to \mathbf{c}x^* + \mathbf{d}$ so that $\psi^{*-1} : x^* \to \mathbf{c}^{-1}x^* - \mathbf{c}^{-1}\mathbf{d}$, and let $\varphi^* : a^* \to \mathbf{c} \cdot a^*$. We may replace β by $\beta\varphi^{*-1}$ since this change does not affect the representation of g and φ. Moreover, if we replace α_1 by $\alpha = \alpha_1\psi^{*-1}$ then α maps $I_1 = \langle 0, \infty)$ onto X and it follows by an easy calculation that $F(a, u) = \alpha(\beta^{-1}(a) + \alpha^{-1}(u))$, and $\psi(x) = \alpha(\mathbf{c}\alpha^{-1}(x) + \mathbf{d})$.

Hence we get a representation for f, g, φ , and ψ as required in Theorem 1 also in the case when $J_1 = \langle 0, \infty)$. The proof of Theorem 1 is now finished.

3. Equations $(g1)$ and $(g2)$

In this section we shall prove a lemma on functions φ, g simultaneously satisfying the two equations (g1) and (g2).

LEMMA 1. *Let A be a real interval. Let $\varphi : A \to A$ and $g : A \times \varphi(A) \to A$ be functions satisfying equations $(g1)$ and $(g2)$ and assume that φ is continuous, φ^3 is not constant, and g is monotonic and continuous in both arguments. Then*

$$g(a,b) = \beta(\beta^{-1}(a) + \beta^{-1}(b)) \quad \textit{for } a \in A, \quad b \in B = \varphi(A), \quad \textit{and}$$
$$\varphi(a) = \beta(\mathbf{c}\beta^{-1}(a)) \quad \textit{for } a \in A.$$

Here $\mathbf{c} \neq 0$ is a constant, β is a monotonic and continuous bijective mapping of a suitable interval I_0 onto A and I_0 satisfies the conventions of Theorem 1.

Proof of Lemma 1. The lemma will be proved by the following steps (3.1)–(3.20).

(3.1) (Denotation). *From now on throughout this section we shall use the operator $\circ$ instead of g. Thus $a \circ b$ simply stands for $g(a, b)$.*

As in the previous section let $B = \varphi(A)$ and $C = \varphi(B)$. Note that B and C are proper intervals (not single points). This follows from the hypothesis that φ is continuous and φ^3 is not constant.

The two equations (g1) and (g2) may obviously be restated as follows.

(3.2) $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$ *whenever* $a \in A, b \in B$,

(3.3) $a \circ (b \circ c) = (a \circ b) \circ c$ *whenever* $a \in A, b \in B, c \in C$.

The next step is proved by an argument almost identical to the one in [1], p. 255–256.

(3.4) *$a \circ b$ is strictly increasing in both arguments.*

Assume there existed u_0 such that for $v_1 < v_2$ we would have $v_1 \circ u_0 > v_2 \circ u_0$ or in other words $v_1 \circ u_0 - v_2 \circ u_0 > 0$. Then each u in A would have this property.

Otherwise there would exist $u \in A$ such that $v_1 \circ u - v_2 \circ u < 0$. For reasons of continuity there would exist a value u_1 in between u_0 and u such that $v_1 \circ u_1 - v_2 \circ u_1 = 0$ contradicting the strict monotonicity. Now let us choose v_1, v_2, u, t in C. Then it follows that $v_2 \circ (u \circ t) < v_1 \circ (u \circ t) = (v_1 \circ u) \circ t < (v_2 \circ u) \circ t$ which is a contradiction. The proof for the second argument is analogous.

(3.5) (Denotation). *Let* $H_0 = C$ *and* $H_{n+1} = H_n \circ C$. *Let* $H = \cup H_n$.

It will be shown that H is a semigroup with respect to the operation $\circ$.

(3.6) *H has the following three properties* (i) $H \subseteq B$, (ii) $B \circ H \subseteq B$, *and* (iii) $a \circ (b \circ c) = (a \circ b) \circ c$ *whenever* $a \in A, b \in B$, *and* $c \in H$.

We prove by induction that H_n has the three properties stated for H. Then the assertion will follow simply because H is the union of all H_n.

First for H_0 the required properties follow immediately from (3.3) and (3.2). Assuming that H_n has the required properties for $a \in A, b \in B, t \in H_n$, and $c \in C$ we see that $a \circ (b \circ [t \circ c]) = a \circ ([b \circ t] \circ c) = (a \circ [b \circ t]) \circ c = ([a \circ b] \circ t) \circ c = [a \circ b] \circ [t \circ c]$ which proves the third property for H_{n+1}. The first property follows since H_n has it, and since $C = H_0$ has the second property. To verify the second property we conclude from $b \circ [t \circ c] = [b \circ t] \circ c$ and from the second property for H_n that $b \circ [t \circ c] \in B$.

(3.7) *H is a semigroup.*

It follows by induction on i that $t \circ x \in H_{n+i}$ for $t \in H_n$ and $x \in H_i$. Hence $H \circ H \subseteq H$. The associativity follows from 3.6 iii).

(3.8) *Let* $h \in H$. *From* $h \circ h = h$ *it follows* $x \circ h = x$ *for all* $x \in A$.

For let $x \in A$. Then $(x \circ h) \circ h = x \circ (h \circ h) = x \circ h$. Hence $x \circ h = x$.

(3.9) *Let $h \in H$. From $h \circ h > h$ it follows $x \circ h > x$ for all $x \in A$.*

For let $x \in A$. Then $(x \circ h) \circ h = x \circ (h \circ h) > x \circ h$. Hence $x \circ h > x$.

(3.10) *Let $h \in H, h \circ h > h$ and let $a \in A$. Define $a_0 = a$ and $a_{n+1} = a_n \circ h$. Then the sequence a_n is strictly increasing and tends towards the upper end of the interval A. Moreover A is open at its upper end.*

From (3.9) we get $a_{n+1} = a_n \circ h > a_n$. Thus the sequence a_n is increasing. Suppose the sequence a_n would have a limit v belonging to the interval A. Then $v \circ h > v$ by (3.9) and $a_n \circ h$ tends to $v \circ h$. But on the other hand $a_n \circ h = a_{n+1}$ which tends to v.

Assertions (3.9)–(3.10) have obvious counterparts with ">" exchanged by "<". In the following we shall freely use not only (3.9)–(3.10) but also any of their counterparts as we need them.

(3.11) (Denotation). *Let P, E, and N denote the following sets of elements of the set A:*

$$P = \{a : a \circ a > a\},$$
$$E = \{e : e \circ e = e\},$$
$$N = \{a : a \circ a < a\}.$$

(3.12) *The set $E \cap H$ contains at most one element e.*

Let $e_1, e_2 \in E \cap H$. By (3.8) we have $x \circ e_1 = x = x \circ e_2$ for an arbitrarily chosen $x \in A$. This implies $e_1 = e_2$.

As a consequence of (3.12) and the fact that C contains more than one element we note

(3.13) *C contains at least one element of P or of N.*

(3.14) *When $H \cap P \neq \emptyset$ the interval B has its upper end-point in common with A. Similarly, when $H \cap N \neq \emptyset$ the interval B has its lower end-point in common with A.*

Let $h \in H \cap P$ and consider the sequence a_i of (3.10) for $a = h$. It follows that $a_i \in H$ and since $H \subseteq B$ we may conclude that B has its upper end-point in common with A. If $H \cap N \neq \emptyset$ the proof is similar.

(3.15) *The interval C has at least one of its end-points in common with the interval A.*

If $A = B$ then $B = C$ and there is nothing to show. Thus we may assume that $C \subset B \subset A$ and that A and B have e.g. their upper end-point in common. This assumption implies that $H \cap N = \emptyset$.

If for all $b \in B$ we have $\varphi(b) \geq b$ it is obvious that $C = \varphi(B)$ will also have its upper end-point in common with A. Hence we may assume that $\varphi(b) < b$ for some $b \in B$. For an element a close enough to the lower end-point of A it follows that $\varphi(a) > a$. Hence there exists an element u such that $\varphi(u) = u$. This implies that $u \in B$ and $\varphi(u) \in C$, i.e. $u \in C$. Moreover we have $\varphi(u \circ u) = \varphi(u) \circ \varphi(u) = u \circ u$ and similarly $\varphi(u \circ u \circ u) = u \circ u \circ u$ etc. so that the entire sequence $u, u \circ u, \ldots$ is contained in C. If $u \in P$ it follows that C extends unto the upper end of the interval A. Since $u \in N$ is impossible we may assume that $u \in E$, i.e. $u \circ u = u$.

Then $u = u \circ u = u \circ u \circ u = \ldots$ is contained in H_i for $i = 0, 1, 2, \ldots$ (see (3.5)). Since each H_i is an interval it follows that $H = \cup H_i$ is an interval. Since $H \cap P \neq \emptyset$ it follows that H extends unto the upper end of A.

On the interval H the operation $\circ$ is associative and hence a quasi-addition: $a \circ b = \beta_1(\beta_1^{-1}(a) + \beta_1^{-1}(b))$ for a, b in H. The function φ maps each H_i into itself and hence it is an endomorphism of H. But then $\varphi|_H$ must have the form $\varphi|_H(h) = \beta_1(\mathbf{c}\beta_1^{-1}(h))$ for a suitable constant $\mathbf{c}$ (see (2.5)). The case $\mathbf{c} = 0$ may be excluded because φ^3 is not constant. But now it follows that $\varphi|_H(H) = H$ and thus $C = H$. We have proved that C extends unto the upper end-point of A.

(3.16) $C \circ C \subseteq C$.

When C contains elements from P and from N it follows that H contains elements arbitrarily close to the upper and to the lower end-point of A. From this it follows that $A = B$ and hence $B = C$. Consequently $C \circ C \subseteq C$.

Thus we may assume that $C \cap N = \emptyset$ or $C \cap P = \emptyset$. Let us consider e.g. the case that $C \cap N = \emptyset$. When C has its lower end-point in common with A from $C \cap P \neq \emptyset$ we may again conclude that H contains elements arbitrarily close to the upper end-point of A and therefore $A = B$ and $B = C$. Hence we may assume that $C \subset A$ and that C has its upper end-point in common with A.

Let now $c_1, c_2 \in C$. Then $c_2 \circ c_2 \geq c_2$ since c_2 is not contained in the set N. It follows that $c_1 \circ c_2 \geq c_1$ and thus $c_1 \circ c_2 \in C$.

In the case that $C \cap P = \emptyset$ the argument is analogous.

(3.17) *The mapping φ is injective.*

Let a, b be elements such that $\varphi(a) = \varphi(b)$. If $C \cap P \neq \emptyset$ we may choose $h \in C$ such that $h \circ h > h$. For a suitable non-negative integer i the elements $a_i = a \circ h \ldots \circ h$ and $b_i = b \circ h \ldots \circ h$ will belong to C. It follows that $\varphi(a_i) = \varphi(b_i)$. Since on C the operation $\circ$ is a quasi-addition and φ is not constant on C we may conclude that $a_i = b_i$. This is only possible if $a = b$. Thus φ is injective.

If $C \cap N \neq \emptyset$ the proof is similar.

(3.18) $B \circ B \subseteq B$.

Let b_1 and b_2 be elements in B and $a = b_1 \circ b_2$. Then $\varphi(a) = \varphi[b_1 \circ b_2] = \varphi(b_1) \circ \varphi(b_2) = c_1 \circ c_2 \in C$, whence $\varphi(a) \in C$. From this it follows that $a \in B$.

(3.19) *The operation* $\circ$ *is associative on* B. *Therefore there exists a monotonic and continuous bijection* $\beta_1 : I_0 \to B$ *such that*

$$b \circ c = \beta_1[\beta_1^{-1}(b) + \beta_1^{-1}(c)].$$

The mapping $\varphi : B \to C$ is an isomorphism with respect to the operation $\circ$. As the operation $\circ$ is associative on C it follows that it must be associative on B.

In a similar way as in (2.5) we may conclude from this:

(3.20) *On* B *the mapping* φ *admits of a representation* $\varphi|_B(b) = \beta_1[\mathbf{c}\beta_1^{-1}(b)]$ *for all* $b \in B$.

To finish the proof of the lemma we consider the two structures $S_1 = [B \supseteq C, \circ]$ and $S_2 = [A \supseteq B, \circ]$ where $\circ$ is a partial operation defined on $B \times C$ or $A \times B$ respectively. The mapping $\chi = \varphi^{-1}$ is a monotonic and continuous isomorphism from S_1 onto S_2.

Let $\beta = \chi\beta_1$ so that $\beta : I_0 \to A$. Let $a \in A, b \in B$, and $a = \chi(a_1), b = \chi(b_1)$ where $a_1 \in B, b_1 \in C$. From the representation of $\circ$ on B as a quasi-addition we obtain

$$\begin{aligned} a \circ b = \chi(a_1) \circ \chi(b_1) &= \chi[a_1 \circ b_1] = \chi[\beta_1(\beta_1^{-1}(a_1) + \beta_1^{-1}(b_1))] \\ &= \chi\beta_1[\beta_1^{-1}\chi^{-1}(a) + \beta_1^{-1}\chi^{-1}(b)] = \beta[\beta^{-1}(a) + \beta^{-1}(b)]. \end{aligned}$$

Since $\chi = \varphi^{-1}$ we have $\varphi = \chi\varphi\,|_B\,\chi^{-1}$. Hence from (3.20) we obtain

$$\begin{aligned} \varphi(a) &= \chi\varphi\,|_B\,\chi^{-1}(a) = \chi\varphi\,|_B\,(a_1) = \chi\beta_1[\mathbf{c}\beta_1^{-1}(a_1)] \\ &= \chi\beta_1[\mathbf{c}\beta_1^{-1}\chi^{-1}(a)] = \beta[\mathbf{c}\beta^{-1}(a)]. \end{aligned}$$

4. Equations $(q1^*)$ and $(q3)$

THEOREM 2. *Let* A *and* X *be real intervals. If condition* (a1) *is assumed for the function* $f : A \times X \to X$ *and continuity in each variable for* $G : A \times A \to A$ *then the complete solution of equations* $(q1^*)$ *and* $(q3)$ *is as follows:*

$$f(a, x) = \alpha(\beta^{-1}(a) + \alpha^{-1}(\psi(x))) = \alpha(\beta^{-1}(a) + \mathbf{c}\alpha^{-1}(x) + \mathbf{d}),$$

$$G(a, b) = \beta(\beta^{-1}(a) + \mathbf{c}\beta^{-1}(b)),$$

$$\psi(x) = \alpha(\mathbf{c}\alpha^{-1}(x) + \mathbf{d}),$$

$$\varphi(a) = \beta(\mathbf{c}\beta^{-1}(a)).$$

Here $\alpha : I_1 \to X$ and $\beta : I_0 \to A$ are strictly monotonic and continuous mappings, **c** *and* **d** *are constants and the cases to be considered for the intervals I_1 and I_0 and the constants are the same as in Theorem* 1.

Proof of Theorem 2. Theorem 2 will be reduced to Theorem 1 by constructing a function g satisfying the hypotheses of Theorem 1.

(4.1) $\varphi[G(a,b)] = G[\varphi(a), \varphi(b)]$ *for* a, b *in* A.

Computing $\psi[f(a, f(b,x))]$ by first applying equation (q3) we get

$$\begin{aligned}\psi[f(a,f(b,x))] &= f[\varphi(a), \psi[f(b,x)]]\\ &= f[\varphi(a), f[\varphi(b), \psi(x)]] = f(G[\varphi(a), \varphi(b)], \psi^2(x)).\end{aligned}$$

On the other hand by using first equation (q1*) we get

$$\psi[f(a, f(b,x)] = \psi[f(G(a,b), \psi(x))] = f[\varphi[G(a,b)], \psi^2(x)].$$

Since f is strictly monotonic in the first variable this proves (4.1).

(4.2) $G[a, G(b,c)] = G[G(a,b), \varphi(c)]$ *for all* a, b, c *in* A.

We compute $f(a, f[b, f(c,x)])$ in two ways first working "from inside outwards" and secondly in the reverse order. We get

$$f(a, f[b, f(c,x)]) = f(a, f[G(b,c), \psi(x)]) = f(G[a, G(b,c)], \psi^2(x)).$$

On the other hand by working "from outside inwards" we get

$$\begin{aligned}f(a, f[b, f(c,x)]) &= f(G[a,b], \psi[f(c,x)])\\ &= f(G[a,b], f[\varphi(c), \psi(x)]) = f(G(G[a,b], \varphi(c)), \psi^2(x)).\end{aligned}$$

Since f is strictly monotonic in the first variable we get (4.2).

(4.3) φ *and* ψ *are strictly monotonic and* ψ *is continuous.*

The proof of (4.3) is similar to the proof of the corresponding assertions in section 2 and we shall not repeat it here.

(4.4) φ *is continuous. In particular* $B = \varphi(A)$ *and* $C = \varphi(B)$ *are intervals.*

Since $G(a,b)$ is continuous in both arguments there is an inverse funtion G_2^{-1} such that $b = G_2^{-1}(a,c)$ if and only if $c = G(a,b)$. It follows from (4.2) that $\varphi(c) = G_2^{-1}[G(a,b), G[a, G(b,c)]]$. Since G is continuous in its second argument it follows that G_2^{-1} is continous in its second argument and so it follows that φ is continuous.

(4.5) (Definition). *For* $a \in A$ *and* $b \in B = \varphi(A)$ *we define a new function with values in* A *by setting* $g(a,b) = G[a, \varphi^{-1}(b)]$.

(4.6) *The new function g satisfies the following pair of functional equations*

(g1) $\varphi[g(a,\varphi(b))] = g[\varphi(a),\varphi^2(b)]$ *and*

(g2) $g[g(a,\varphi(b)),\varphi^2(c)] = g[a,g(\varphi(b),\varphi^2(c))]$ *for all* $a,b,c \in A$.

Applying the definition of g and (4.1) we get

$$\begin{aligned}\varphi[g(a,\varphi(b))] &= \varphi[G(a,\varphi^{-1}(\varphi(b)))] = G[\varphi(a),\varphi[\varphi^{-1}(\varphi(b))]] \\ &= G[\varphi(a),\varphi^{-1}(\varphi^2(b))] = g[\varphi(a),\varphi^2(b)]\end{aligned}$$

which proves (g1).

Again using the definition, equation (g1), and assertion (4.2) we may transform the righthand side of (g2) as follows:

$$\begin{aligned}g[a,g(\varphi(b),\varphi^2(c))] &= g[a,\varphi(g(b,\varphi(c)))] = G[a,\varphi^{-1}\varphi(g(b,\varphi(c)))] \\ &= G[a,g(b,\varphi(c))] = G[a,G(b,c)] = G[G(a,b),\varphi(c)] \\ &= G[G(a,\varphi^{-1}(\varphi(b))),\varphi^{-1}(\varphi^2(c))] = g[g(a,\varphi(b)),\varphi^2(c)]\end{aligned}$$

which proves (g2).

(4.7) *$g(a,b)$ is strictly monotonic in both arguments.*

Since φ is strictly monotonic so is φ^{-1}. Hence the assertion follows if it can be shown that $G(a,b)$ is strictly monotonic in both variables. But this follows since

$$G(a,b) = f_1^{-1}[f(a,f(b,x)),\psi(x)].$$

(4.8) *$g(a,b)$ is continuous in both arguments.*

For the first argument this follows from the hypothesis that $G(a,b)$ is continuous in the first argument. For the second argument it follows from the continuity of $G(a,b)$ in the second argument and from the fact that the inverse function φ^{-1} of φ is continuous.

It is now possible to finish the proof of Theorem 2. From the lemma proved in the previous section we may conclude that $g(a,b)$ is a quasi-addition, i.e. $g(a,b) = \beta(\beta^{-1}(a) + \beta^{-1}(b))$ for $a \in A$, $b \in B$ and that $\varphi(a) = \beta(\mathbf{c}\beta^{-1}(a))$ for $a \in A$. It is therefore obvious that we may extend the definition of the function g to $A \times A$ in such a way that $g(a,b) = \beta[\beta^{-1}(a) + \beta^{-1}(b)]$ for $a,b \in A$. If we do this φ satisfies the equation $\varphi(g(a,b)) = g(\varphi(a),\varphi(b))$ for arbitrary a,b in A, i.e. it satisfies equation (q2).

It follows that f,g,φ , and ψ satisfy the hypotheses of Theorem 1 and so it is easy to reduce the proof of Theorem 2 to Theorem 1.

References

[1] J. Aczél, *Lectures on Functional Equations and their Application*, Academic Press, 1966.

[2] P. M. Kogge, *Parallel Solutions of Recurrence Problems*, IBM J. Res. Development, March 1974, 138–148.

[3] A. Krapež, *Generalized Associativity on Groupoids*, Publ. Math. Beograd, 28 (42), (1980), 105–112.

[4] F. Radó, *Equations fonctionnelles caractérisantes les nomogrammes avec trois échelles rectilignes*, Mathematica Cluj, Vol. 1 (24), 1, 1959, pp. 143–166.

[5] J. M. Targonski, *Recurrences with companion and their parallel evaluation*, PhD thesis, Universitat de les Illes Balears, Spain, 1992.

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Received February 4, 1994.