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# ON SOME PROPERTIES OF SOLUTIONS OF THE EQUATION $x^{(n)} + ax = 0$

## 1. Introduction

The differential equation $x^{(2)} + ax = 0$ for $|a| = 1$ with initial conditions $(x_0, x_1, x_0', x_1') = (1, 0, 0, 1)$ has the fundamental sequence of solutions $(\cos t, \sin t)$ for $a = 1$ and $(\cosh t, \sinh t)$ for $a = -1$. These solutions satisfy, in particular, the equalities $x_0^2 + x_1^2 \operatorname{sgn} a = 1$, $x_1(t+u) = x_1(t)x_0(u) + x_0(t)x_1(u)$.

In this paper we generalize some of results of J. Mikusiński [1], [2]; we present some properties of solutions $x_0, x_1, \ldots, x_{n-1}$ of the equation

$$(1) \qquad x^{(n)} + ax = 0 \; a \neq 0, \; n \geq 2,$$

satisfying the conditions

$$(2) \qquad x_s^{(p)}(0) = r^p \delta_s^p, \; p, s = 0, 1, \ldots, n-1,$$

where $r = \sqrt[n]{|a|}$ and $\delta_s^p$ is Kronecker's delta.

## 2. General solution, sequence of particular solutions, trigonometric identities

The equation (1) has $n$ linearly independent solutions $x_k^* = e^{a_k t}$, $k = -, 1, \ldots, n-1$, $a_k = r(\cos b_k + i \sin b_k)$, $nb_k = 2k\pi + \arg(-a)$. Thus the general solution of (1) is an $n$-parameter class of functions $x^*(t) = \sum_{k=0}^{n-1} C_k x_k^*$, where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants.

Define for (1) the sequence $x_0, x_1, \ldots, x_{n-1}$ of solutions satisfying the conditions (2). Such a sequence is unique and infinitely differentiable.

From this definitions it follows that

$$(3) \qquad x_0 = \frac{1}{n} \sum_{k=0}^{n-1} e^{a_k t}, \quad x_s = \frac{-r^s}{a} x_0^{(n-s)}, \quad s = 0, 1, \ldots, n-1,$$

since

$$\sum_{k=0}^{n-1} a_k^m = a_0^m \sum_{k=0}^{n-1}\left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right)^k = -an\delta_m^n, \quad m = 1,2,\ldots,n.$$

So the sequence of real solutions of (1), satisfying (2), is

$$x_s = \frac{-r^s}{an} re \sum_{k=0}^{n-1} a_k^{n-s} e^{a_k t}$$

which, by (3), becomes

$$x_s = \frac{1}{n}\sum_{k=0}^{n-1} e^{rt\cos b_k}\cos(rt\sin b_k - sb_k) \tag{4}$$

and, besides, the following relations hold

$$\sum_{k=0}^{n-1} e^{rt\cos b_k}\sin(rt\sin b_k - sb_k) = 0, \quad s = 0,1,\ldots,n-1. \tag{5}$$

If $n = 2$ and $a = -4$ the relations (4) and (5), for $s = 0,1$, are of the form $x_s = (1-s)\cosh t + s\sinh t$ and

$$\sum_{k=0}^{1} e^{2t\cos b_k}\sin(2t\sin b_k - sb_k) = 0,$$

respectively.

The $p$th derivative, $p = 0,1,\ldots,n-1$, of the solution (4) is

$$x_s^{(p)} = \frac{r^p}{n}\sum_{k=0}^{n-1} e^{rt\cos b_k}\cos(rt\sin b_k + (p-s)b_k).$$

Hence, by (2), we get $\sum_{k=0}^{n-1}\cos((p-s)b_k) = n\delta_s^p$.

### 3. Evenness and oddness of the solutions (4)

For $\frac{n+1}{2} \in N$ the solutions (4) can neither be even nor odd, because each of them satisfies the equation (1). On the other hand, for $\frac{n}{2} \in N$,

$$x_0 = \frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1}(\cosh(rt\cos b_k)\cos(rt\sin b_k)).$$

Hence, by (3), $x_0, x_2, \ldots, x_{n-2}$ are even and $x_1, x_3, \ldots, x_{n-1}$ are odd. The evenness and oddness of $x_s$ for $\frac{n}{2} \in N$ also result from (4), which can be written in the form

$$x_s = \frac{1}{n}\sum_{k=0}^{\frac{n}{2}-1}(e^{rt\cos b_k}\cos(rt\sin b_k - sb_k) + (-1)^s e^{rt\cos b_k}\cos(rt\sin b_k + sb_k)), \quad s = 0,1,\ldots,n-1.$$

**4. Properties of the solutions (4) in the neighbourhood of $t = 0$**

By (2), we can observe that

1. if $\frac{n}{2} \in N$, then at $t = 0$; $x_0$ has extremum equal to 1 (minimum for $a < 0$ and maximum for $a > 0$), $x_2, x_4, \ldots, x_{n-2}$ have minimum equal to 0 and $x_1, x_3, \ldots, x_{n-1}$ have an inflexion point,

2. if $\frac{n-1}{2} \in N$, then at $t = 0$ $x_0$ has an inflexion point, $x_1$ has neither extremum nor an inflexion point, $x_2, x_4, \ldots, x_{n-1}$ have minimum equal to 0 and $x_3, x_5, \ldots, x_{n-2}$ have an inflexion point.

**5. Wronskian for equation (1)**

If $W(t)$ denotes the Wronskian of the solutions (4), then $W'(t) = 0$ and $W(0) = r^{\binom{n}{2}}$. Hence we have, for every $t \in R$, the identity

$$\begin{vmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_0' & x_1' & \ldots & x_{n-1}' \\ \ldots & \ldots & \ldots & \ldots \\ x_0^{(n-1)} & x_1^{(n-1)} & \ldots & x_{n-1}^{(n-1)} \end{vmatrix} = r^{\binom{n}{2}}$$

which for $n = 2$ and $|a| = 1$ takes the form $x_0^2 + x_1^2 \operatorname{sgn} a = 1$.

**6. Functions depending on the sum of arguments**

A function $f(t)$ satisfying the equation (1) and the conditions $f^{(s)}(0) = c_s$, $s = 0, 1, \ldots, n-1$, is of the form $f(t) = \sum_{k=0}^{n-1} c_k r^{-k} x_k(t)$. For given $u$ and for $s = 0, 1, \ldots, n-1$ the function $g(t) = x_0(t+u)$ satisfies $g^{(s)}(0) = x_0^{(s)}(u)$. From this it follows that

$$x_0(t+u) = \sum_{k=0}^{n-1} r^{-k} x_0^{(k)}(u) x_k(t).$$

For $s = 0, 1, \ldots, n-1$ we differentiate this equation $n-s$ times with respect to $t$ and $u$ and obtain the relations

$$(6) \qquad x_s(t+u) = \frac{-r^s}{a} \sum_{k=0}^{n-1} r^{-k} x_0^{(k)}(u) x_k^{(n-s)}(t),$$

and

$$(7) \qquad x_s(t+u) = \frac{-r^s}{a} \sum_{k=0}^{n-1} r^{-k} x_0^{(n-s+k)}(u) x_k(t)$$

implying

$$x_s(2t) = \frac{-r^s}{a} \sum_{k=0}^{n-1} r^{-k} x_0^{(k)}(t) x_k^{(n-s)}(t)$$

and

$$x_s(2t) = \frac{-r^s}{a} \sum_{k=0}^{n-1} r^{-k} x_0^{(n-s+k)}(t) x_k(t),$$

respectively. In particular, both these equalities for $n = 2$, $a = -2$, $s = 0$ and $t\sqrt{2} = q$ take the form $\cosh 2q = \cosh^2 q + \sinh^2 q$.

## 7. Functions depending on the difference of arguments

If $\frac{n}{2} \in N$, then (6) and (7) imply

$$(8) \quad x_s(t-u) = \frac{-r^s}{a} \sum_{k=0}^{n-1} (-1)^k r^{-k} x_0^{(k)}(u) x_k^{(n-s)}(t)$$

and

$$(9) \quad x_s(t-u) = \frac{-r^s}{a} \sum_{k=0}^{n-1} (-1)^{k+s} r^{-k} x_0^{(n-s+k)}(u) x_k(t),$$

respectively. If we put $u = t$ and $s = 0$ in (8), (9), then, by (2), we get the equality

$$\sum_{k=0}^{n-1} (-1)^k r^{-k} x_0^{(k)}(t) x_k(t) = 1$$

which, for $n = 2$ and $a = 1$, has the form $\cos^2 t + \sin^2 t = 1$.

## 8. Power series for the solution (4)

The functions (4) satisfy the problem (1), (2), hence their power series expansion is of the form

$$x_s(t) = r^s \sum_{k=0}^{\infty} \frac{(-a)^k}{(kn+s)!} t^{kn+s}, \quad s = 0, 1, \ldots, n-1.$$

These series are convergent for every $t \in R$.

### References

[1] J. Mikusiński, *Trigonometry of the differential equation* $x^{(3)} + x = 0$, Wiadomości Mat. 2 (1959), 207–227. (in Polish).

[2] J. Mikusiński, *Trigonometry of the differential equation* $x^{(4)} + x = 0$, Wiadomości Mat. 4 (1960), 73–84. (in Polish).

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