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A FIXED POINT THEOREM FOR m-WEAK** COMMUTING MAPPINGS

1. Introduction

Let be R^+ the set of non-negative reals, N the set of positive integers and (X,d) a complete metric space. Consider the set F of all real functions $f : R^+ \to R^+$ satisfying the following properties:

(i) f is upper semi-continuous,
(ii) f is non-decreasing in each coordinate variables,
(iii) $f(t) < t$ for any $t > 0$.

THEOREM A [1]. *Let $S,T : X \to X$ be continuous. Then S, T have a common fixed point w if and only if there exist two self-mappings A, B of X and a function $f \in F$ such that*

(1.1) $A(X) \cup B(X) \subset S(X) \cap T(X)$,

(1.2) *both A and B commute with S and T,*

(1.3) $$d(Ax,By) \le f(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{1}{2}[d(Sx,By)+d(Ty,Ax)]\}), \quad x,y \in X.$$

Further, w is the unique common fixed point of A,B,S,T.

2. THEOREM B [2]. *Let A,B,S,T be four self mappings of (X,d) such that*

(2.1) $A^2(X) \subset T^2(X)$ *and* $B^2(X) \subset S^2(X)$,

(2.2) $$d(A^2x,B^2y) \le f(\max\{d(S^2x,T^2y), d(S^2x,A^2x), d(T^2y,B^2y), \tfrac{1}{2}[d(S^2x,B^2y)+d(T^2y,A^2x)]\}), \quad x,y \in X,$$

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where f satisfies (i), (ii), (iii). *If one of A, B, S, T is continuous and if A and B weak** commute with S and T, respectively, then A, B, S, T have common fixed point z. Further, z is the unique common fixed point of A, B, S, T.*

The intent of the present paper is to improve Theorems A, B with the use of concept of m-weak** commuting pair of mappings, by modifying and extending the definition of weak** commuting mapping introduced in [4] as defined below.

DEFINITION 1. Two self-maps A, S of (X, d) are called m-weak** commuting, if $A(X) \subset S(X)$ and

(iv) $d(A^m S^m x, S^m A^m x) \leq d(A^m Sx, SA^m x) \leq d(AS^m x, S^m Ax) \leq d(S^m x, A^m x)$ for all $x \in X$, and $m \in N$.

Clearly, two commuting mappings also commute m-weak**, but not necessarily conversely as it is shown in the following example.

EXAMPLE 1. Let $X = [0, 1]$ with euclidean metric d and let A, B, S and T be defined, for all $x \in X$, as $Ax = \frac{x}{x+2}$, $Bx = \frac{x}{x+6}$, $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$, respectively. Then $A(X) = [0, \frac{1}{3}] \subset [0, \frac{1}{2}] = S(X)$ and

$$\begin{aligned}
&d(A^m S^m x, S^m A^m x)\\
&= \frac{x}{(2^m - 1)x + 2^{2m}} - \frac{x}{(2^m - 1)2^m x + 2^{2m}}\\
&= \frac{x^2(2^{2m} - 2^{m+1} + 1)}{[(2^m - 1)x + 2^{2m}][(2^m - 1)2^m x + 2^{2m}]}\\
&\leq \frac{x^2(2^{m+1} - 2^m - 1)}{[(2^m - 1)x + 2^{m+1}][2(2^{m-1} - 1)x + 2^{m+1}]}\\
&= \frac{x}{(2^m - 1)x + 2^{m+1}} - \frac{x}{2(2^m - 1)x + 2^{m+1}} = d(A^m Sx, SA^m x)\\
&\leq \frac{x^2(2^m - 1)}{(x + 2^{m+1})(2^m x + 2^{m+1})} = \frac{x}{x + 2^{m+1}} - \frac{x}{2^m x + 2^{m+1}}\\
&= d(AS^m x, S^m Ax) = \frac{x^2(2^m - 1)}{(x + 2^{m+1})(2x + 2^m)} \leq \frac{x^2}{(x + 2^m)(2x + 2^m)}\\
&\leq \frac{(2m - 1)x^2}{2^m[(2m - 1)x + 2^m]} = \frac{x}{2^m} - \frac{x}{(2m - 1)x + 2^m} = d(A^m x, S^m x).
\end{aligned}$$

Hence, we conclude that $d(A^m S^m x, S^m A^m x) \leq d(A^m Sx, SA^m x) \leq d(AS^m x, S^m Ax) \leq d(A^m x, S^m x)$ for any $x \in X$. But, for any $x \neq 0$ we have

$$AS^m x = \frac{x}{x + 2^{m+1}} > \frac{x}{2^m x + 2^{m+1}} = S^m Ax.$$

3. THEOREM 1. *Let A, B, S, T be four self-mappings of (X, d) such that*

(3.1) $A^m(X) \subseteq T^m(X),\ B^m(X) \subseteq S^m(X)$,

(3.2) $$d(A^m x, B^m y) \leq f(\max\{d(S^m x, T^m y), d(S^m x, A^m x), d(T^m y, B^m y), \tfrac{1}{2}[d(S^m x, B^m y) + d(T^m y, A^m x)], \tfrac{1}{2}[d(S^m x, B^m y) + d(S^m x, T^m x)]\}),$$

for all $x, y \in X$, $m \in N$, where f satisfies (i), (ii), (iii). *If one of A, B, S, T is continuous and if A and B m-weak** commute with S and T, respectively, then A, B, S, T have unique common fixed point z.*

Proof. Let x_0 be an arbitrary point of X and x_1, x_2 in X such that $A^m x_0 = T^m x_1$, $B^m x_1 = S^m x_2$. This can be done, since (3.1) holds. According to Fisher [3], we can inductively define a sequence

(3.3) $$A^m x_0, B^m x_1, A^m x_2, B^m x_3, \ldots, A^m x_{2n}, B^m x_{2n+1}, \ldots$$

such that $A^m x_{2n} = T^m x_{2n+1}$, $B^m x_{2n+1} = S^m x_{2n+2}$ for each integer $n \in N \cup \{0\}$. Employing the method of proof due to Singh and Meade [5], we state that (3.3) is a Cauchy sequence and thus it converges to a point z. Suppose that S is continuous. Since the sequences $\{A^m x_{2n}\} = \{T^m x_{2n+1}\}$ and $\{B^m x_{2n-1}\} = \{S^m x_{2n}\}$ converge also to z, we have that the sequence $\{SA^m x_{2n}\}$ converges to Sz. Besides, A being weak** commuting with S, we deduce

$$\begin{aligned} d(A^m S x_{2n}, Sz) &\leq d(A^m S x_{2n}, SA^m x_{2n}) + d(SA^m x_{2n}, Sz) \\ &\leq d(S^m x_{2n}, A^m x_{2n}) + d(SA^m x_{2n}, Sz) \end{aligned}$$

which implies that $\{S^{m+1} x_{2n+1}\}$ converges to Sz, as $n \to \infty$.

Now, using (3.2) and the fact that $\{S^{m+1} x_{2n+1}\}$ converges to Sz, we have

$$\begin{aligned} d(A^m S x_{2n}, B^m x_{2n+1}) \leq f(\max\{&d(S^{m+1} x_{2n}, T^m x_{2n+1}), \\ &d(S^{m+1} x_{2n}, A^m S x_{2n}), d(T^m x_{2n+1}, B^m x_{2n+1}), \\ &\tfrac{1}{2}[d(S^{m+1} x_{2n}, B^m x_{2n+1}) + d(T^m x_{2n+1}, A^m S x_{2n})], \\ &\tfrac{1}{2}[d(S^{m+1} x_{2n}, B^m x_{2n+1}) + d(S^{m+1} x_{2n}, T^m x_{2n+1})]\}), \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Sz, z) &\leq f(\max\{d(Sz, z), d(Sz, Sz), d(z, z), \tfrac{1}{2}[d(Sz, z) + d(z, Sz)], \\ &\tfrac{1}{2}[d(Sz, z) + d(Sz, z)]\}) \leq f(\max\{d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)\}) \\ &\leq f(d(Sz, z)) \leq d(Sz, z) \end{aligned}$$

which implies $Sz = z$, by property (iii), and so $S^m z = z$. Now,

$$\begin{aligned} d(A^m z, B^m x_{2n+1}) \leq f(\max\{&d(S^m z, T^m x_{n+1}), d(S^m z, A^m z), \\ &d(T^m x_{2n+1}, B^m x_{2n+1}), \tfrac{1}{2}[d(S^m z, B^m x_{2n+1}) + d(T^m x_{2n+1}, A^m z)], \\ &\tfrac{1}{2}[d(S^m z, B^m x_{2n+1}) + d(S^m z, T^m x_{2n+1})]\}). \end{aligned}$$

Letting $n \to \infty$ and using (iii), we have

$$d(A^m z, z) \le f(\max\{d(z,z), d(z, A^m z), d(z,z), \tfrac{1}{2}[d(z,z) + d(z, A^m z)], \tfrac{1}{2}[d(z,z) + d(z,z)]\}) \le f(\max\{0, d(z, A^m z), 0, \tfrac{1}{2}d(z, A^m z), 0\}) \le f(d(A^m z, z)) < d(A^m z, z),$$

a contradiction, and therefore $A^m z = S^m z = z$. Since the range of T^m contains the range of A^m, let z' be a point in X such that $T^m z' = z$. Then using (3.2), we have

$$\begin{aligned} d(z, B^m z') &= d(A^m z', B^m z') \\ &\le f(\max\{d(S^m z, T^m z'), d(S^m z, A^m z), d(T^m z', B^m z'), \\ &\quad \tfrac{1}{2}[d(S^m z, B^m z') + d(S^m z, A^m z')], \tfrac{1}{2}[d(S^m z, B^m z') + d(S^m z, T^m z')]\}) \\ &\le f(\max\{d(z,z), d(z,z), d(z, B^m z')], \\ &\quad \tfrac{1}{2}[d(z, B^m z') + d(z,z)], \tfrac{1}{2}[d(z, B^m z') + d(z,z)]\}) \\ &\le f(\max\{0, 0, d(z, B^m z'), \tfrac{1}{2}d(z, B^m z')\}) \le f(d(z, B^m z')) < d(z, B^m z') \end{aligned}$$

which implies $z = B^m z'$, by property (iii). Since B is m-weak** commuting with T, we have $B^m T^m z' = T^m B^m z'$ which implies $B^m z = T^m z$. Using again (3.2) and (iii), we deduce $B^m z = T^m z = z$. Since A m-weak** commutes with S, we have $S^m A z = A S^m z$ and $S^m A z = A^{m+1} z = Az$. Now

$$\begin{aligned} d(Az, z) &= d(A^{m+1} z, B^m z) \\ &\le f(\max\{d(S^m Az, T^m z), d(S^m Az, A^{m+1} z), d(T^m z, B^m z), \\ &\quad \tfrac{1}{2}[d(S^m Az, B^m z) + d(T^m z, A^{m+1} z)], \tfrac{1}{2}[d(S^m Az, B^m z) + d(S^m Az, T^m z)]\}) \\ &\le f(\max\{d(Az, z), d(Az, Az), d(z,z), \\ &\quad \tfrac{1}{2}[d(Az, z) + d(z, Az)], \tfrac{1}{2}[d(Az, z) + d(Az, z)]\}) \\ &\le f(\max\{d(Az, z), 0, 0, d(Az, z), d(Az, z)\}) \le f(d(Az, z)) < d(Az, z) \end{aligned}$$

which implies $Az = z$, by property (iii). Therefore $Az = Sz = z$. Using (3.2), (iii) and m-weak** commutativity of B, T, one deduces $Tz = Bz = z$. Therefore, z is a common fixed point of A, B, S, T.

Analogous proof can be given, if one supposes the continuity of T instead of S.

Now we suppose the continuity of A. Then the sequence $\{AS^m x_{2n}\}$ converges to Az. Since A m-weak** commutes with S, we have

$$\begin{aligned} d(S^m A x_{2n}, Az) &\le d(S^m A x_{2n}, AS^m x_{2n}) + d(AS^m x_{2n}, Az) \\ &\le d(S^m x_{2n}, A^m x_{2n}) + d(AS^m x_{2n}, Az) \end{aligned}$$

which implies that the sequence $\{S^m A x_{2n}\}$ converges to Az, as $n \to \infty$.

Using (3.2) and properties (i), (ii), (iii), and observing that $\{A^{m+1} x_{2n}\}$ converges also to Az, one proves that $Az = z$. As above, one shows that

$T^m z = B^m z = z$. Since the range of S^m contains the range of B^m, let z'' be a point of X such that $S^m z'' = z$. Using again (3.2), we have

$$\begin{aligned} d(A^m z'', z) = d(A^m z'', B^m z) \leq \\ f(\max\{d(S^m z'', T^m z), d(S^m z'', A^m z''), d(T^m z, B^m z), \\ \tfrac{1}{2}[d(S^m z'', B^m z) + d(T^m z, A^m z'')], \tfrac{1}{2}[d(S^m z'', B^m z) + d(S^m z'', T^m z)]\}) \\ \leq f(\max\{d(z,z), d(z, A^m z''), d(z,z), \tfrac{1}{2}[d(z,z) + d(z, A^m z'')], \\ \tfrac{1}{2}[d(z,z) + d(z,z)]\}) \leq f(d(z, A^m z'')) < d(z, A^m z'') \end{aligned}$$

which implies $A^m z'' = z$, by property (iii). Since A m-weak** commutes with S, we have $d(A^m S^m z'', S^m A^m z'') \leq d(S^m z'', A^m z'') = d(z,z) = 0$ and therefore $S^m z = S^m A^m z'' = A^m S^m z'' = A^m z = z$. Since A m-weak** commutes with S, we have $A^m Sz = SA^m z$ and so $A^m Sz = S^{m+1} z = Sz$. Now,

$$\begin{aligned} d(Sz, z) &= d(A^m Sz, B^m z) \\ &\leq f(\max\{d(S^{m+1}z, T^m z), d(S^{m+1}z, A^m Sz), d(T^m z, B^m z) \\ &, \tfrac{1}{2}[d(S^{m+1}z, B^m z) + d(T^m z, A^m Sz)], \tfrac{1}{2}[d(S^{m+1}z, B^m z) + d(S^{m+1}z, T^m z)]\}) \\ &\leq f(\max\{d(Sz,z), d(Sz,Sz), d(z,z), \tfrac{1}{2}[d(Sz,z) + d(z,Sz)], \\ &\tfrac{1}{2}[d(Sz,z) + d(Sz,z)]\}) \leq f(\max\{d(Sz,z), 0, 0, d(Sz,z), d(Sz,z)\}) \\ &\leq f(d(Sz,z)) < d(Sz,z) \end{aligned}$$

which implies $Sz = z$, by property (iii) and so $Az = Sz = z$. Since B m-weak** commutes with T, we have $B^m Tz = TB^m z$ and $T^m Bz = BT^m z$ which yields $B^m Tz = T^{m+1} z = Tz$ and $T^m Bz = B^{m+1} z = Bz$, respectively. Further

$$\begin{aligned} d(z, Tz) &= d(A^m z, B^m Tz) \\ &\leq f(\max\{d(S^m z, T^{m+1} z), d(S^m z, A^m z), d(T^{m+1} z, B^m Tz), \\ &\tfrac{1}{2}[d(S^m z, B^m Tz) + d(T^{m+1} z, A^m z)], \tfrac{1}{2}[d(S^m z, B^m Tz) + d(S^m z, T^{m+1} z)]\}) \\ &\leq f(\max\{d(z,Tz), d(z,z), d(Tz,Tz), \tfrac{1}{2}[d(z,Tz) + d(Tz,z)], \\ &\tfrac{1}{2}[d(z,Tz) + d(z,Tz)]\}) \leq f(\max\{d(z,Tz), 0, 0, d(z,Tz), d(z,Tz)\}) \\ &\leq f(d(z,Tz)) < d(z,Tz) \end{aligned}$$

which implies $Tz = z$, by property (iii). Similarly, we can prove that $Bz = z$ and so $Tz = Bz = z$, we have therefore proved that z is again a common fixed point of A, B, S, T.

If the mapping B is continuous instead of A, then the proof that z is again a common fixed point of A, B, S, T is similar. Using (3.2), the uniqueness of z is easily proved.

EXAMPLE 2. Let X be the subset of R^2 defined by $X = \{F, G, H, I, J\}$, where $F \equiv (0,0)$, $G \equiv (\frac{1}{4}, 0)$, $H \equiv (0,1)$, $I \equiv (\frac{1}{6}, 0)$, $J \equiv (-1, 0)$. Let

$A, B, S, T : X \to X$ be given by

$$\begin{array}{lll} A(F) = A(G) = G, & A(H) = F, & A(I) = A(J) = H, \\ B(F) = B(G) = G, & B(H) = F, & B(I) = B(J) = H, \\ S(F) = S(G) = G, & S(H) = F, & S(I) = S(J) = G, \\ T(F) = T(G) = G, & T(H) = F, & T(I) = T(J) = G, \end{array}$$

Further $T^3(X) = G = A^3(X)$, $S^3(X) = G = B^3(X)$.

By routine calculation one can easily verify the weak** commutativity of pairs $\{A, S\}$ and $\{B, T\}$. Then A, B, S, T satisfy condition (3.2) for all $x, y \in X$ and G is the unique common fixed point of A, B, S, T.

However A, B, S, T do not satisfy conditions (1.3), (2.2). For otherwise, choosing $x = G$, $y = J$, we would have

$$d(A(G), B(J)) \le f(\max\{d(S(G), T(J)), d(S(G), A(G)), d(T(J), B(J)), \tfrac{1}{2}[d(S(G), B(J)) + d(T(J), A(G))]\}),$$

i.e., $d(G, H) \le f(\max\{d(G, G), d(G, G), d(G, H), \frac{1}{2}[d(G, H) + d(G, G)]\})$,

hence, $\frac{\sqrt{17}}{4} \le f(\max\{0, 0, \frac{\sqrt{17}}{4}, \frac{\sqrt{17}}{8}\})$ and $\frac{\sqrt{17}}{4} \le f(\frac{\sqrt{17}}{4})$, contradiction to the required condition (iii). Similarly

$$d(A^2(G), B^2(J)) \le f(\max\{d(S^2(G), T^2(J)), d(S^2(G), A^2(G)), d(T^2(G), B^2(J)), \tfrac{1}{2}[d(S^2(G), B^2(J)) + d(T^2(J), A^2(G))]\}),$$

$$d(G, F) \le f(\max\{d(G, G), d(G, G), d(G, F), \tfrac{1}{2}[d(G, F) + d(G, G)]\})$$

or $\frac{1}{4} \le f(\max(0, 0, \frac{1}{4}, \frac{1}{8})) = f(\frac{1}{4})$, a contradiction.

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