DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 3 1995

Stanisław Stoiński

L_α-ALMOST PERIODIC FUNCTIONS

Let X_0 be the set of functions defined on the whole real axis and taking finite real values.

Let us put for $x \in X_0$

$$L_\alpha(t,\delta;x) = \sup_{\substack{u_1,u_2 \in \langle t-\delta, t+\delta\rangle \\ u_1 \neq u_2}} \frac{|x(u_1)-x(u_2)|}{|u_1-u_2|^\alpha} \quad \text{for } t \in (-\infty,+\infty),$$

where $\delta > 0$, $0 < \alpha \le 1$.

We say that $t_0 \in (-\infty,+\infty)$ is an α-singular point of x if $L_\alpha(t_0,\delta;x) = +\infty$ for every $\delta > 0$.

Let us write

$$X_0^\alpha = \{x \in X_0 : \text{there are no } \alpha\text{-singular points of } x\}.$$

X_0^α is a linear set.

Let us put for $x \in X_0^\alpha$

$$L_\alpha(x) = \sup_{-\infty<t<\infty} \{|x(t)| + \lim_{\delta\to 0} L_\alpha(t,\delta;x)\}.$$

A sequence (x_n), where $x_n \in X_0^\alpha$ for $n = 1,2,\ldots$, is called L_α-convergent to $x_0 \in X_0^\alpha$ iff for an arbitrary $\varepsilon > 0$ there exists $N > 0$ such that for $n > N$ $L_\alpha(x_n - x_0) \le \varepsilon$. The limit x_0 of the sequence (x_n) which is L_α-convergent is uniquely defined.

We say that $x \in X_0$ satisfies the condition (L_α) iff for every $t_0 \in (-\infty,\infty)$ there exists a neighbourhood of t_0 for which the function $L_\alpha(\cdot,1;x)$ is bounded.

Let us write

$$\widetilde{X}_0^\alpha = \{x \in X_0 : x \text{ satisfies the condition } (L_\alpha)\}.$$

Key words and phrases: almost periodic function, Hölder norm
1991 Mathematics Subject Classification: Primary 42A75

We say that $x \in X_0^\alpha$ is an L_α-bounded function iff $L_\alpha(x) < \infty$.

A set $E \subset (-\infty, \infty)$ is called relatively dense iff there is a positive number l such that in every open interval $(\gamma, \gamma + l)$, $\gamma \in (-\infty, \infty)$ there is at least one element of the set E.

Let $x \in \widetilde{X}_0^\alpha$. If for $\varepsilon > 0$ there is $L_\alpha(x - x_\tau) \leq \varepsilon$, where $x_\tau(t) \equiv x(t+\tau)$, then the number $\tau \in (-\infty, \infty)$ is called L_α, ε-almost period of the function x. Let us denote by $E_\alpha\{\varepsilon; x\}$ the set of L_α, ε-almost periods of x.

A function $x \in \widetilde{X}_0^\alpha$ is called almost periodic in the sense of Hölder (L_α-a.p.) iff for each $\varepsilon > 0$ the set $E_\alpha\{\varepsilon; x\}$ is relatively dense. For $\alpha = 1$ we obtain an almost periodic function in the sense of Lipschitz (L-a.p.).

Every L_α-a.p. function is a Bohr's a.p. function (see (1)). Note that for every $x \in X_0^{\alpha''}$ we have $L_{\alpha'}(x) \leq L_{\alpha''}(x)$ providing that $0 < \alpha' < \alpha'' \leq 1$, and so every $L_{\alpha''}$-a.p. function x is also $L_{\alpha'}$-a.p.

THEOREM 1. *If x is an L_α-a.p. function, then x is L_α-bounded.*

Proof. For any $t \in (-\infty, \infty)$ there exists an L_α, ε-almost period $\tau \in (-t, -t+l)$, where $l = l(\varepsilon)$ is such that for $\delta > 0$ we have

$$\begin{aligned} L_\alpha(t, \delta; x) &\leq L_\alpha(t, \delta; x - x_\tau) + \sup_{0<t'<l} L_\alpha(t', \delta; x) \\ &\leq L_\alpha(t, \delta, x - x_\tau) + M, \end{aligned}$$

such l exists because of the relative density of the set $E_\alpha\{\varepsilon; x\}$. From the above we obtain

$$L_\alpha(x) \leq L_\alpha(x - x_\tau) + \sup_{-\infty<t<\infty} |x(t)| + M < \infty,$$

because x is bounded and x satisfies the condition (L_α), what completes the proof.

Let us observe that any L-a.p. function x is also V-a.p.(see [3]).

To prove this, fix x and $\varepsilon > 0$ and choose the relatively dense set $E_1\{\varepsilon/10; x\}$. For every $\tau \in E_1\{\varepsilon/10; x\}$ and for every $t \in (-\infty, \infty)$ we have

$$|x(t+\tau) - x(t)| \leq \frac{\varepsilon}{10}$$

and

$$\lim_{\delta \to 0} L_1(t, \delta; x - x_\tau) \leq \frac{\varepsilon}{10}.$$

Hence there exists a positive integer $n' = n'(\varepsilon, t)$ such that

$$L_1(t, \delta_{n'}; x - x_\tau) \leq \frac{\varepsilon}{5}, \tag{1}$$

where $\delta_{n'} \leq 5/4$. For any function $x \in X_0$ denote by

$$V_a^b x = \sup_P \sum_{i=0}^{n-1} |x(u_{i+1}) - x(u_i)|,$$

where $P : a = u_0 < u_1 < \ldots < u_n = b$, the Jordan variation of x on the interval $\langle a, b\rangle$. From (1) it follows that for every $t \in (-\infty, \infty)$

$$V_{t-1}^{t+1}(x - x_\tau) < \frac{2}{5}\varepsilon(1 + \delta_{n'}).$$

Therefore

$$V(x - x_\tau) = \sup_{-\infty<t<\infty} \{|x(t+\tau) - x(t)| + V_{t-1}^{t+1}(x - x_\tau)\} \leq \varepsilon,$$

i.e. $E_V\{\varepsilon; x\} \supset E_1\{\varepsilon/10; x\}$, where $E_V\{\varepsilon; x\}$ is the relatively dense set of V, ε-almost periods of x. Hence x is V-a.p.

Now we will build a V-a.p. function that is not L-a.p. Let us put

$$x = x_1 + x_2,$$

where

$$x_1(u) = \begin{cases} \arcsin(u - 4k) & \text{for } u \in \langle 4k-1, 4k+1), \\ \arcsin(-u + 4k + 2) & \text{for } u \in \langle 4k+1, 4k+3), \end{cases}$$

$$x_2(u) = \begin{cases} \arcsin(\sqrt{2}u - 4k) & \text{for } u \in \langle (4k-1)/\sqrt{2}, (4k+1)/\sqrt{2}), \\ \arcsin(\sqrt{2}u + 4k + 2) & \text{for } u \in \langle (4k+1)/\sqrt{2}, (4k+3)/\sqrt{2}), \end{cases}$$

$k = 0, \pm 1, \pm 2, \ldots$

Because V-a.p. functions x_1 and x_2 are V-continuous (see [3]), so the summ $x_1 + x_2$ is V-a.p. For $t_0 = \sqrt{2}/2$ we have $L_1(t_0, 1; x) = +\infty$, and so x is not L-a.p.

We say that $x \in X_0^\alpha$ is an L_α-continuous function iff for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in (-\infty, \infty)$, $|h| < \delta$, we have $L_\alpha(x - x_\tau) \leq \varepsilon$, where $x_h(u) \equiv x(u+h)$. For $\alpha = 1$ we have an L-continuous function.

THEOREM 2. *If the derivative x' of $x \in X_0^1$ is uniformly continuous and bounded on the whole real axis, then x is L-continuous.*

Proof. Choose $M > 0$ such that $|x'(u)| \leq M$ for every $u \in (-\infty, \infty)$, and observe that $|x(u+h) - x(u)| \leq Mh$. Because x' is uniformly continuous, so, for an arbitrary $\varepsilon > 0$, there exists $\delta' \in (0, \varepsilon/(4M))$ such that for every $u \in (-\infty, \infty)$ and $|h| < \delta'$ we have

$$\left| \frac{x(u+h) - x(u)}{h} - x'(u) \right| < \frac{\varepsilon}{4}.$$

Hence we obtain

$$L_1(x - x_h) \le \sup_{-\infty<t<\infty} |x(t+h) - x(t)| + \sup_{-\infty<t<\infty} \lim_{\delta\to 0} L_1(t,\delta; x - x_h)$$

and

$$\sup_{-\infty<t<\infty} |x(t+h) - x(t)| \le M|h| \le \frac{\varepsilon}{4},$$

$$L_1(t; x - x_h) \le \sup_{-\infty<u<\infty} |x'(u+h) - x'(u)| \\ + 2 \sup_{-\infty<u<\infty} \left| \frac{x(u+h) - x(u)}{h} - x'(u) \right| \le \frac{3}{4}\varepsilon$$

for every $t \in (-\infty, \infty)$, where $0 < \delta < \delta'/2$ and $|h| < \delta'$. Therefore $L_1(x - x_h) \le \varepsilon$ for every $|h| < \delta'$ what gives the L-continuity of x.

THEOREM 3. *If x is such an L-a.p. function that the derivative x' is continuous, then x is L-continuous.*

Proof. For arbitrary $\tau \in E_1\{\varepsilon/6; x\}$ choose $l = l(\varepsilon)$ that for every $h \in (-\infty, \infty)$ we have

$$L_1(x - x_h) \le \frac{\varepsilon}{3} + \sup_{-\infty<t<\infty} |x(t+h) - x(t)| + \lim_{\delta\to 0} \sup_{0<t<l} L_1(t,\delta; x - x_h).$$

(such an $l = l(\varepsilon)$ exists because of the relative density of $E_1\{\varepsilon/6; x\}$). Because x is a Bohr's a.p. function, so x is uniformly continuous, i.e. there exists $\delta' > 0$ such that

$$\sup_{-\infty<t<\infty} |x(t+h) - x(t)| \le \frac{\varepsilon}{6} \quad \text{for } |h| < \delta'.$$

Because x' is uniformly continuous on the interval $\langle -1, l+1 \rangle$, then there exists $\delta'' > 0$ such that for every $u \in \langle -1, l+1 \rangle$ we have

$$|x'(u+h) - x'(u)| < \frac{\varepsilon}{6} \quad \text{for } |h| < \delta''.$$

Then for $|h| < \delta_0 = \min(\delta', \delta''/2)$ we have $L_1(x - x_h) \le \varepsilon$. This means that x is L-continuous and ends the proof.

For $x \in X_0$ let us denote by

$$p_\alpha(u,v) = \frac{x(u+v) - x(u)}{|v|^\alpha}, \tag{2}$$

where $u \in (-\infty, \infty)$, $v \in \langle -1, 1 \rangle \setminus \{0\}$, $0 < \alpha \le 1$.

THEOREM 4. *Assume that x is such an L_α-a.p. function for which the function p_α given by (2) is continuous with respect to $u \in (-\infty, \infty)$, uniformly continuous with respect to v. Then x is L_α-continuous.*

Proof. For any $\tau \in E_\alpha\{\varepsilon/6; x\}$ and for $|h| < \delta'$, where $\delta' = \delta'(\varepsilon) > 0$, pick $l = l(\varepsilon)$ such that we have

$$L_\alpha(x - x_h) \le \frac{\varepsilon}{2} + \lim_{\delta\to 0}\ \sup_{0<t<l} L_\alpha(t, \delta; x - x_h)$$

(such $l = l(\varepsilon) > 0$ exists because of the relative density of $E_\alpha\{\varepsilon/6; x\}$). Since the function p_α is uniformly continuous on the interval $\langle -\delta, l + \delta\rangle$, where $0 < \delta \le 1/2$, and uniformly with respect to v, then for every $t \in (0, l)$, we obtain

$$L_\alpha(t, \delta; x - x_h) \le \frac{\varepsilon}{2} \quad \text{for } |h| < \delta'',$$

where $\delta'' = \delta''(\varepsilon) > 0$. Hence, for $|h| < \min(\delta', \delta'')$ we have $L_\alpha(x - x_h) \le \varepsilon$, i.e. x is L_α-continuous.

THEOREM 5. *The set of all L-a.p. functions x, y for which derivatives x', y' are continuous is a vector space.*

We prove Theorem 5 in the same way as for Bohr's a.p. functions (see [1] pp. 202–204) using the following:

LEMMA. *For an L-a.p. function x and for an arbitrary $\varepsilon > 0$ there exist numbers $\delta > 0$, $\omega > 0$ such that for every $h \in (0, \delta\rangle$ in every open interval of the length ω there exists an L_1, ε-almost period τ of the function x such that $\tau = kh$, where k is an integer, if x is L-continuous.*

THEOREM 6. *The set of all L_α-a.p. functions is L_α-closed.*

Proof. Fix a sequence (x_n) of L_α-a.p. functions L_α-converging to x. For an arbitrary $\varepsilon > 0$ there exists n_0 such that $L_\alpha(x_{n_0} - x) \le \varepsilon/3$. Note that for any $\tau \in E_\alpha\{\varepsilon/3; x_{n_0}\}$ we have

$$L_\alpha(x - x_\tau) \le \frac{2}{3} + L_\alpha(x_{n_0\tau} - x_{n_0}),$$

where $x_{n_0\tau}(t) \equiv x_{n_0}(t + \tau)$. Hence $L_\alpha(x - x_\tau) \le \varepsilon$, and so x is L_α-a.p.

THEOREM 7. *If the sequence (x_n), where $x_n \in X_0^1$ are L_{α_n}-a.p. functions for $n = 1, 2, \ldots$, $0 < \alpha_n < 1$, $\alpha_n \to 1$, is L_1-convergent to a function $x \in \widetilde{X}_0^1$, then x is L-a.p.*

Proof. For an arbitrary $\varepsilon > 0$ and for any $u_1, u_2 \in \langle t - \delta, t + \delta\rangle$, where $t \in (-\infty, \infty)$, $\delta > 0$, $u_1 \ne u_2$, we have for $n > N_1$ the following inequality

$$\frac{|(x - x_\tau)(u_1) - (x - x_\tau)(u_2)|}{|u_1 - u_2|} < \frac{\varepsilon}{2} + \frac{|(x - x_\tau)(u_1) - (x - x_\tau)(u_2)|}{|u_1 - u_2|^{\alpha_n}}$$

and hence we obtain

$$L_1(x - x_\tau) \le \frac{\varepsilon}{2} + L_{\alpha_n}(x - x_\tau) \quad \text{for } n > N_1. \tag{3}$$

Because there exists $N_2 > 0$ such that for $n > N_2$ we have $L_1(x - x_n \leq \varepsilon/6$, for $\tau \in E_{\alpha_n}\{\varepsilon/6; x_n\}$, where $n > N_2$, we see that

$$L_{\alpha_n}(x - x_\tau) \leq 2L_1(x - x_n) + L_{\alpha_n}(x_n - x_{n\tau}) \leq \frac{\varepsilon}{2}, \tag{4}$$

where $x_{n\tau}(u) \equiv x_n(u + \tau)$.

From (3) and (4) it follows now that $L_1(x - x_\tau) \leq \varepsilon$, i.e. x is L-a.p. and completes the proof.

THEOREM 8. *Let x be a Bohr's a.p. function for which the derivative x' is continuous. Then the function x is L-a.p. iff the derivative x' is uniformly continuous.*

Proof. Sufficiency. Assume that x' is uniformly continuous on $(-\infty, \infty)$. It is known (see [1]) that $E\{\varepsilon; x\} \subset E\{\beta(\varepsilon); x'\}$, where $E\{\varepsilon; x\}$ is the relatively dense set of ε-almost periods of x and $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then x' is a Bohr's a.p. function. For $\tau \in E\{\varepsilon; x\}$ we have $L_1(x - x_\tau) \leq 3\varepsilon + \beta(\varepsilon)$, i.e. x which satisfies the condition (L_1) is L-a.p.

Necessity. We assume that x is L-a.p. From Theorem 3 it follows that x is L-continuous, i.e. for an arbitrary $\varepsilon > 0$ there exists $\Delta > 0$ such that for $|h| < \Delta$ we have $L_1(x - x_h) \leq \varepsilon/3$, where $x_h(u) \equiv x(u + h)$. Hence for every $u \in (-\infty, \infty)$ there exists $\delta' = \delta'(\varepsilon, h, u) > 0$ such that for every v, $0 < |v| \leq 2\delta'$ we have

$$\left| \frac{x(u+v) - x(u)}{v} - \frac{x_h(u+v) - x_h(u)}{v} \right| < \frac{\varepsilon}{3} \tag{5}$$

for $|h| < \Delta$. But for every $u \in (-\infty, \infty)$ and h such that $|h| < \Delta$ we have

$$\begin{aligned} |x'(u) - x'_h(u)| \leq & \left| x'(u) - \frac{x(u+v) - x(u)}{v} \right| \\ & + \left| \frac{x(u+v) - x(u)}{v} + \frac{x_h(u+v) - x_h(u)}{v} \right| \\ & + \left| \frac{x_h(u+v) - x_h(u)}{v} - x'_h(u) \right| \\ = & \; A(u,v) + B(h,u,v) + C(h,u,v). \end{aligned}$$

For v, $0 < |v| \leq \delta''$, where $\delta'' = \delta''(\varepsilon, u)$, we have $A(u, v) < \varepsilon/3$ and for v, $0 < |v| \leq \delta'''$, where $\delta''' = \delta'''(\varepsilon, u, h) > 0$, we obtain $C(h, u, v) < \varepsilon/3$. Then, using (5), we see that

$$|x'(u) - x'_h(u)| < \varepsilon \quad \text{for } 0 < |v| \leq \delta_0 = \min(2\delta', \delta'', \delta'''),$$

uniformly with respect to $u \in (-\infty, \infty)$, but this precisely gives the uniform continuity.

For example the function $x(u) = \sin u + \sin(\sqrt{2}u)$ for $u \in (-\infty, \infty)$ is L-a.p.

COROLLARY. *If x is an L-a.p. function for which the derivative x'' is uniformly continuous, then the derivative x' is L-a.p.*

THEOREM 9. *If x is a Bohr's a.p. function for which*

$$p_\alpha(u, v) \to 0 \quad \textit{as } v \to 0,$$

uniformly with respect to $u \in (-\infty, \infty)$, where $0 < \alpha \leq 1$, then x is L_α-a.p.

Proof. For an arbitrary $\varepsilon > 0$ and for $\tau \in E\{\varepsilon/3; x\}$, where $E\{\varepsilon/3; x\}$ is the relatively dense set of $\varepsilon/3$-almost periods of x, we have

$$\sup_{-\infty<t<\infty} |x(t+\tau) - x(t)| \leq \frac{\varepsilon}{3}$$

and for every $t \in (-\infty, \infty)$

$$L_\alpha(t, \delta; x - x_\tau) \leq \frac{2}{3} \quad \text{for } 0 < \delta < \delta',$$

where $\delta' = \delta'(\varepsilon)$. Hence $L_\alpha(x - x_\tau) \leq \varepsilon$. It is easily seen that x satisfies the condition (L_α). Therefore x is L_α-a.p.

Now we will build an L_α-a.p. function, where $0 < \alpha < 1$, which is not L-a.p.

Let us put

$$x(u) = \varphi(|\sin u + \sin(\sqrt{2}u)|) \quad \text{for } u \in (-\infty, \infty),$$

where φ is a concave φ-function (see [2]) such that

$$\varphi(u) = o(u^\alpha) \quad \text{for } 0 < \alpha < 1 \text{ and } \frac{\varphi(u)}{u} \to \infty \quad \text{as } u \to 0+.$$

Then x is the Bohr's a.p. function and for $u, v \in (-\infty, \infty)$, $v \neq 0$, we obtain

$$\frac{|x(u+v) - x(u)|}{|v|^\alpha} \leq \frac{1}{|v|^\alpha}[\varphi(|v|) + \varphi(\sqrt{2}|v|)] \to 0 \quad \text{as } v \to 0,$$

uniformly with respect to $u \in (-\infty, \infty)$. By the Theorem 9 it follows that x is L_α-a.p. For $t_0 = 0$ we have $L_1(t_0, 1; x) = +\infty$, and so x is not L-a.p.

It is known (see [1]) that if the indefinite integral of a Bohr's a.p. function is bounded, then this integral is a Bohr's a.p. function. The following property is true:

THEOREM 10. *If x is a Bohr's a.p. function and the indefinite integral*

$$F(u) = \int_{u_0}^{u} x(v)dv \quad \textit{for } u \in (-\infty, \infty)$$

is bounded, then F is an L-a.p. function.

P r o o f. Since x is a bounded function and for an arbitrary $t \in (-\infty, \infty)$ we have $L_1(t, 1; F) \leq M$, where M is a constant, we see that $F \in \widetilde{X}_0^\alpha$. It is known (see [1], p. 29) that for an arbitrary $\varepsilon > 0$ there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that $\varepsilon' < \varepsilon/3$ and every ε'-almost period of x is an $\varepsilon/3$-almost period of F. Let us denote by $E\{\varepsilon'; x\}$ the relatively dense set of ε'-almost periods of the function x. Because x is uniformly continuous, so for an arbitrary $\varepsilon > 0$ there exists $\delta' > 0$ such that for $\tau \in E\{\varepsilon'; x\}$ we have

$$\sup_{-\infty<t<\infty} |F(t+\tau) - F(t)| \leq \frac{\varepsilon}{3}$$

and for every $t \in (-\infty, \infty)$

$$L_1(t, \delta; F - F_\tau) \leq \frac{\varepsilon}{3} + \varepsilon' \quad \text{for } 0 < \delta < \frac{\delta'}{2}.$$

Hence we have $L_1(F - F_\tau) \leq \varepsilon$, and so $E\{\varepsilon'; x\} \subset E_1\{\varepsilon; F\}$, i.e. F is L-a.p.

References

[1] B. M. L e v i t a n, *Almost periodic functions*, Moscow 1953 (in Russian).

[2] J. M u s i e l a k, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer, Berlin 1983.

[3] S. S t o i ń s k i, *Real-valued functions almost periodic in variation*, Functiones et Approximatio 22 (1993), 141–148.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
ul. Matejki 48/49
60-769 POZNAŃ, POLAND

Received December 6, 1993.