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GLOBAL IN TIME EXISTENCE OF SOLUTIONS TO THE CONSTITUTIVE MODEL OF BODNER-PARTOM WITH ISOTROPIC HARDENING

1. Introduction and statement of the problem and result

We study a system of equations modelling the nonelastic deformation of metals. This system has been proposed by R. S. Bodner and Y. Partom in [3]. We show that in the one-dimensional case the initial boundary-value problem generated by this system has global in time solutions to all sufficiently small initial data. The quasi-static case in $\mathbb{R}^3$ without the hardening has been studied by P. Le Tallec in [6] and by myself in [4]. In this case the system of equations to the constitutive model of Bodner-Partom has global in time solutions for large initial data. In [6] the author has used the theory of monotone operators and in [4] the existence result has been shown with energy estimates. The uni-axial dynamical case with a growth condition for the constitutive function has been studied by H. D. Alber in [2]. In this work L^∞-bounds are derived for solutions of the Neumann boundary-value problem. These estimates imply the global in time existence of large solutions.

Let $\Omega = (0, L) \subset \mathbb{R}$ and $u : \Omega \times [0, \infty) \to \mathbb{R}^3$ be the displacement field. Then the equations are

$$\text{(P)} \qquad \begin{aligned} \rho\partial_t^2 u(x,t) &= \operatorname{div} S(x,t) \\ E(x,t) &= \frac{1}{2}\left(\nabla u(x,t) + (\nabla u(x,t))^T\right) \\ S(x,t) &= \mathcal{D}(E(x,t) - E^n(x,t)) \\ s(x,t) &= S(x,t) - \frac{1}{3}(\operatorname{tr} S(x,t)) \cdot I \\ \partial_t E^n(x,t) &= \mathcal{G}\Big(\frac{|s(x,t)|}{z(x,t)}\Big) \cdot \frac{s(x,t)}{|s(x,t)|} \\ \partial_t z(x,t) &= m(z^1 - z(x,t)) \cdot |s(x,t)| \cdot |\partial_t E^n(x,t)|. \end{aligned}$$

We study the system (P) with the Dirichlet boundary condition

$$(\mathrm{D}) \qquad u(0,t) = u(L,t) = 0,$$

or with the Neumann boundary condition

$$(\mathrm{N}) \qquad S(0,t)\cdot n_- = S(L,t)\cdot n_+ = 0 \quad \text{where } n_\pm = (\pm 1, 0, 0)$$

and with initial conditions

$$(\mathrm{I}) \qquad \begin{aligned} u(x,0) &= u^0(x), & \partial_t u(x,0) &= u^1(x),\\ E^n(x,0) &= E^{n,0}(x), & z(x,0) &= z^0(x). \end{aligned}$$

Here $\rho > 0$ is the mass density, which we assume to be constant; $S(x,t)$ is the stress field; $s(x,t)$ is the stress deviator; I is the identity matrix; $E(x,t)$ is the strain field; $E^n(x,t)$ is the inelastic part of the strain; $\mathcal{D} = (d_{ijkl})_{i,j,k,l=1,2,3}$ is the elasticity tensor, which we assume to be constant, symmetric and positive definite, i.e.

$$(\mathcal{D}) \qquad \begin{aligned} &\forall i,j,k,l = 1,2,3 \quad \begin{aligned} d_{ijkl} &= d_{jikl},\\ d_{ijkl} &= d_{klij}, \end{aligned}\\ &\exists d > 0\ \forall \eta \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \sum_{i,j,k,l=1}^{3} d_{ijkl}\eta_{ij}\eta_{kl} \ge d \sum_{i,j=1}^{3} \eta_{ij}^2; \end{aligned}$$

and $z(x,t)$ is an internal state variable called the hardening; m, z^1 are positive material parameters. Properties of this system depend on the nonlinear function $\mathcal{G}$. In this paper we assume that $\mathcal{G}$ satisfies the following

Assumptions:

(A1) $\mathcal{G} \in C^2([0,\infty);[0,\infty)), \quad \mathcal{G}(0) = 0;$

(A2) $\forall p \in \mathbb{R}_+\ \mathcal{G}'(p) \ge 0$

(A3) $\exists g^* > 0\ \exists g_1, g_2 > 0\ \forall p < g^*$

$$g_1\mathcal{G}'(p)\cdot p \le \mathcal{G}(p) \le g_2\mathcal{G}'(p)\cdot p.$$

The assumption (A3) satisfies for example all polynomials. The properties of the function $\mathcal{G}$ are important only for small arguments, because we consider the problem (P) for small initial data and the hardening is a strongly positive L^∞-function. The last statement follows directly form equation (P6). The integration of this equation gives

$$(1.1) \quad z(x,t) = z^0(x) + (z^1 - z^0(x))\exp\left\{-\int_0^t |s(x,\tau)|\cdot|\partial_t E^n(x,\tau)|\,d\tau\right\}$$

and with the assumptions

$$(\mathrm{A4}) \qquad \forall x \in (0,L) \quad z^0(x) \le z^1,$$

$$(\mathrm{A5}) \qquad \forall x \in (0,L) \quad \inf_{x\in(0,L)} z^0(x) \ge z^* > 0$$

we have, that the hardening is a bounded function greater than zero

$$0 < z^* \leq z(x,t) \leq z^1 \quad \text{for all } (x,t). \tag{1.2}$$

We define for tensors $R = (r_{ij})_{i,j=1,2,3}$, $S = (s_{ij})_{i,j=1,2,3}$ the product

$$R \cdot S = \sum_{i,j=1}^{3} r_{ij} s_{ij} \quad \text{and} \quad |R| = (R \cdot R)^{1/2}. \tag{1.3}$$

By $\mathbf{H}^i(\Omega; \mathbb{R}^k)$ we denote the usual Sobolev space of functions defined on Ω with values in $\mathbb{R}^k$, with the norm $\|\cdot\|_i$. The norm and the scalar product in the space $\mathbf{L}^2(\Omega; \mathbb{R}^k)$ we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively.

The main result is

THEOREM 1.1. *There exist sufficiently small positive constants C_1, C_2 with the following property*:

Assume that the initial data satisfy:

$$u^0 \in \mathbf{H}^2(\Omega; \mathbb{R}^2), \quad u^1 \in \mathbf{H}^1(\Omega; \mathbb{R}^3), \tag{1.4}$$

$$E^{n,0} \in \mathbf{H}^1(\Omega; \mathbb{R}^9), \quad z^0 \in \mathbf{H}^1(\Omega; \mathbb{R}_+) \tag{1.5}$$

with $E^{n,0}(x)$ symmetric, with $\operatorname{tr} E^{n,0}(x) = 0$ for almost all $x \in (0, L)$ and with the assumptions (A4) *and* (A5). *Moreover suppose that the compatibility conditions*

$$u^0(x) = 0, \quad u^1(x) = 0 \tag{1.6}$$

for the Dirichlet boundary-value problem or

$$\frac{1}{2}(\nabla u^0(x) + (\nabla u^0(x))^T) = E^{n,0}(x) \tag{1.7}$$

for the Neumann boundary-value problem hold for $x = 0$ and $x = L$. Finally suppose that

$$\|u^0\|_2 + \|u^1\|_1 + \|E^{n,0}\|_1 \leq C_1, \tag{1.8}$$

$$\|\partial_x z^0\| \leq C_2. \tag{1.9}$$

Then there exists a global solution

$$(u, S, E^n, z) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3 \times (\mathbb{R}^9)^2 \times \mathbb{R}_+$$

of the problems (P) + (D) + (I) *or* (P) + (N) + (I) *with*

$$\forall T > 0 \quad u \in \mathbf{H}^2(\Omega \times [0,T]; \mathbb{R}^3),$$
$$S, E^n \in \mathbf{H}^1(\Omega \times [0,T]; \mathbb{R}^9),$$
$$z \in \mathbf{H}^1(\Omega \times [0,T]; \mathbb{R}_+).$$

The proof of this result is based on energy estimates for a sequence of approximate solutions. The sequence is constructed using the Galerkin method. The energy estimates are proven using a method similar to those used by H. D. Alber in [1].

2. The sequence of approximate solutions

We will approximate a solution of the problem (P) by a sequence of C^2–solutions of approximate problems using the Galerkin method. First we should choose a basis in the space $\mathbb{L}^2(\Omega;\mathbb{R}^3)$. For the problems (P) + (D) + (I) we choose the basis as follows:

Let $\{\tilde{f}_l\}_{l=1}^{\infty}$ be a system of orthonormal, complete in $\mathbb{L}^2(\Omega;\mathbb{R})$ eigenfunctions for the boundary-value problem

$$\begin{aligned} &\frac{d^2}{dx^2}\tilde{f}_l(x) + \lambda_l \tilde{f}_l(x) = 0, \\ &\tilde{f}_l(0) = \tilde{f}_l(L) = 0. \end{aligned} \tag{2.1}$$

The functions $f_l(x) = a_l \tilde{f}_l(x)$ with suitable vector $a_l \in \mathbb{R}^3$ are orthonormal and complete in $\mathbb{L}^2(\Omega;\mathbb{R}^3)$. For the systems (P) + (N) + (I) we construct the basis similarly, but now the functions $\tilde{f}_l$ are solutions of the following boundary-value problem:

$$\begin{aligned} &\frac{d^2}{dx^2}\tilde{f}_l(x) + \lambda_l \tilde{f}_l(x) = 0, \\ &\frac{d}{dx}\tilde{f}_l(0) = \frac{d}{dx}\tilde{f}_l(L) = 0. \end{aligned} \tag{2.2}$$

Now we want to approximate the solution (u, S, E^n, z) of the problem (P) by a sequence $\{(u_k, S_k, E_k^n, z_k)\}_{k=1}^{\infty}$, where u_k is the linear combination

$$u_k(x,t) = \sum_{l=1}^{k} a_{lk}(t) f_l(x) \tag{2.3}$$

of the functions $f_1, \ldots, f_k$. The other components satisfy the equations

$$E_k = \frac{1}{2}(\nabla u_k + (\nabla u_k)^T), \tag{2.4}$$

$$S_k = \mathcal{D}(E_k - E_k^n), \tag{2.5}$$

$$(\rho \partial_t^2 u_k, f_l) + (S_k, \nabla f_l) = 0 \quad \text{for } l = 1, \ldots, k, \tag{2.6}$$

$$s_k = S_k - \frac{1}{3}(\operatorname{tr} S_k) I, \tag{2.7}$$

$$\partial_t E_k^n = \mathcal{G}_k\left(\frac{|s_k|}{z}\right)\frac{s_k}{|s_k|}, \tag{2.8}$$

$$\partial_t z = m(z^1 - z)|s_k| \cdot \mathcal{G}_k\left(\frac{|s_k|}{z}\right). \tag{2.9}$$

Here the function $\mathcal{G}_k : [0,\infty) \to [0,\infty)$ is defined as follows:

$$\mathcal{G}_k(p) = \chi(kp)\mathcal{G}(p) \tag{2.10}$$

where $\chi \in C^2(\mathbb{R})$ is a cut-off function with $\chi(p) = 0$ for $p < \frac{1}{2}$, $\chi(p) = 1$ for

$p > 1$, $0 \le \chi \le 1$, $\chi' \ge 0$ and $\chi_0 \cdot p \cdot \chi'(p) \le \chi(p)$ for all $p \in \mathbb{R}_+$, (χ_0 is a positive constant). The function χ is introduced to regularize the singular behavior for the right hand sides of the equations (P5) and (P6) at the point $s = 0$. We note that for all k the function $\mathcal{G}_k$ satisfies the assumptions (A1)–(A3) because we have the following estimates

$$p\mathcal{G}_k'(p) = kp\chi'(kp)\mathcal{G}(p) + \chi(kp)p\mathcal{G}'(p) \ge \frac{1}{g_2}\chi(kp)\mathcal{G}(p) = \frac{1}{g_2}\mathcal{G}_k(p),$$

$$p\mathcal{G}_k'(p) \le \frac{1}{\chi_0}\chi(kp)\mathcal{G}(p) + \frac{1}{g_1}\chi(kp)\mathcal{G}(p) = \left(\frac{1}{\chi_0} + \frac{1}{g_1}\right)\mathcal{G}_k(p)$$

for all $p < g^*$.

The necessary initial conditions are

$$(2.11)\qquad \begin{aligned} &(u_k(0), f_l) = (u^0, f_l),\\ &\partial_t u_k(0), f_l) = (u^1, f_l), \quad l = 1, \ldots, k,\\ &E_k^n(0) = E^{n,0},\\ &z_k(0) = z^0. \end{aligned}$$

The next lemma proves existence of solutions of the approximate problems (2.4)–(2.9) with the initial conditions (2.11).

LEMMA 2.1. *Let us suppose that*

$$(2.12)\qquad \begin{aligned} &u^0 \in \mathbb{H}^2(\Omega; \mathbb{R}^3), \quad u^1 \in \mathbb{H}^1(\Omega; \mathbb{R}^3),\\ &E^{n,0} \in C^2(\Omega; \mathbb{R}^9), \quad z^0 \in C^2(\Omega; \mathbb{R}_+) \end{aligned}$$

with $E^{n,0}(x)$ *symmetric and with* $\operatorname{tr} E^{n,0}(x) = 0$ *for all* $x \in (0, L)$. *Moreover, let the assumptions* (A4) *and* (A5) *hold. Finally for the Dirichlet boundary-value problem assume that*

$$(2.14)\qquad u^0(0) = u^0(L) = u^1(0) = u^1(L) = 0$$

and for the Neumann boundary-value problem

$$\nabla u^0(0) = E^{n,0}(0) = \nabla u^0(L) = E^{n,0}(L) = 0.$$

Then there exists $T_k > 0$ *and a unique solution*

$$(u_k, S_k, E_k^n, z_k) \in C^2(\Omega \times [0, T_k); \mathbb{R}^3 \times (\mathbb{R}^9)^2 \times \mathbb{R}_+)$$

of (2.4)–(2.9) *with initial conditions* (2.11). *Moreover for all* $(x, t) \in \Omega \times [0, T_k)$ $S_k(x, t), E_k^n(x, t)$ *are symmetric and* $\operatorname{tr} E_k^n(x, t) = 0$, *and for the Neumann boundary-value problem we have*

$$(2.15)\qquad S_k(x, t) = E_k(x, t) = E_k^n(x, t) = 0 \quad \text{for } x = 0 \text{ and } x = L.$$

Proof. We transform the equations (2.4)–(2.9) into a first order system of ordinary differential equations. Let us denote $b_{lk}(t) = a_{lk}'(t)$ for $l = 1, \ldots, k$. Thus we have

$$\frac{d}{dt}a_{lk}(t) = b_{lk}(t),$$

$$\rho\sum_{j=1}^{k}(f_j, f_l)\frac{d}{dt}b_{jk}(t) + \sum_{j=1}^{k}\Big(\frac{1}{2}\mathcal{D}(\nabla f_j + (\nabla f_j)^T), \nabla f_l\Big)a_{jk}(t) =$$

$$(2.16) \qquad = (\mathcal{D}E_k^n(t), \nabla f_l), \quad l = 1, \ldots, k,$$

$$\frac{d}{dt}E_k^n(t) = \mathcal{G}_k\Big(\frac{|s_k(t)|}{z_k(t)}\Big)\frac{s_k(t)}{|s_k(t)|},$$

$$\frac{d}{dt}z_k(t) = m(z_1 - z_k(t))|s_k(t)|\mathcal{G}_k\Big(\frac{|s_k(t)|}{z_k(t)}\Big),$$

where $s_k(t) = S_k(t) - \frac{1}{3}(\operatorname{tr} S_k(t))I$ and

$$S_k(t) = \sum_{j=1}^{k}\frac{1}{2}\mathcal{D}(\nabla f_j + (\nabla f_j)^T)a_{jk}(t) - \mathcal{D}E_k^n(t).$$

The system (2.16) has the following form

$$(2.17) \qquad \frac{d}{dt}W_k(t) = \mathcal{F}(W_k(t)),$$

where $W_k(t)=(a_{1k}(t), \ldots, a_{kk}(t), b_{1k}(t), \ldots, b_{kk}(t), E_k^n(t), z_k(t))$. The function

$$\mathcal{F} : \mathcal{A} \to (\mathbb{R}^k)^2 \times C^2([0,L];\mathbb{R}^9) \times C^2[0,L];\mathbb{R}_+)$$

where $\mathcal{A} = \{(a_{1k}, \ldots, a_{kk}, b_{1k}, \ldots, b_{kk}, E_k^n, z_k) \in (\mathbb{R}^k)^2 \times C^2([0,L];\mathbb{R}^9) \times C^2([0,L];(\frac{1}{2}z^*, \infty)\}$ is twice continuous differentiable, because by definition in (2.10) the function χ vanishes in the neighborhood of zero. The assumptions for the initial data imply that $W_k(0) \in \mathcal{A}$. Therefore it follows from the usual theory of ordinary differential equations in Banach spaces, that there exist $T_k > 0$ and a unique solution $W_k \in C^2([0,T_k);\mathcal{A})$ of (2.17) with initial conditions (2.11). Moreover, it follows that the solution can be continued as long as it stays in $\mathcal{A}$. Now we prove the properties of this solution.

The assumption ($\mathcal{D}1$) for the elasticity tensor $\mathcal{D}$ implies that $S_k(x,t)$ and $s_k(x,t)$ are symmetric for all $(x,t) \in \Omega \times [0,T_k)$. Thus we obtain

$$(2.18) \qquad \frac{d}{dt}E_k^n(t) = \frac{d}{dt}(E_k^n(t))^T.$$

Hence we have that $(u_k, S_k, (E_k^n)^T, z_k)$ is a solution for the problem (2.4)–(2.9) with the same initial conditions (2.11). Since the solution is unique, it follows that $E_k^n(x,t) = (E_k^n(x,t))^T$ for all $(x,t) \in \Omega \times [0,T_k)$. From the equation (2.8) we obtain an ordinary differential equation for $\operatorname{tr} E_k^n(t)$

$$(2.19) \qquad \frac{d}{dt}(\operatorname{tr} E_k^n(t)) = 0$$

with an initial condition $\operatorname{tr} E_k^n(0) = \operatorname{tr} E^{n,0} = 0$. From the uniqueness of solutions of linear differential equations we have $\operatorname{tr} E_k^n(x,t) = 0$ for all $(x,t) \in \Omega \times [0,T_k)$.

To prove (2.15), note that

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\mathcal{D}(E_k(x,t) - E_k^n(x,t))\cdot(E_k(x,t) - E_k^n(x,t)\Big]\\ = S_k(x,t)\cdot\partial_t E_k(x,t) - S_k(x,t)\cdot\partial_t E_k^n(x,t)\\ = S_k(x,t)\cdot\partial_t E_k(x,t) - s_k(x,t)\cdot\partial_t E_k^n(x,t) \le S_k(x,t)\cdot\partial_t E_k(x,t).\end{aligned}$$

The boundary condition in (2.2) and the assumption (2.14) for the initial data yield

$$E_k(0,t) = E_k(L,t) = 0 \quad \text{for all } t \in [0,T_k) \tag{2.20}$$

and time-differentiation of (2.20) gives

$$\partial_t E_k(0,t) = \partial_t E_k(L,t) = 0 \quad \text{for all } t \in [0,T_k).$$

Thus for $x = 0$ and $x = L$ we obtain

$$\begin{aligned}\frac{1}{2}\mathcal{D}(E_k(x,t) - E_k^n(x,t))\cdot(E_k(x,t) - E_k^n(x,t))\\ \le \frac{1}{2}\mathcal{D}(E_k(x,0) - E_k^n(x,0))\cdot(E_k(x,0) - E_k^n(x,0)) = 0\end{aligned}$$

because the initial data satisfies (2.14). From this inequality, from (2.20) and from assumptions $(\mathcal{D})$ follows (2.15). The proof of this lemma is complete. ∎

3. Energy estimates

In this section we prove energy estimates for a solution (u_k, S_k, E_k^n, z_k) constructed in the last section. These estimates are very important to show that the sequence of approximate solutions $\{(u_k, S_k, E_k^n, z_k)\}_{k=1}^{\infty}$ contains a subsequence, which converges to a solution of the problem (P). For simplicity we usually drop in this section the index k and assume that $(u, S, E^n, z) \in C^2(\Omega \times [0,T))$ is a solution of the system (2.4)–(2.9) with initial conditions (2.11). Let us define now the energy function

$$\mathcal{E}(t) = \mathcal{E}_0(t) + \mathcal{E}_t(t) + \mathcal{E}_x(t), \tag{3.1}$$

where

$$\mathcal{E}_0(t) = \frac{1}{2}(\rho\partial_t u(t), \partial_t u(t)) + \frac{1}{2}(S(t), E(t) - E^n(t)), \tag{3.2}$$

$$\mathcal{E}_t(t) = \frac{1}{2}(\rho\partial_t^2 u(t), \partial_t^2 u(t)) + \frac{1}{2}(\partial_t S(t), \partial_t E(t) - \partial_t E^n(t)), \tag{3.3}$$

$$\mathcal{E}_x(t) = \frac{1}{2}(\rho\partial_t\partial_x u(t), \partial_t\partial_x u(t)) + \frac{1}{2}(\partial_x S(t), \partial_x E(t) - \partial_x E^n(t)). \tag{3.4}$$

Note that the assumptions $(\mathcal{D})$ imply

$$\mathcal{E}_0(t) \geq \frac{\rho}{2}\|\partial_t u(t)\|^2 + \frac{d}{2}\|(E - E^n)(t)\|^2, \tag{3.5}$$

$$\mathcal{E}_t(t) \geq \frac{\rho}{2}\|\partial_t^2 u(t)\|^2 + \frac{d}{2}\|\partial_t(E - E^n)(t)\|^2, \tag{3.6}$$

$$\mathcal{E}_x(t) \geq \frac{\rho}{2}\|\partial_t\partial_x u(t)\|^2 + \frac{d}{2}\|\partial_x(E - E^n)(t)\|^2. \tag{3.7}$$

LEMMA 3.1. *Let* $(u, S, E^n, z) \in C^2(\Omega\times[0,T))$ *be a solution of the problem* (2.4)–(2.9) *with the initial conditions* (2.11). *Then*

$$\frac{d}{dt}\mathcal{E}_0(t) = -\int_0^L \mathcal{G}_k\left(\frac{|s(t)|}{z(t)}\right)|s(t)|\,dx \leq 0, \tag{3.8}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_t(t) = &-\int_0^L \mathcal{G}_k'\left(\frac{|s(t)|}{z(t)}\right)\frac{(\partial_t|s(t)|)^2}{z(t)}\,dx \\ &-\int_0^L \mathcal{G}_k\left(\frac{|s(t)|}{z(t)}\right)\left(\frac{|\partial_t s(t)|}{|s(t)|} - \frac{(\partial_t|s(t)|)^2}{|s(t)|}\right)dx \\ &-\int_0^L \mathcal{G}_k'\left(\frac{|s(t)|}{z(t)}\right)\frac{\partial_t z(t)}{z^2(t)}|s(t)|\partial_t|s(t)|\,dx,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_x(t) = &-\int_0^L \mathcal{G}_k'\left(\frac{|s(t)|}{z(t)}\right)\frac{(\partial_x|s(t)|)^2}{z(t)}\,dx \\ &-\int_0^L \mathcal{G}_k\left(\frac{|s(t)|}{z(t)}\right)\left(\frac{|\partial_x s(t)|^2}{|s(t)|} - \frac{(\partial_x|s(t)|)^2}{|s(t)|}\right)dx \\ &-\int_0^L \mathcal{G}_k'\left(\frac{|s(t)|}{z(t)}\right)\frac{\partial_x z(t)}{z^2(t)}|s(t)|\partial_x|s(t)|\,dx\end{aligned} \tag{3.10}$$

or alternatively

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_x(t) = &-\int_0^L \frac{z(t)}{\mathcal{G}_k'\left(\frac{|s(t)|}{z(t)}\right)}(\partial_x|\partial_t E^n(t)|)^2\,dx \\ &-\int_0^L \frac{|s(t)|}{\mathcal{G}_k\left(\frac{|s(t)|}{z(t)}\right)}(|\partial_x\partial_t E^n(t)|^2 - (\partial_x|\partial_t E^n(t)|)^2)\,dx \\ &+\int_0^L \frac{|s(t)|}{z^2(t)}\partial_x z(t)\partial_x|\partial_t E^n(t)|\,dx.\end{aligned} \tag{3.11}$$

Proof.

$$\frac{d}{dt}\mathcal{E}_0(t) = \frac{d}{dt}\left\{\frac{1}{2}(\rho\partial_t u, \partial_t u) + \frac{1}{2}(S, E - E^n)\right\}$$
$$= (\rho\partial_t^2 u, \partial_t u) + (S, \partial_t E) - (S, \partial_t E^n).$$

From the symmetry of S and from $\operatorname{tr}\partial_t E^n = 0$ we obtain

$$\frac{d}{dt}\mathcal{E}_0(t) = \left(\rho\partial_t^2 u, \sum_{j=1}^{k} a'_{jk} f_j\right) + (S, \nabla\partial_t u) - (s, \partial_t E^n)$$
$$= \sum_{j=1}^{k} a'_{jk}\{(\rho\partial_t^2 u, f_j) + (S, \nabla f_j)\} - \int_0^L |s||\partial_t E^n|\,dx.$$

(3.8) follows from this equality and from (2.6). For the part $\mathcal{E}_t(t)$ of the energy function we have

$$\frac{d}{dt}\mathcal{E}_t(t) = \frac{d}{dt}\left\{\frac{1}{2}(\rho\partial_t^2 u, \partial_t^2 u) + \frac{1}{2}(\partial_t S, \partial_t(E - E^n))\right\}$$
$$= (\rho\partial_t^3 u, \partial_t^2 u) + (\partial_t S, \partial_t^2 E) - \partial_t S, \partial_t^2 E^n)$$
$$= (\rho\partial_t^3 u, \partial_t^2 u) + (\partial_t S, \nabla\partial_t^2 u) - (\partial_t s, \partial_t^2 E^n)$$
$$= \sum_{j=1}^{k} a''_{jk}\{(\rho\partial_t^3 u, f_j) + (\partial_t S, \nabla f_j)\} - (\partial_t s, \partial_t^2 E^n)$$
$$= \sum_{j=1}^{k} a''_{jk}\frac{d}{dt}\{(\rho\partial_t^2 u, f_j) + (S, \nabla f_j)\} - (\partial_t s, \partial_t^2 E^n).$$

The equation (2.6) yields

$$\frac{d}{dt}\mathcal{E}_t(t) = -(\partial_t s, \partial_t^2 E^n). \tag{3.12}$$

Now we compute $\partial_t^2 E^n$ from (2.8)

$$\partial_t^2 E^n = \mathcal{G}'_k\left(\frac{|s|}{z}\right)\left(\frac{\partial_t|s|}{z} - \frac{\partial_t z\cdot|s|}{z^2}\right)\frac{s}{|s|} + \mathcal{G}_k\left(\frac{|s|}{z}\right)\left(\frac{\partial_t s}{|s|} - \frac{s}{|s|^2}\partial_t|s|\right) \tag{3.13}$$

and insert the result into (3.12). (3.9) follows from this and from the equality

$$\partial_t s\cdot s = \partial_t|s|\cdot|s|. \tag{3.14}$$

To prove (3.8) and (3.9) it was not important that the basis $\{f_l\}_{l=1}^{\infty}$ is chosen very special. The equalities are true for all orthonormal and complete systems in $\mathbf{L}^2(\Omega;\mathbb{R}^3)$. The proof of (3.10) and (3.11) needs first properties of the functions f_l.

$$\frac{d}{dt}\mathcal{E}_x(t) = \frac{d}{dt}\left\{\frac{1}{2}(\rho\partial_t\partial_x u, \partial_t\partial_x u) + \frac{1}{2}(\partial_x S, \partial_x(E - E^n))\right\}$$
$$= (\rho\partial_t^2\partial_x u, \partial_t\partial_x u) + (\partial_x S, \partial_t\partial_x E) - (\partial_x S, \partial_x\partial_t E^n)$$
$$= \sum_{l=1}^{k}\sum_{j=1}^{k}\rho a_{lk}'' a_{jk}'\left(\frac{d}{dx}f_l, \frac{d}{dx}f_j\right) + \sum_{j=1}^{k} a_{jk}'\left(\partial_x S, \frac{d}{dx}\nabla f_j\right) - (\partial_x s, \partial_x\partial_t E^n)$$
$$= \sum_{j=1}^{k} a_{jk}'\lambda_j(\rho\partial_t^2 u, f_j) + \sum_{l=1}^{k}\sum_{j=1}^{k}\rho a_{lk}'' a_{jk}' f_l \frac{d}{dx}f_j\Big|_0^L +$$
$$+ \sum_{j=1}^{k} a_{jk}'\lambda_j(S, \nabla f_j) + \sum_{j=1}^{k} a_{jk}' S \cdot \frac{d}{dx}\nabla f_j\Big|_0^L - (\partial_x s, \partial_x\partial_t E^n).$$

The boundary terms vanish because for the Dirichlet boundary-value problem the following equalities

$$f_l(0) = f_l(L) = 0 \quad \text{and} \quad \frac{d}{dx}\nabla f_l(0) = \frac{d}{dx}\nabla f_l(L) = 0 \quad \text{for all } l,$$

hold and for the Neumann boundary-value problem we have

$$\frac{d}{dx}f_l(0) = \frac{d}{dx}f_l(L) = 0 \quad \text{for all } l \text{ and } S(0,t) = S(L,t) = 0.$$

Thus we obtain

$$\frac{d}{dt}\mathcal{E}_x(t) = \sum_{j=1}^{k} a_{jk}'\lambda_j\{(\rho\partial_t^2 u, f_j) + (S, \nabla f_j)\} - (\partial_x s, \partial_x\partial_t E^n).$$

Again the equation (2.6) implies that

$$\frac{d}{dt}\mathcal{E}_x(t) = -(\partial_x s, \partial_x\partial_t E^n). \tag{3.15}$$

To prove (3.10) we compute $\partial_x\partial_t E^n$ from (2.8) and insert the result into (3.15). Now we write the right hand side of (3.15) in another form

$$-(\partial_x s, \partial_x\partial_t E^n) = -\left(\partial_x\left(\frac{|s|}{|\partial_t E^n|}\partial_t E^n\right), \partial_x\partial_t E^n\right)$$
$$= -\left(\frac{|s|}{|\partial_t E^n|}\partial_x\partial_t E^n, \partial_x\partial_t E^n\right)$$
$$- (\partial_x|s|, \partial_x|\partial_t E^n|) + \left(\frac{|s|}{|\partial_t E^n|}\partial_x|\partial_t E^n|, \partial_x|\partial_t E^n|\right).$$

Compute $\partial_x|s|$ from (2.8) and insert the result into this equation to obtain the equality (3.11). ∎

We cannot conclude directly from the last lemma that the energy function is decreasing, because (3.9), (3.10) or (3.11) contain terms, which do not have a sign. These terms must be estimated. To do this we need informations about the derivatives $\partial_t z$ and $\partial_x z$. The properties of the time-derivative $\partial_t z$ are described by the equation (2.9).

We introduce the notations

$$F_0(t) = \int_0^L \mathcal{G}_k\left(\frac{|s(t)|}{z(t)}\right)|s(t)|\,dx,$$

$$F_1(t) = \int_0^L \frac{\mathcal{G}_k'(\frac{|s(t)|}{z(t)})}{z(t)}(\partial_x|s(t)|)^2\,dx,$$

$$F_2(t) = \int_0^L \frac{z(t)}{\mathcal{G}_k'(\frac{|s(t)|}{z(t)})}\cdot(\partial_x|\partial_t E^n|)^2\,dx.$$

LEMMA 3.2. *Let (u, S, E^n, z) be a solution of the problem (2.4)–(2.9) with initial conditions (2.11). Moreover suppose that assumptions (A4) and (A5) hold, and for all $(x,t) \in \Omega \times [0,T)$ we have*

$$|s(x,t)| \le g^* z(x,t). \tag{3.16}$$

Then the last term in the equality (3.9) can be estimated as follows

$$\begin{aligned} &-\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)\frac{\partial_t z}{z^2}|s|\partial_t|s|\,dx \\ &\qquad\le \frac{1}{2}\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)\frac{(\partial_t|s|)^2}{z}\,dx + C_t\mathcal{E}(t)\cdot F_0(t), \end{aligned}\tag{3.17}$$

where $C_t = C_t(d, z^*, z^1, m, g_1, \mathcal{G}(g^*), c^s)$ is a positive constant and c^s is the positive constant from the Sobolev inequality.

Proof. From the equation (2.9) and the Cauchy–Schwartz inequality we obtain

$$\begin{aligned} &-\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)\frac{\partial_t z}{z^2}|s|\partial_t|s|\,dx \\ &\qquad\le -\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)m(z^1-z)|s|\mathcal{G}_k\left(\frac{|s|}{z}\right)\frac{|s|}{z^2}\partial_t|s|\,dx \\ &\qquad\le \frac{1}{2}\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)\frac{(\partial_t|s|)^2}{z}\,dx \\ &\qquad\quad+\frac{1}{2}\int_0^L \mathcal{G}_k'\left(\frac{|s|}{z}\right)m^2(z^1-z)^2\frac{|s|^4}{z^3}\mathcal{G}_k^2\left(\frac{|s|}{z}\right)dx. \end{aligned}$$

Using assumptions (A1)–(A5) and (3.16) we infer that the right hand side of the last inequality can be estimated as follows

$$\frac{1}{2}\int_0^L \mathcal{G}_k'\Big(\frac{|s|}{z}\Big)\frac{(\partial_t|s|)^2}{z}\,dx+\frac{1}{2g_1}\int_0^L\frac{m^2(z_1-z)^2}{z^2}\mathcal{G}_k^3\Big(\frac{|s|}{z}\Big)|s|^3\,dx$$

$$\le\frac{1}{2}\int_0^L \mathcal{G}_k'\Big(\frac{|s|}{z}\Big)\frac{(\partial_t|s|)^2}{z}\,dx+\frac{1}{2g_1}\mathcal{G}_k^2(g^*)\sup_{x\in(0,L)}\frac{m^2(z_1-z)^2}{z^2}\sup_{x\in(0,L)}|s|^2\cdot F_0(t).$$

This lemma follows now from (1.2), (2.7), the definition of the energy function and the Sobolev inequality. ∎

To estimate the last terms in (3.10) and (3.11) we need an information about the derivative $\partial_x z$. Differentiation of (1.1) yields

$$\begin{aligned}(3.18)\qquad \partial_x z=\partial_x z^0\Big(1-\exp\Big\{-\int_0^t|s|\mathcal{G}_k\Big(\frac{|s|}{z}\Big)d\tau\Big\}\Big)\\ -(z^1-z^0)\exp\Big\{-\int_0^t|s|\mathcal{G}_k\Big(\frac{|s|}{z}\Big)d\tau\Big\}\cdot\int_0^t\partial_x\Big[|s|\cdot\mathcal{G}_k\Big(\frac{|s|}{z}\Big)\Big]d\tau.\end{aligned}$$

The next lemma derives a L^2-estimation for the function $\partial_x z$.

LEMMA 3.3. *There exists a positive constant C_z with the following property: Let (u,S,E^n,z) be a solution of* (2.4)–(2.9) *with initial conditions* (2.11) *and assume that*

$$|s(x,t)|\le g^*z(x,t)\quad \textit{for all } (x,t)\in\Omega\times[0,T)$$

and the assumptions (A4) *and* (A5) *hold. Then we obtain the following estimation*

$$\int_0^L|\partial_x z(t)|^2\,dx\le C_z\Big(\int_0^t[F_0(\tau)+F_1(\tau)+F_2(\tau)]\,d\tau\Big)^2+2\|\partial_x z^0\|^2$$

and $C_z=C_z(g_1,g_2,z^*,z^1,m,c^s,\chi_0)$.

Proof. From (3.18) we obtain

$$\int_0^L|\partial_x z|^2\,dx\le 2\int_0^L|\partial_x z^0|^2\,dx+2(z^1-z^*)^2\int_0^L\Big(\int_0^t\partial_x\Big[|s|\cdot\mathcal{G}_k\Big(\frac{|s|}{z}\Big)\Big]d\tau\Big)^2dx$$

$$\leq 2\|\partial_x z^0\|^2 + 4(z^1 - z^*)\int_0^L \left\{\left(\int_0^t \partial_x|s|\mathcal{G}_k\left(\frac{|s|}{z}\right)d\tau\right)^2 + \left(\int_0^t |s|\partial_x\mathcal{G}_k\left(\frac{|s|}{z}\right)d\tau\right)^2\right\}dx$$

$$\leq 2\|\partial_x z^0\|^2 + 4(z^1 - z^*)\int_0^L \left\{\left[\int_0^t \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,d\tau\right]\left[\int_0^t \mathcal{G}_k\left(\frac{|s|}{z}\right)\frac{(\partial_x|s|)^2}{|s|}\,d\tau\right] + \left[\int_0^t \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,d\tau\right]\cdot\left[\int_0^t \frac{|s|}{\mathcal{G}_k\left(\frac{|s|}{z}\right)}(\partial_x|\partial_t E^n|)^2\right]\right\}dx$$

$$\leq 2\|\partial_x z^0\|^2 + 4(z^1 - z^*)\max\left\{g_2, \frac{1}{\chi_0} + \frac{1}{g_1}\right\} \times \int_0^t \sup_{x\in(0,L)}\left(\mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\right)d\tau \int_0^t (F_1 + F_2)\,d\tau.$$

To end this proof we find out that $\sup_{x\in(0,L)}\mathcal{G}_k\left(\frac{|s|}{z}\right)|s|$ can be estimated by the functions F_0, F_1 and F_2. The Sobolev inequality yields

$$\sup_{x\in(0,L)}\mathcal{G}_k\left(\frac{|s|}{z}\right)|s| \leq c^s\left\{\int_0^L \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,dx + \int_0^L \left[\left|\partial_x\mathcal{G}_k\left(\frac{|s|}{z}\right)\right||s| + \left|\partial_x|s|\right|\mathcal{G}_k\left(\frac{|s|}{z}\right)\right]dx\right\} \tag{3.19}$$

$$\leq c^s\left\{F_0 + \frac{1}{2}\int_0^L (\partial_x|\partial_t E^n|)^2\frac{|s|}{\mathcal{G}_k\left(\frac{|s|}{z}\right)}\,dx + \frac{1}{2}\int_0^L \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,dx + \frac{1}{2}\int_0^L \frac{\mathcal{G}_k\left(\frac{|s|}{z}\right)}{|s|}(\partial_x|s|)^2\,dx + \frac{1}{2}\int_0^L \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,dx\right\}$$

$$\leq c^s\left\{F_0 + \frac{1}{2}\left(\frac{1}{\chi_0} + \frac{1}{g_1}\right)F_2 + \frac{1}{2}F_0 + \frac{1}{2}g_2F_1 + \frac{1}{2}F_0\right\}$$

$$= c^s\left\{2F_0 + \frac{1}{2}\left(\frac{1}{\chi_0} + \frac{1}{g_1}\right)F_2 + \frac{1}{2}g_2F_1\right\}. \blacksquare$$

Now we are ready to estimate the terms without sign in (3.10) and (3.11). ∎

LEMMA 3.4. *There exist positive constants C_x, C'_x with the following property:*

Let (u, S, E^n, z) be a solution of (2.4)–(2.9) with initial conditions (2.11). Moreover suppose that

$$|s(x,t)| \le g^* z(x,t) \quad \text{for all } (x,t) \in \Omega \times [0,T)$$

and the assumptions (A4) and (A5) hold. Then

$$\text{(3.20)} \qquad \text{a)} \ -\int_0^L \mathcal{G}'_k\left(\frac{|s(t)|}{z(t)}\right)\frac{\partial_x z(t)}{z^2(t)}|s(t)|\partial_x|s(t)|\,dx \le \|\partial_x z(t)\| C_x(F_0(t)+F_1(t)+F_2(t)),$$

$$\text{(3.21)} \qquad \text{b)} \ \int_0^L \frac{|s(t)|}{z^2(t)}\partial_x z(t)\partial_x|\partial_t E^n(t)|\,dx \le \|\partial_x z(t)\| C'_x(F_0(t)+F_1(t)+F_2(t)).$$

The constants C_x and C'_x depend on z^*, χ_0, g_1, g_2 and c^s.

Proof. Using Cauchy–Schwartz' inequality we obtain

$$\begin{aligned}
&-\int_0^L \mathcal{G}'_k\left(\frac{|s|}{z}\right)\frac{\partial_x z}{z^2}|s|\partial_x|s|\,dx \\
&\le \sqrt{\int_0^L |\partial_x z|^2\,dx}\,\sqrt{\int_0^L \left(\mathcal{G}'_k\left(\frac{|s|}{z}\right)\right)^2 \frac{|s|^2}{z^4}(\partial_x|s|)^2\,dx} \\
&\le \|\partial_x z\|\left\{\frac{1}{2}\sup_{x\in(0,L)}\frac{\mathcal{G}'_k(\frac{|s|}{z})|s|^2}{z^3} + \frac{1}{2}\int_0^L \mathcal{G}'_k\left(\frac{|s|}{z}\right)\frac{(\partial_x|s|)^2}{z}\,dx\right\} \\
&\le \|\partial_x z\|\left\{\frac{1}{2(z^*)^2}\left(\frac{1}{\chi_0}+\frac{1}{g_1}\right)^{-1}\sup_{x\in(0,L)}\mathcal{G}_k\left(\frac{|s|}{z}\right)|s| + \frac{1}{2}F_1\right\}.
\end{aligned}$$

We insert the estimation (3.19) into the last inequality and obtain a). Similarly

$$\begin{aligned}
\int_0^L \frac{|s|}{z^2}\partial_x z\partial_x|\partial_t E^n|\,dx &\le \sqrt{\int_0^L |\partial_x z|^2\,dx}\,\sqrt{\int_0^L \frac{|s|^2}{z^4}(\partial_x|\partial_t E^n|)^2\,dx} \\
&\le \|\partial_x z\|\left\{\frac{1}{2}\sup_{x\in(0,L)}\frac{\mathcal{G}'_k(\frac{|s|}{z})|s|^2}{z^5} + \frac{1}{2}\int_0^L \frac{z}{\mathcal{G}'_k(\frac{|s|}{z})}(\partial_x|\partial_t E^n|)^2\,dx\right\}
\end{aligned}$$

$$\leq \|\partial_x z\| \left\{ \frac{1}{2(z^*)^4} \left(\frac{1}{\chi_0} + \frac{1}{g_1} \right)^{-1} \sup_{x \in (0,L)} \mathcal{G}_k\left(\frac{|s|}{z} \right) |s| + \frac{1}{2} F_2 \right\}.$$

The statement of this lemma follows again from (3.19). ■

Lemmas 3.1–3.4 produced L^2–bounds for the space and time-derivatives of the elastic strain field $E - E^n$, but we need estimates for derivatives of the strain field E, too. This problem solves the next lemma.

LEMMA 3.5. *There eixsts a positive constants* C_E, C_F *with the following property: Let* (u, S^n, E^n, z) *be a solution of* (2.4)-(2.9) *with initial conditions* (2.11). *Moreover assume that assumptions* (A4) *and* (A5) *hold and*

$$\forall (x,t) \in \Omega \times [0,T) \quad |s(x,t)| \leq g^* z(x,t).$$

Then

a) $\|E(t)\|^2 + \|E^n(t)\|^2 + \|\nabla u(t)\|^2 \leq C_E\left\{ t \int_0^t F_0(\tau) d\tau + \|E^{n,0}\|^2 + \mathcal{E}(t) \right\},$

b) $\|\partial_x E(t)\|^2 + \|\partial_x E^n(t)\|^2 + \|\partial_x \nabla u(t)\|^2$

$$\leq C_F \left\{ t \int_0^t F_1(\tau)\, d\tau + \int_0^t \|\partial_x z(\tau)\|^2\, d\tau \right.$$

$$\left. + \int_0^t \int_0^L \frac{\mathcal{G}_k\left(\frac{|s(\tau)|}{z(\tau)}\right)}{|s(\tau)|} (|\partial_x s(\tau)|^2 - (\partial_x |s(\tau)|)^2)\, dx + \|\partial_x E^{n,0}\|^2 + \mathcal{E}(t) \right\},$$

c) $\|\partial_t E(t)\|^2 + \|\partial_t E^n(t)\|^2 \leq \left(\frac{4}{\rho} + \frac{2}{d} \right) \mathcal{E}(t),$

d) $\|u(t)\|^2 \leq \frac{4}{\rho} t \int_0^t \mathcal{E}(\tau)\, d\tau + 2\|u^0\|^2,$

e) $\|\partial_t z(t)\|^2 \leq m^2 (z^1 - z^*)^2 \sup_{x \in (0,L)} \mathcal{G}_k\left(\frac{|s|}{z} \right) |s(t)| \cdot F_0(t).$

The constants C_E, C_F *depend on* $z^*, g_1, g_2, g^*, \chi_0, \sup_{\xi \leq g^*} \mathcal{G}'(\xi)$.

Proof. First we prove the estimation for $\|E^n(t)\|$.

$$\|E^n\|^2 = \int_0^L \Big| \int_0^t \partial_t E^n\, d\tau + E^n(0)|^2\, dx \leq 2t \int_0^t \int_0^L |\partial_t E^n|^2\, dx\, dt + 2\|E^{n,0}\|^2$$

(3.22)

$$= 2t \int_0^t \int_0^L \mathcal{G}_k^2\left(\frac{|s|}{z} \right) dx\, dt + 2\|E^{n,0}\|^2$$

$$\leq 2t \int_0^t \left\{ \sup_{x\in(0,L)} \frac{\mathcal{G}_k(\frac{|s|}{z})}{|s|} \int_0^L \mathcal{G}_k\left(\frac{|s|}{z}\right)|s|\,dx \right\} dt + 2\|E^{n,0}\|^2$$

$$\leq 2g_2 \cdot z^1 \sup_{\xi\leq g^*} \mathcal{G}_k'(\xi)\cdot t \int_0^t F_0\,dt + 2\|E^{n,0}\|^2.$$

The definition of the function $\mathcal{G}_k$ yields

$$\mathcal{G}_k'(\xi) \leq \left(\frac{g_2}{\chi_0}+1\right)\mathcal{G}'(\xi) \quad \text{for all } \xi. \tag{3.23}$$

Using (3.1)–(3.4) we obtain

$$\|E(t)\|^2 \leq 2\|E^n(t)\|^2 + 2\|(E-E^n)(t)\|^2 \leq 2\|E^n(t)\|^2 + \frac{4}{d}\mathcal{E}(t). \tag{3.24}$$

The estimates (3.22), (3.23), (3.24) and the inequality

$$\|\nabla u(t)\|^2 \leq 2\|E(t)\|^2 \tag{3.25}$$

prove the part a) of this lemma. To prove b) we first show again the estimation for the derivative $\partial_x E^n$.

$$\|\partial_x E^n\|^2 = \int_0^L \left| \int_0^t \partial_x\partial_t E^n\,dt + \partial_x E^n(0)\right|^2 \tag{3.26}$$

$$\leq 2t \int_0^t\int_0^L |\partial_x\partial_t E^n|^2\,dx\,dt + 2\|\partial_x E^{n,0}\|^2.$$

Now, from the equation (2.6) we calculate $\partial_x\partial_t E^n$

$$\partial_x\partial_t E^n = \mathcal{G}_k'\left(\frac{|s|}{z}\right)\left(\frac{\partial_x|s|}{z} - \frac{\partial_x z\cdot|s|}{z^2}\right)\frac{s}{|s|} + \mathcal{G}_k\left(\frac{|s|}{z}\right)\left(\frac{\partial_x s}{|s|} - \frac{s}{|s|^2}\partial_x|s|\right)$$

and insert the result into (3.26). Thus we have

$$\|\partial_x E^n\|^2 \leq 2\|\partial_x E^{n,0}\|^2 + 2t\int_0^t \left\{ 4\int_0^L \left(\mathcal{G}_k'\left(\frac{|s|}{z}\right)\right)^2 \frac{(\partial_x|s|)^2}{z^2}\,dx \right.$$

$$+ 4\int_0^L \left(\mathcal{G}_k'\left(\frac{|s|}{z}\right)\right)^2 \frac{(\partial_x z)^2}{z^4}|s|^2\,dx$$

$$\left. + 2\int_0^L \mathcal{G}_k^2\left(\frac{|s|}{z}\right)\left(\frac{\partial_x s}{|s|} - \frac{s}{|s|^2}\partial_x|s|\right)^2 dx \right\} dt.$$

A simple calculation yields

$$\left|\partial_x s - \partial_x|s|\frac{s}{|s|}\right|^2 = |\partial_x s|^2 - (\partial_x|s|)^2.$$

Using this and the assumptions (A1)-(A5) we have

$$\tag{3.27} \|\partial_x E^n\|^2 \le 2\|\partial_x E^{n,0}\|^2 + 2t\Big\{ \frac{4}{z^*} \sup_{\xi \le g^*} \mathcal{G}_k'(\xi) \int_0^t F_1\,dt + \frac{4}{(z^*)^2}\Big(\frac{1}{\chi_0} + \frac{1}{g_1}\Big)^2 \mathcal{G}_k^2(g^*) \int_0^t \|\partial_x z\|^2\,dt + \int_0^t\int_0^L \frac{\mathcal{G}_k\big(\frac{|s|}{z}\big)}{|s|}(|\partial_x s|^2 - (\partial_x|s|))^2 dx\,dt \Big\}.$$

Similarly as in (3.24) and in (3.25) we can show the estimations for the functions $\partial_x E$ and $\partial_x \nabla u$. The estimation b) follows from this and from (3.23) and (3.27). To prove the part c) of this lemma note that

$$\|\partial_t E(t)\|^2 \le \|\nabla \partial_t u\|^2 \le \frac{2}{\rho}\mathcal{E}(t).$$

The statement d) is obtained by integration of $\partial_t u$ and the part e) follows directly from the equation (2.9). ∎

4. Existence of a solution

In this section we prove Theorem 1.1. Let denote by $\mathcal{E}(t;k)$ the energy function calculated for the solution (u_k, S_k, E_k^n, z_k). We will show, that for sufficiently small initial data this function is decreasing. The last statement points out that for all k $\mathcal{E}(t;k) \le \mathcal{E}(0;k)$. To prove Theorem 1.1 we need uniform bounds for the energy norm of the sequence of approximate solutions, and therefore we should estimate the sequence $\mathcal{E}(0;k)$.

LEMMA 4.1. *Let the initial data $u^0, u^1, E^{n,0}, z^0$ satisfy the hypotheses of Lemma* 2.1 *and (u_k, S_k, E_k^n, z_k) be the solution of* (2.4)–(2.9) *with initial conditions* (2.11) *obtained in this lemma. Then we have*

$$\tag{4.1} \mathcal{E}(0;k) \to \mathcal{E}(0) \quad \textit{for } k \to \infty,$$

where $\mathcal{E}(0)$ denotes the energy function calculated for the initial data u^0, u^1, $E^{n,0}$, z^0.

Proof. The system $\{f_l\}_{l=1}^{\infty}$ is a basis in $\mathbf{L}^2(\Omega, \mathbb{R}^3)$, hence

$$\Big\| u^1 - \sum_{l=1}^{k} (u^1, f_l) f_l \Big\| \to 0 \quad \text{for } k \to \infty.$$

Let us denote $(u^1, f_l) = \alpha_l$. We will show that the last statement is true for the norm $\|\cdot\|_1$, too.

$$\tag{4.2} 0 \le \Big(\frac{d}{dx}u^1 - \frac{d}{dx}\sum_{l=1}^{k}\alpha_l f_l, \frac{d}{dx}u^1 - \frac{d}{dx}\sum_{l=1}^{k}\alpha_l f_l\Big) = \Big\|\frac{d}{dx}u^1\Big\|^2 - \sum_{l=1}^{k}\lambda_l \alpha_l^2.$$

(The boundary terms vanish, because $u^1(0) = u^1(L) = f_l(0) = f_l(L) = 0$ for the Dirchlet boundary-value problem and $\frac{d}{dx}f_l(0) = \frac{d}{dx}f_l(L) = 0$ for the Neumann boundary-value problem). The inequality (4.2) implies that the sequence $\sum_{l=1}^{k} \lambda_l \alpha_l^2$ converges. Now we show that the sequence $w_k = \frac{d}{dx}(u^1 - \sum_{l=1}^{k} \alpha_l f_l)$ is a Cauchy sequence in $\mathbb{L}^2(\Omega;\mathbb{R}^3)$:

$$(w_n - w_m, w_n - w_m) = \left(\frac{d}{dx}\sum_{l=m+1}^{n} \alpha_l f_l, \frac{d}{dx}\sum_{l=m+1}^{m} \alpha_l f_l\right) = \sum_{l=m+1}^{n} \lambda_l \alpha_l^2.$$

Thus we obtain that

$$\left\|u^1 - \sum_{l=1}^{k} \alpha_l f_l\right\|_1 \to 0 \quad \text{and}$$

(4.3)

$$\sum_{l=1}^{k} \lambda_l \alpha_l^2 \to \left\|\frac{d}{dx}u^1\right\|^2 \quad \text{for } k \to \infty.$$

From (2.11) and (4.3) we have

$$\|\partial_t u_k(0) - u^1\|_1^2 = \sum_{l=k+1}^{\infty} (1+\lambda_l)\alpha_l^2 \to 0 \quad \text{for } k \to \infty. \tag{4.4}$$

Similarly we prove that

$$\|u_k(0) - u^0\|_1 \to 0 \quad \text{for } k \to \infty. \tag{4.4}$$

Moreover

$$\left\|\frac{d^2}{dx^2}u_k(0) - \frac{d^2}{dx^2}u^0\right\|^2 = \sum_{l=k+1}^{\infty} \left(\frac{d^2}{dx^2}u^0, f_l\right) f_l \to 0 \quad \text{for } k \to \infty, \tag{4.6}$$

while $u^0 \in \mathbb{H}^2(\Omega;\mathbb{R}^3)$. The statements (4.4), (4.5) and (4.6) yield

$$\mathcal{E}_0(0;k) \to \mathcal{E}(0)$$
$$\mathcal{E}_x(0;k) \to \mathcal{E}_x(0)$$
$$(\partial_t S_k(0), \partial_t(E_k - E_k^n)(0)) \to (\partial_t S(0), \partial_t(E - E^n)(0)) \quad \text{for } k \to \infty,$$

where

$$\partial_t S(0) = \mathcal{D}\left(\frac{1}{2}(\nabla u^1 + (\nabla u^1)^T\right) - \mathcal{G}\left(\frac{|s(0)|}{z^0}\right)\frac{s(0)}{|s(0)|}\right),$$
$$s(0) = S(0) - \frac{1}{3}(\operatorname{tr} S(0))I,$$
$$S(0) = \mathcal{D}\left(\frac{1}{2}(\nabla u^0 + (\nabla u^0)^T) - E^{n,0}\right).$$

At the end of the proof of this lemma, note that (2.13) and (2.1) yield

$$(\partial_t^2 u_k(0), f_l) = (S_k(0), \nabla f_l) = -(\operatorname{div} S_k(0), f_l). \tag{4.7}$$

From (4.6) the sequence $\operatorname{div} S_k(0)$ converges in $\mathbb{L}^2(\Omega;\mathbb{R}^3)$ to $\operatorname{div} S(0) = \rho\partial_t^2 u(0)$. The last statement and (4.7) complete the proof. ∎

The next lemma proves that for sufficiently small initial data the energy function $\mathcal{E}(t;k)$ is a decreasing function.

LEMMA 4.2. *Let the initial data $u^0, u^1, E^{n,0}, z^0$ satisfy the hypotheses of Lemma 2.1 and (u_k, S_k, E_k^n, z_k) be the solution of (2.4)–(2.9) with initial conditions (2.11) obtained in Lemma 2.1. Moreover assume that z^0 and $\mathcal{E}(0)$ satisfy the following conditions:*

There exists a positive constant γ such that

$$C_t(\mathcal{E}(t)+\gamma) < \frac{1}{2} \qquad \text{for } t=0, \tag{4.8}$$

$$(c^s)^2(\mathcal{E}(t)+\gamma) < (z^* g^*)^2 \qquad \text{for } t=0, \tag{4.9}$$

$$\{C_z(4\mathcal{E}(t)+4\gamma)^2 + 2\|\partial_x z(t)\|^2\}^{1/2}(C_x + C_x') < \frac{1}{2} \qquad \text{for } t=0. \tag{4.10}$$

Then there exists $k_0 > 0$ such that for all $k \ge k_0$

$$\mathcal{E}(t;k) + \frac{1}{4}\int_0^t [F_0(\tau;k) + F_1(\tau;k) + F_2(\tau;k)]\,d\tau \le \mathcal{E}(0;k) \le \mathcal{E}(0)+\gamma \tag{4.11}$$

where $F_i(\tau;k)$ denotes the function $F_i(\tau)$ calculated for (u_k, S_k, E_k^n, z_k).

Proof. Since (u_k, S_k, E_k^n, z_k) is a C^2–solution of (2.4)–(2.9), it follows that there exists the largest T with $0 < T \le T_k$ such that (4.8)–(4.10) hold for all $t \in [0,T)$. Now from Lemma 4.1 we choose k_0 so large that

$$\mathcal{E}(0;k) \le \mathcal{E}(0)+\gamma \quad \text{for all } k \ge k_0.$$

From lemmas (3.1)–(3.4) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t;k) \le\; & -F_0(t;k) - \frac{1}{2}\int_0^L \mathcal{G}_k'\left(\frac{|s_k(\tau)|}{z_k(\tau)}\right)\frac{(\partial_t|s_k(\tau)|)^2}{z_k(\tau)}\,d\tau \\ & + C_t\mathcal{E}(t;k)\cdot F_0(t;k) - \frac{1}{2}F_1(t;k) - \frac{1}{2}F_2(t;k) \\ & + \frac{1}{2}\|\partial_x z_k(t)\|(C_x + C_x')(F_0(t;k) + F_1(t;k) + F_2(t;k)).\end{aligned}$$

Now for a fixed $k \ge k_0$ and for $t \in [0,T)$ we have the following estimation

$$\frac{d}{dt}\mathcal{E}(t;k) \le -\frac{1}{4}F_0(t;k) - \frac{1}{4}F_1(t;k) - \frac{1}{4}F_2(t;k) \tag{4.12}$$

where we use that (4.11) gives for $t \in [0,T)$

$$\|\partial_x z_k(t)\|(C_x + C'_x) < \frac{1}{2}$$

and (4.9) implies

$$|s_k(t)| < g^* z^*,$$

thus

$$|s_k(t)| < g^* z_k(t).$$

We integrate the inequality (4.12) and obtain (4.11) for $t \in [0,T)$. Using (4.11) we show that (4.8)–(4.10) are satisfied at the point T, hence $T = T_k$. ∎

The last lemma and Lemma 3.5 yield that the solution (u_k, S_k, E^n_k, z_k) obtained in Lemma 2.1 for initial data satisfying (4.8)–(4.10) and for $k \geq k_0$ exists for all time i.e. $T_k = \infty$. Now we can start with the proof of Theorem 1.1.

Proof of Theorem 1.1. We first assume that the initial data u^0, u^1, $E^{n,0}$, z^0 satisfy the hypotheses of Lemma 2.1. We choose constants C_1, C_2, such that the inequalities (4.8)–(4.10) are satisfied. Thus for $k \geq k_0$ and for all $T > 0$ there exists a solution $(u_k, S_k, E^n_k, z_k) : \Omega \times [0,T) \to \mathbb{R}^3 \times (\mathbb{R}^9)^2 \times \mathbb{R}_+$.

The last lemma implies that

$$\mathcal{E}(t;k) \leq \mathcal{E}(0;k) \leq \mathcal{E}(0) + \gamma.$$

From Lemmas 3.4 and 3.5 we obtain that there exists a positive constant M, such that

$$\|u_k\|_{2,T} + \|S_k\|_{1,T} + \|E^n_k\|_{1,T} + \|z\|_{1,T} \leq M,$$

where $\|\cdot\|_{i,T}$ denotes the norm in the space $\mathbb{H}^i(\Omega \times [0,T))$. Using Rellich's selection theorem for every integer N we can select a subsequence $(u_{N_k}, S_{N_k}, E^n_{N_k}, z_{N_k})$, which converges in $V^N = \mathbb{H}^1(\Omega \times [0,N);\mathbb{R}^3) \times \mathbb{L}^2(\Omega \times [0,N);\mathbb{R}^9)^2 \times \mathbb{L}^2(\Omega \times [0,N);\mathbb{R}_+)$. The diagonal sequence converges in V^T for all $T > 0$. We denote the subsequence by (u_k, S_k, E^n_k, z_k) again and the limes by (u, S, E^n, z). Moreover we can assume that $u_k(x,t) \to u(x,t), S_k(x,t) \to S(x,t), E^n_k(x,t) \to E^n(x,t), z_k(x,t) \to z(x,t)$ for $k \to \infty$ and for almost all $(x,t) \in \Omega \times [0,T)$, and that

$$(u_k, S_k, E^n_k, z_k) \rightharpoonup (u, S, E^n, z) \quad \text{for } k \to \infty$$

in the space $\mathbb{H}^2(\Omega \times [0,T);\mathbb{R}^3) \times \mathbb{H}^1(\Omega \times [0,T);\mathbb{R}^9)^2 \times \mathbb{H}^1(\Omega \times [0,T);\mathbb{R}_+)$.

We will prove now that (u, S, E^n, z) is a solution of the problem (P). It is clear that the equations (P2), (P3) and (P4) are satisfied. We establish $\varphi \in C_0^\infty(\mathbb{R}_+)$, then

$$\int_0^\infty (\rho\partial_t^2 u(t), \varphi(t)f_l)\,dt$$

$$= -\int_0^\infty (\rho\partial_t u(t), \partial_t\varphi(t)f_l)\,dt = -\lim_{k\to\infty}\int_0^\infty (\rho\partial_t u_k(t), \partial_t\varphi(t)f_l)\,dt$$

$$= \lim_{k\to\infty}\int_0^\infty (\rho\partial_t^2 u_k(t), \varphi(t)f_l)\,dt = -\lim_{k\to\infty}\int_0^\infty (S_k(t), \varphi(t)\nabla f_l)\,dt$$

$$= \lim_{k\to\infty}\int_0^\infty (\operatorname{div} S_k(t), \varphi(t)f_l)\,dt = \int_0^\infty (\operatorname{div} S(t), \varphi(t)f_l)\,dt.$$

Thus for every $\psi \in C_0^\infty(\Omega \times (0,\infty))$

$$(\rho\partial_t^2 u, \psi)_{L^2(\Omega\times(0,\infty))} = (\operatorname{div} S, \psi)_{L^2(\Omega\times(0,\infty))}$$

and the equation (P1) is satisfied. The nonlinear equations (P5) and (P6) are satisfied, because the sequence $(u_k, S_k, E_k^n, z_k)(x,t)$ converges for almost all $(x,t) \in \Omega \times [0,T)$. For the Dirichlet boundary-value problem $u_k \in \mathbf{H}^1([0,T); \mathbf{H}_0^1(\Omega;\mathbb{R}^3))$ and $u_k \to u$ in this space, hence $u \in H^1([0,T);$ $\mathbf{H}_0^1(\Omega;\mathbb{R}^3))$ and the boundary condition (D) is satisfied. For the Neumann boundary-value problem we have $S_k(0,t) = S_k(L,t) = 0$ and $S_k \rightharpoonup S$ in $\mathbf{H}^1(\Omega \times [0,T);\mathbb{R}^9)$. This proves the condition (N). Lemma 4.1 yields that the solution (u, S, E^n, z) satisfies initial conditions (I). This proves Theorem 1.1 for initial data satisfying the hypotheses of Lemma 2.1.

To prove the theorem for initial data, which satisfy (2.12) but not (2.14) we define new initial conditions as follows

$$\begin{aligned}
\tilde{u}^0 &= u^0(x) + \frac{1}{2L}\left[\frac{d}{dx}u^0(0)(x-L)^2 - \frac{d}{dx}u^0(L)x^2\right],\\
\tilde{u}^1 &= u^1,\\
\tilde{E}^{n,0} &= E^{n,0} + \frac{1}{L}[E^0(0)(x-L) - E^0(L)x],\\
\tilde{z}^0 &= z^0,
\end{aligned}$$

where $E^0 = \frac{1}{2}(\nabla u^0 + (\nabla u^0)^T)$. For these conditions we find a solution $(\tilde{u}, \tilde{S}, \tilde{E}^n, \tilde{z})$ of the problem (P) and

$$\begin{aligned}
u(x,t) &= \tilde{u}(x,t) + u^0(x) - \tilde{u}^0(x),\\
S(x,t) &= \tilde{S}(x,t),\\
E^n(x,t) &= \tilde{E}^n(x,t) + E^{n,0}(x) - \tilde{E}^{n,0}(x),\\
z(x,t) &= \tilde{z}(x,t)
\end{aligned}$$

is a solution of the problem (P) for the initial data $u^0, u^1, E^{n,0}, z^0$.

Finally to prove the theorem for $E^{n,0}(x) \in \mathbb{H}^1(\Omega;\mathbb{R}^9)$ and $z^0 \in \mathbb{H}^1(\Omega;\mathbb{R}_+)$ we approximate these functions by a sequence $\{E_k^{n,0}, z_k^0\}_{k=1}^{\infty} \subset C^2(\Omega)$ and repeat exactly the approximation process described in the first part of this proof. The proof of Theorem 1.1 is complete. ∎

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