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ON THE SETS $C_V C_V$ IN THE GROUP $SL(n, K)$

In [3] has been proved that

(1) if $A \in SL(n, K)$ is not a scalar and K has at least four elements, then A is a commutator of $SL(n, K)$.

In a present paper we shall prove that

(2) if $A \in SL(n, K)$ is not a scalar and $n < |K| - 1$ or $\operatorname{char} K = 0$, then there exists class C_V such that $A \in C_V C_{V^{-1}}$, where C_V denotes the conjugacy class of $V \in SL(n, K)$.

Observe that the condition $A \in C_V C_{V^{-1}}$ implies that A is a commutator. Hence if $n < |K| - 1$ or $\operatorname{char} K = 0$, then the second theorem is stronger than the first one.

We will give a condition for which there exists $V \in SL(n, K)$ such that $SL(n, K) = C_V C_V$. That is an answer to the question: whether there exists C_V such that $C_V C_V = SL(n, K)$, (see [2]. p. 66). Simultaneously we give a new proof of the equality $PSL(n, K) = C_V C_V$, different from that in [2].

The notation is standard. In addition we shall denote by $SL_1(n, K)$ the subgroup of matrices of determinant equal to $\pm$ 1. We shall use the following theorem.

THEOREM 1 (see [1]). *If* $V, W \in GL(n, K)$, $V = \operatorname{diag}(v_1, \ldots, v_n)$, $W = \operatorname{diag}(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ *for* $i \neq j$, $\det A = \det VW$ *and* $A \notin Z(GL(n, K))$, *then the matrix equation* $X^{-1}VXY^{-1}WY = A$ *has a solution* $X, Y \in GL(n, K))$ *such that* $\det X$, $\det Y$ *are arbitrary elements of* K^*.

COROLLARY 1. *If* $V = \operatorname{diag}(v_1, \ldots, v_n)$, $W = \operatorname{diag}(w_1, \ldots, w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ *for* $i \neq j$, *then*

(3) $SL(n, K) = C_V C_W \cup Z(SL(n, K))$ *for* $V, W \in SL(n, K)$,

(4) $PSL(n, K) = C_V C_W \cup \{E\}$ *for* $V, W \in PSL(n, K)$.

LEMMA 1. *If* $V = \operatorname{diag}(1, b, \ldots, b^{n-1})$, *where* $b \in K^*$ *is a primitive n-th root of unity, then for arbitrary element k of* K^* *the equation*

$$b^{j-1}E = VXVX^{-1}, \quad j = 1, \ldots, n$$

for arbitrary element k of K^* *has a solution X such that* $\det X = k$.

The proof follows by an easy observation that the equation $V^{-1}b^{j-1}X = XV$ has the solution

$$X = \begin{bmatrix} X_j & 0 \\ 0 & X_{n-k} \end{bmatrix}, \quad \text{where } X_j$$

$$= \begin{bmatrix} 0 & & x_{1j} \\ & \cdot^{\cdot^{\cdot}} & \\ x_{j1} & & 0 \end{bmatrix}, \quad X_{n-j} = \begin{bmatrix} 0 & & x_{j+1n} \\ & \cdot^{\cdot^{\cdot}} & \\ x_{nj+1} & & 0 \end{bmatrix}.$$

LEMMA 2.
(i) *If* $V = \operatorname{diag}(v_1, v_1^{-1}, \ldots, v_i, v_i^{-1})$, $v_s \in K^*$ *or*
(ii) *if* $V = \operatorname{diag}(1, v_1, v_1^{-1}, \ldots, v_i, v_i^{-1})$, $v_s \in K^*$,

then for arbitrary element k of K^* *the equation* $VXVX^{-1} = E$ *has a solution X such that* $\det X = k$.

The equation $V^{-1}X = XV$ has the solution $X = \operatorname{diag}(F_1, \ldots, F_i)$ for the case (i) and $X = \operatorname{diag}(1, F_1, \ldots, F_i)$ for the case (ii), where

$$F_s = \begin{bmatrix} 0 & x_{2s-12s} \\ x_{2s2s-1} & 0 \end{bmatrix} \quad \text{for } s = 1, \ldots, i.$$

Now the statement of Lemma 2 is evident.

THEOREM 2. *If K is a field with* $\operatorname{char} K = 0$ *or* $K = F_q$ *and* $n < |F_q| - 1$, *then there are classes* C_V *such that*

(5) $SL(n, K) - Z(SL(n, K)) \subseteq C_V C_V$ *and*
(6) $SL(n, K) - Z(SL(n, K)) \cup \{E\} \subseteq C_V C_{V^{-1}}$.

Proof. It follows from the assumptions that there are matrices $V = \operatorname{diag}(v_1, \ldots, v_n)$ such that $v_i \neq v_j$ for $i \neq j$. Hence for $W = V$ we receive (5) and for $W = V^{-1}$ we obtain (6), by Corollary 1.

THEOREM 3.
(i) *If* $n | q - 1$, $n = 2s + 1$ *or* $n | q - 1$, *q is even, then there exists* $C_V \subset SL(n, q)$ *such that* $SL(n, q) = C_V C_V$.
(ii) *If* $n | q - 1$, *q is odd and* $n = 2s$, *then there exists* $C_V \subset SL_1(n, q)$ *such that* $SL(n, q) = C_V C_V$.

Proof. (i) The matrix $V = \operatorname{diag}(1, a^1, \ldots, (a^i)^{n-1})$, where $ni = q - 1$, a^i is a primitive n-th root of unity, fulfills the assumptions of Corollary 1. Hence for $W = V$ it follows $SL(n,q) \subseteq C_V C_V \cup Z(SL(n,q))$, by Corollary 1. It is easy to see that $\det V = 1$. Therefore $Z(SL(n,q)) \subset C_V C_V$, by Lemma 1.

(ii) Observe that $\det V = -1$. The rest of the proof is same as in (i).

THEOREM 4. *Let K be an algebraically closed field.*

(i) *If $n = 2s+1$, then there exists $C_V \subset SL(n,K)$ such that $SL(n,K) = C_V C_V$.*

(ii) *If $n = 2s$, then there exists $C_V \subset SL_1(n,K)$ such that $SL(n,K) = C_V C_V$.*

Proof. (i) For $W = V = \mathrm{diag}(1, e, \ldots, e^{n-1})$, e-primitive root of unity,

$$SL(n,K) \subseteq C_V C_V \cup Z(SL(n,K)), \tag{7}$$

by Corollary 1. Since $n = 2s+1$, $\det V = 1$. Hence $Z(SL(n,K)) \subseteq C_V C_V$, by Lemma 1. Therefore $SL(n,K) = C_V C_V$, by (7).

(ii) Since $n = 2s$, $\det V = -1$. Hence $C_V \subset SL_1(n,K)$. Complete the proof as in (i).

THEOREM 5.

(i) *If K is a field with* $\mathrm{char}\, K = 0$ *or*

(ii) *If $K = F_q$, q-odd and $n < q-1$ or*

(iii) *if n, q are even and $n < q$, then there exists classes C_V such that $C_V C_V = PSL(n,K)$. (cf [2]).*

Proof. By the assumption of cases (i)-(iii) there are $v_1, \ldots, v_i \in K$ such that $2i = n$ elements $v_1, \ldots, v_i, v_1^{-1}, \ldots, v_i^{-1}$ are different and $v_s \neq 1$ for $s = 1, \ldots i$. If we take

$$V = \mathrm{diag}(v_1, v_1^{-1}, \ldots, v_i, v_i^{-1}) \quad \text{in the even case,}$$
$$V = \mathrm{diag}(1, v_1, v_1^{-1}, \ldots, v_i, v_i^{-1}) \quad \text{in odd case}$$

and $W = V^{-1}$ accordingly, then we receive $PSL(n,K) \subseteq C_V C_{V^{-1}} \cup \{E\}$, by (4). Now by Lemma 2 it follows that $C_V = C_{V^{-1}}$ and $E \in C_V C_V$. Therefore $PSL(n,K) = C_V C_V$.

References

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