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ON SOME PROPERTIES OF ALGEBRAS OF LMC-ALGEBRA VALUED FUNCTIONS

Definitions and notations

Let A be a commutative locally m-convex algebra (lmc-algebra) over the field C of complex numbers. In this paper we shall assume that A has not a unit element. Let $\mathcal{P}(\Lambda) = \{p_\lambda \mid \lambda \in \Lambda\}$ be a family of seminorms defining the topology in A denoted by $T(\mathcal{P})$. We shall assume that $T(\mathcal{P})$ is a Hausdorff topology, in other words, if we have for an element x in A $p_\lambda(x) = 0$ for all $\lambda \in \Lambda$, then $x = 0$. Furthermore, we assume that the family $\mathcal{P}$ is directed i.e. it is closed under taking a maxima of two of its members.

We shall denote by $\Delta(A)$ the set of all non-trivial continuous C-homomorphisms on A. The set $\Delta(A)$ will be equipped with the relative $\sigma(A', A)$-topology. With this topology $\Delta(A)$ will be called the carrier space of $(A, T(\mathcal{P}))$. For a given element $x \in A$ we shall define a function $\widehat{x}$ on the carrier space $\Delta(A)$ by an equation $\widehat{x}(\tau) = \tau(x)$, $\tau \in \Delta(A)$. Furthermore we shall denote $\widehat{A} = \{\widehat{x} \mid x \in A\}$. Obviously $\widehat{A} \subset C(\Delta(A))$ (= the set of all continuous C-valued functions defined on $\Delta(A)$). If I is an ideal of A , then a hull of I denoted by $h(I)$ is defined as $h(I) = \{\tau \in \Delta(A) \mid \widehat{x}(\tau) = 0, x \in I\}$. Correspondingly a kernel $k(E)$ of a subset E of $\Delta(A)$ is defined by $k(E) = \{x \in A \mid \widehat{x}(\tau) = 0,\ \tau \in E\}$ and for an empty set $\emptyset$ we shall define $k(\emptyset) = A$.

For a completely regular space X denote by $C(X, A)$ the set of all continuous A-valued functions defined on X. If $A =$ C (= the field of complex numbers), then we shall denote $C(X, C) = C(X)$. Algebraic operations in $C(X, A)$ will be defined by a pointwise manner. If $x \in A$, then we shall denote by f_x the constant function $f_x(t) = x$, $t \in X$. Thus f_0 is a zero element of $C(X, A)$.

Let $\mathcal{K}$ be a compact cover of X which is closed under a finite union. For given $K \in \mathcal{K}$ and $\lambda \in \Lambda$ we shall define a seminorm $p_{(K,\lambda)}$ on $C(X, A)$ by an equation

$$p_{(K,\lambda)}(f) = \sup_{t\in K} p_\lambda(f(t)), \quad f \in C(X,A).$$

Denote by $\mathcal{P}(\mathcal{K},\Lambda) = \{p_{(K,\lambda)} \mid K \in \mathcal{K}, \lambda \in \Lambda\}$. The family $\mathcal{P}(\mathcal{K},\lambda)$ defines a locally m-convex topology on $C(X,A)$ denoted by $T(\mathcal{K},\mathcal{P})$. Let $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$. We shall denote by $V_{(K,\lambda)}(\epsilon)$ the set $V_{(K,\lambda)}(\epsilon) = \{f \in C(X,A) \mid p_{(K,\lambda)}(f) < \epsilon\}$. Obviously the sets $V_{(K,\lambda)}$, $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$ form a subbase of neighbourhoods at f_0. Let t be a point of X and I an ideal of A. We shall define the ideal $J_{(t,I)}$ of $C(X,A)$ by $J_{(t,I)} = \{f \in C(X,A) \mid f(t) \in I\}$.

In this paper we shall study the ideal structure of the lmc-algebra $(C(X,A),T(\mathcal{K},\mathcal{P}))$. Especially we shall extend some results of [2].

1. Auxiliary results

First we shall consider the structure of $(A,T(\mathcal{P}))$. Since we assume that A has not a unit element we can naturally adjoin the unit element to A by a usual way. (See for ex. [5]). Denote by $A(e)$ the algebra with an adjoint unit. Elements of $A(e)$ will be denoted by (x,α) , where $x \in A$ and $\alpha \in C$. Thus, $(0,1)$ is the unit element of $A(e)$ and we shall denote $(0,1) = e$. Let $\lambda \in \Lambda$. We shall define a seminorm q_λ on $A(e)$ by $q_\lambda((x,\alpha) = p_\lambda(x) + |\alpha|$, $(x,\alpha) \in A(e)$. Denote by $\mathcal{Q}(\Lambda) = \{q_\lambda \mid \lambda \in \Lambda\}$. The family $\mathcal{Q}(\Lambda)$ defines a locally m-convex topology on $A(e)$ and we shall denote this topology by $T(\mathcal{Q})$.

The mapping $x \mapsto (x,0)$ from $(A,T(\mathcal{P}))$ into $(A(e),T(\mathcal{Q}))$ is a semi-isometric homomorphism in the sense of [1]. Namely we have $p_\lambda(x) = q_\lambda((x,0))$, $x \in A$ and $\lambda \in \Lambda$. Obviously $A_0 = \{(x,0) \mid x \in A\}$ is a closed maximal ideal of $(A(e),T(\mathcal{Q}))$. Since $(A,T(\mathcal{P}))$ and $(A_0,T(\mathcal{Q}))$ are semi-isometrically isomorphic (see [1]) they can be identified as lmc-algebras. Therefore $(A,T(\mathcal{P}))$ can be considered as a closed maximal ideal of $(A(e),T(\mathcal{Q}))$. It now follows from this that $(C(X,A),T(\mathcal{K},\mathcal{P}))$ can be considered as a closed ideal of $(C(X,A(e)),T(\mathcal{K},\mathcal{Q}))$.

If I is an ideal of A we shall say that I is regular, if there is an element $u \in A$ such that $ux - x \in I$ for all $x \in I$. The element u will be called an identidy in A modulo I.

LEMMA 1.1. *Let I be a closed ideal of a lmc-algebra $(A,T(\mathcal{P}))$ (A with or without unit). Let I_0 be a closed regular proper ideal of $(I,T(\mathcal{P}))$. Then I_0 is also a regular ideal of A and furthermore there is a closed proper regular ideal I_1 of $(A,T(\mathcal{P}))$ such that $I_0 = I_1 \cap I$.*

Proof. To prove that I_0 is also an ideal of A it suffices to show that $xy \in I_0$ for all $x \in I_0$ and $y \in A$. Let u be an identity in I modulo I_0. Now for any $x \in I_0$ and $y \in A$ we have uy and $xy \in I$. Thus $xy = x(uy)-(u(xy)-xy) \in I_0$

for all $x \in I_0$ and $y \in A$. Obviously u is also an identity in A modulo I_0 and thus I_0 is a regular ideal of A.

Let $I_1 = \{x \in A \mid ux \in I_0\}$. We shall show that I_1 is a closed ideal of $(A, T(\mathcal{P}))$ and $I_0 = I_1 \cap I$. Let $\{x_\alpha \mid \alpha \in \Gamma\}$ be a net in I_1 for which $x_\alpha \to x$ for some $x \in A$. Then we have $ux_\alpha \to ux$ and $ux_\alpha \in I_0$. Thus $ux_0 \in I_0$ and so $x \in I_1$. It is easy to see that I_1 is a regular ideal of A. Clearly $I_0 \subset I_1 \cap I$. If $x \in I_1 \cap I$, then $ux \in I_0$ and $ux - x \in I_0$ and thus $x = ux - (ux - x) \in I_0$ and we can see that $I_1 \cap I \subset I_0$ and therefore $I_0 = I_1 \cap I$.

COROLLARY 1.1 *Each closed proper regular ideal J of $(C(X, A), T(\mathcal{K}, \mathcal{P}))$ is a $C(X)$-module. In other words we have $gf \in J$ for all $g \in C(X)$ and $f \in J$.*

Proof. Since $(C(X, A), T(\mathcal{K}, \mathcal{P}))$ can be considered as a closed ideal of $(C(X, A(e)), T(\mathcal{K}, \mathcal{Q}))$ by Lemma 1.1 there is a closed ideal J_1 of $(C(X, A(e)), T(\mathcal{K}, \mathcal{Q}))$ such that $J = J_1 \cap C(X, A)$. Now for any $f \in J \subset J_1$ and $g \in C(X) \subset C(X, A(e))$ we have $gf \in J_1$. Since fg belongs to $C(X, A)$, we have $gf \in J$.

Denote by $L(C(X), A)$ the linear hull of the sets $C(X)$ and A. Thus $L(C(X), A) = \{\sum_{i=1}^{n} g_i f_{x_i} \mid g_i \in C(X), x_i \in A, n \in N\}$.

LEMMA 1.2. *$L(C(X), A)$ is a dense subset of $(C(X, A), T(\mathcal{K}, \mathcal{P}))$.*

Proof. This result can be shown similarly as the corresponding result for the compact open topology. See [3] Theorem 2.3.1.

2. On the ideal structure of $(C(X, A), T(\mathcal{K}, \mathcal{P}))$

In this chapter we shall extend some results of [2] in such a sense that A has not a unit element. Let t be a point of X and J an ideal of $C(X, A)$. We shall denote by $I(t) = cl(\{f(t) \mid f \in J\}$ where cl means the closure operation in A with respect to the topology $T(\mathcal{P})$.

THEOREM 2.1. *Let J be a proper closed regular ideal of $(C(X, A), T(\mathcal{K}, \mathcal{Q}))$. Then there is at least one point $t \in X$ such that $I(t)$ is a proper closed regular ideal of $(A, T(\mathcal{P}))$.*

Proof. It is easy to see that $I(t)$ is either a proper closed regular ideal of $(A, T(\mathcal{P}))$ or otherwise $I(t) = A$. Suppose that $I(t) = A$ for all $t \in X$. Thus for $x \in A$, $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$ and any $t \in K$ there is a function $f^{(t)} \in J$ such that

$$p_\lambda(f^{(t)}(t) - x) = p_\lambda(f^{(t)}(t) - f_x(t)) < \epsilon.$$

So by the continuity of $f^{(t)}$ and f_x there is a neighbourhood $U(t)$ of t for which

$$p_\lambda(f^{(t)}(s) - f_x(s)) < \epsilon \text{ for all } s \in U(t). \tag{2.1}$$

By the compactness of K, there are $t_1, t_2, \ldots, t_n$ in K such that (2.1) holds true for $t = t_1, t_2, \ldots, t_n$ and by Lemma 2.1.1 of [3] there are functions $\alpha_i \in C(X)$, $i = 1, 2, \ldots, n$ such that $0 \leq \alpha_i(t) \leq 1$ for all $t \in X$ and $i = 1, 2, \ldots, n$, $supp\alpha_i \subset U_i$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i(t) = 1$ for all $t \in K$. Put $F_{(K,\lambda)} = \sum_{i=1}^{n} \alpha_i f_i$ by Corollary 1.1 $F_{(K,\lambda)} \in J$. Obviously

$$p_\lambda(F_{(K,\lambda)} - f_x) < \epsilon.$$

Since $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$ were choosen arbitrarily we infer that all constant functions f_x are in J. The Corollary 1.1 implies now $L(C(X), A) \subset J$. By Lemma 1.2 we obtain $C(X, A) = J$ - a contradiction proving our assertion.

Next we shall characterisize maximal regular ideals and a carrier space of the algebra $(C(X, A), T(\mathcal{K}, \mathcal{P}))$. Let $t \in X$ and $\tau \in \Delta(A)$. Denote by $\phi_{(t,\tau)}$ the mapping from $C(X, A)$ into C defined by

$$\phi_{(t,\tau)}(f) = \tau(f(t)), \; f \in C(X, A).$$

It is easy to see that $\phi \in \Delta(C(X, A))$.

LEMMA 2.1. *If N is a closed regular maximal ideal of $(C(X, A), T(\mathcal{K}, \mathcal{P}))$, then there are unique points $t \in X$ and $\tau \in \Delta(A)$ such that*

$$N = ker\phi_{(t,\tau)} = \{f \in C(X, A) \mid \phi_{(t,\tau)}(f) = 0\}.$$

Proof. By Theorem 2.1 there is a point t in X such that $I(t) = cl(\{f(t) \mid f \in N\})$ is a proper closed regular ideal of $(A, T(\mathcal{P}))$. This implies that there is $\tau \in \Delta(A)$ with $I(t) \subset ker\tau$. Now, if $f \in N$, then $f(t) \in I(t) \subset ker\tau$ so that $N \subset ker\phi_{(t,\tau)}$. By the maximality of N we have $N = ker\phi_{(t,\tau)}$.

Next we shall show that the points t and τ are unique. Suppose that $N = ker\phi_{(t_0,\tau_0)} = ker\phi_{(t_1,\tau_1)}$. This implies that $\phi_{(t_0,\tau_0)} = \phi_{(t_1,\tau_1)}$. Now we have $\tau_0(x) = \tau_0(f_x(t_0)) = \phi_{(t_0,\tau_0)}(f_x) = \phi_{(t_1,\tau_1)}(f_x) = \tau_1(f_x(t_1)) = \tau_1(x)$ for all $x \in A$. Thus we can see that $\tau_0 = \tau_1$. If $t_0 \neq t_1$, then by completely regularity of X there is a function $g \in C(X)$ for which $g(t_0) = 1$ and $g(t_1) = 0$. Since $\tau_0 = \tau_1$ there is an element y in A such that $\tau_0(y) = \tau_1(y) = 1$. If we now choose $f = gf_y$, then $f \in C(X, A)$ and $\phi_{(t_0,\tau_0)}(f) = 1$ and $\phi_{(t_1,\tau_1)}(f) = 0$ which contradicts with an assumption that $\phi_{(t_0,\tau_0)} = \phi_{(t_1,\tau_1)}$.

COROLLARY 2.1. *Each closed regular maximal ideal N of $(C(X, A), T(\mathcal{K}, \mathcal{P}))$ is of the form $N = J_{(t,ker\tau)}$ for some $t \in X$ and $\tau \in \Delta(A)$.*

Denote by φ the mapping from $X \times \Delta(A)$ into $\Delta(C(X, A))$ defined by

$$\varphi(t, \tau) = \phi_{(t,\tau)}, \quad (t, \tau) \in X \times \Delta(A). \tag{2.2}$$

THEOREM 2.2. *The mapping φ defined in (2.2) is a bijection from $X \times \Delta(A)$ onto $\Delta(C(X, A))$ for which the inverse function φ^{-1} is automatically continuous. Furthermore, φ is continuous, if $\Delta(A)$ is locally equicontinuous.*

Proof. By the proof of Lemma 2.1 φ is an injection from $X \times \Delta(A)$ into $\Delta(C(X,A))$. To prove the surjectivity let $\phi \in \Delta(C(X,A))$ be arbitrary. Now $\ker \phi$ is a closed regular maximal ideal of $(C(X,A), T(\mathcal{K},\mathcal{P}))$. So by Lemma 2.1 there are unique points $t \in X$ and $\tau \in \Delta(A)$ such that $ker\phi = ker\phi_{(t,\tau)}$. But this implies that $\phi = \phi_{(t,\tau)} = \varphi(t,\tau)$.

To prove the continuity of the inverse mapping φ^{-1} we can use a similar method that Prolla has used in [6] for non-Archimedean function algebras. So let $\{\phi_\alpha \mid \alpha \in \Gamma\}$ be a net in $\Delta(C(X,A))$ such that $\phi_\alpha \to \phi$ for some $\phi \in \Delta(C(X,A))$. Since φ is a surjection there are elements $t_\alpha \in X$, $\alpha \in \Gamma$, $t \in X$, $\tau_\alpha \in \Delta(A)$, $\alpha \in \Gamma$ and $\tau \in \Delta(A)$ such that $\phi_\alpha = \phi_{(t_\alpha,\tau_\alpha)}$ and $\phi = \phi_{(t,\tau)}$. Let f be an element of $C(X,A)$ for which $\phi(f) = 1$. Since $\phi_\alpha \to \phi$ there is $\alpha_0 \in \Gamma$ such that $\phi_\alpha(f) > 0$ for all $\alpha > \alpha_0$. Let $g \in C(X)$ be arbitrary. Now, if $\alpha > \alpha_0$, then

$$g(t_\alpha) = \frac{g(t_\alpha)\tau_\alpha(f(t_\alpha))}{\tau_\alpha(f(t_\alpha))} = \frac{\phi_\alpha(gf)}{\phi_\alpha(f)}.$$

Since $\phi_\alpha(f) \to 1$ we therefore have

$$g(t_\alpha) \to \phi(gf) = \tau(g(t)f(t)) = g(t)\tau(f(t)) = g(t)\phi(f) = g(t) \quad \text{for all } g \in C(X).$$

Now the topology of X coincide with the weak topology in X generated by $C(X)$. So we can see that $t_\alpha \to t$.

By similar methods it can be shown that $\widehat{x}(\tau_\alpha) \to \widehat{x}(\tau)$ for all $x \in A$. Since the topology of $\Delta(A)$ coincide with the weak topology in $\Delta(A)$ generated by $\widehat{A}$ we can see that $\tau_\alpha \to \tau$. Thus from the condition $\phi_\alpha \to \phi$ it follows that $(t_\alpha, \tau_\alpha) \to (t,\tau)$ which implies that $\varphi^{-1}(\phi_\alpha) \to \varphi^{-1}(\phi)$ and we have shown that φ^{-1} is continuous.

The continuity of φ depends on the continuity of the mapping $\widehat{f} : X \times \Delta(A) \mapsto C$ defined by $\widehat{f}(t,\tau) = \widehat{f}(\phi_{(t,\tau)}) = \tau(f(t))$, $(t,\tau) \in X \times \Delta(A)$. But this map is continuous, if $\Delta(A)$ is locally equicontinuous. (See for ex. [4]). Thus φ is continuous, if $\Delta(A)$ is locally equicontinuous.

COROLLARY 2.2. *The carrier space $\Delta(C(X,A))$ of $(C(X,A), T(\mathcal{K},\mathcal{P}))$ is homeomorphic to $X \times \Delta(A)$, if $\Delta(A)$ is locally equicontinuous.*

Next we shall give a description of all proper closed regular ideals of $(C(X,A), T(\mathcal{K},\mathcal{P}))$.

THEOREM 2.3. *If J is a proper closed regular ideal of $(C(X,A), T(\mathcal{K},\mathcal{P}))$, then there is a subset E of X and a family $\{I(t) \mid t \in E\}$ of proper closed regular ideals of $(A, T(\mathcal{P}))$ such that $J = \cap_{t\in E} J_{(t,I(t))}$.*

Proof. For a given point $t \in X$ we put $I(t) = cl(\{f(t) \mid f \in J\})$. Now it is easy to see that $I(t)$ is either a proper closed regular ideal of

$(A, T(\mathcal{P}))$ or otherwise $I(t) = A$. Denote by $E = \{t \in X \mid I(t) \neq A\}$. By Theorem 2.1 E is non empty. Obviously $J \subset \cap_{t \in E} J_{(t,I(t))}$. If $t \in X \sim E =$ the complement of E in X, then $J_{(t,I(t))} = C(X, A)$ from which it follows that $\cap_{t \in X} J_{(t,I(t))} = \cap_{t \in E} J_{(t,I(t))}$. Now let $f \in \cap_{t \in X} J_{(t,I(t))}$ be arbitrary. Furthermore let $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$ be arbitrary. It follows from the definition of $I(t)$ that for each $t_0 \in K$ there is a function $f_{t_0} \in J$ such that $p_\lambda(f_{t_0}(t_0) - f(t_0)) < \frac{\epsilon}{2}$. Since the functions f_{t_0} and f are continuous at t_0 there is a neighbourhood $U(t_0) \subset X$ of t_0 such that

$$p_\lambda(f_{t_0}(t) - f(t)) < \epsilon \text{ for all } t \in U(t_0). \tag{2.3}$$

Thus for each $t \in K$ there is a function $f_t \in J$ and an open neighbourhood $U(t_0)$ of t such that (2.3) holds. Now the family $\{U(t) \mid t \in K\}$ forms an open cover of the compact set K and thus there is a finite subcover U_1, U_2 ,..., U_n of it. By using a similar method that was used in the proof of Theorem 2.1 we can show that there is a function $F_{(K,\lambda)} \in J$ such that $p_{(K,\lambda)}(F_{(K,\lambda)} - f) < \epsilon$. Since $K \in \mathcal{K}$, $\lambda \in \Lambda$ and $\epsilon > 0$ were chosen arbitrarily we can see as in the proof of Theorem 2.1 that $f \in cl(J) = J$ which completes the proof.

We shall say that $(A, T(\mathcal{P}))$ has a property of spectral synthesis, if $k(h(I)) = I$ for all closed ideal I of $(A, T(\mathcal{P}))$.

COROLLARY 2.2. *$(C(X, A), T(\mathcal{K}, \mathcal{P}))$ has a property of spectral synthesis if and only if $(A, T(\mathcal{P}))$ has this property.*

Proof. See [2] Corollary 2.4.

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