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SYMMETRIES OF SECTIONAL CURVATURE ON 3-MANIFOLDS

1. Introduction

Let M be a smooth 3-manifold and g a pseudoriemannian metric tensor on M . Let $G_2(M)$ denote the Grassmannian bundle with fibers $G_2(T_x M)$, the space of (2-dimensional) planes in the tangent space $T_x M$ at a point $x \in M$. Observe that each $G_2(T_x M)$ may be regarded as a (real) algebraic variety, diffeomorphic to the (real) projective plane \mathbb{P}^2 . As in [1], we shall regard the sectional curvature K_x at each point $x \in M$ as a rational mapping of algebraic varieties $G_2(T_x M) \rightarrow \mathbb{R}$, or a rational function for short. The group of all automorphisms of $G_2(T_x M)$ is isomorphic to $PGL_3 \equiv PGL_3(\mathbb{R})$, the group of projective automorphisms of \mathbb{P}^2 . We may then ask: what is the largest subgroup of PGL_3 which leaves K_x invariant as a rational function? We shall refer to this group as the *symmetry group* of K at x .

Throughout this paper, we shall concentrate on the Lorentzian case, giving complete details. On the other hand, we shall merely summarize the Riemannian case, omitting details and leaving them for the reader to supply.

Thus, in Section 2, we determine the possible symmetry groups for Lorentzian sectional curvature, finding them to be

$$PGL_3, \quad PO_2, \quad PO_1^1, \quad PHT, \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad 1,$$

the first case characterizing constant curvature at a point. For this, we use the analysis of canonical forms for selfadjoint operators on a Minkowskian

1991 *Mathematics Subject Classification*: Primary 53C50; Secondary 53B30, 53C30.

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³ Partially supported by DGICYT-Spain.

(Lorentzian) vector space [10, pp. 261–262]. For comparison, we list the corresponding results for Riemannian sectional curvature.

In Section 3, we study the existence of naturally reductive spaces in which the symmetry group of K is the same at every point. All *continuous* symmetry groups (see Theorem 2.2) can be realized on compact models, but there are no naturally reductive models of constant negative curvature or with any *discrete* symmetry group. We do not seek a “neat” classification here, so questions about effectiveness of actions are of no relevance to us, for example. (Cf. [2, p. 58f] for a list of isometry classes of simply connected Riemannian homogeneous spaces of dimension 3.)

Finally, Section 4 contains some particular examples. In most cases, we are able to obtain models which are trivial circle bundles over surfaces. We defer to another paper [3] the exhibition of more explicit forms of metric tensors on these spaces; the techniques used there are not directly related to those used here. We also defer to other papers [3, 4] the exhibition of examples which are not naturally reductive, including all left-invariant Lorentzian metric tensors on 3-dimensional Lie groups.

Our Lorentzian metric tensors will have signature $+- -$. In some cases, we shall have to distinguish among the possible orderings $+- -$, $-+-$, $--+$. (To convert to the other signature convention $++-$, see [10, p. 92].) Thus a vector v is *timelike* if $g(v, v) > 0$, *lightlike* or *null* if $g(v, v) = 0$, *spacelike* if $g(v, v) < 0$, and *causal* if $g(v, v) \geq 0$.

If A is a matrix regarded as a linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$, then the induced mapping $\bigwedge^2 A: \bigwedge^2 \mathbb{R}^n \rightarrow \bigwedge^2 \mathbb{R}^n$ is given by the matrix classically called the *second compound* of A , the matrix whose entries are the determinants of 2×2 submatrices of A in an appropriate ordering [7, Sec. 7.2].

Throughout, we shall regard the Riemann tensor R_{ijkl} as a quadratic form on $\bigwedge^2 TM$ and thus, via the Plücker embedding, on $G_2(M)$; cf. [1]. Then the sectional curvature appears as a rational function on $G_2(M)$ in the form of a quotient of two quadratic functions:

$$K = \frac{R}{\bigwedge^2 g}.$$

Also recall that the associated tensor R_{kl}^{ij} represents the curvature operator $\bar{R}: \bigwedge^2 TM \rightarrow \bigwedge^2 TM$ in local coordinates. Note that if R and \bar{R} are written as matrices with respect to the same local coordinates, then $R = (\bigwedge^2 g)\bar{R}$.

We denote the Lorentz group in $(n = p + q)$ dimensions of signature (p, q) by $O_p^q = O_p^q(\mathbb{R})$, thus the (usual) orthogonal group by $O_n = O_n(\mathbb{R})$. Projectivization of any group of linear transformations is indicated by a prefixed P ; for example $PGL_3 = GL_3/\{aI : 0 \neq a \in \mathbb{R}\} \cong SL_3$.

We thank Graham Hall for pointing out to us the symmetry of horocyclic translations, and for reminding us that eigenvalue degenerations produce symmetries. Most of the results in Section 2 were previously presented by Parker at the A.M.S. Winter Meetings in 1984 and 1985, and at the Bolyai Colloquium on Differential Geometry in 1984. Parker thanks Cordero and the Departamento at Santiago for their extraordinary hospitality during his visits.

2. Canonical forms and symmetry groups

We begin by finding canonical forms for the Riemann tensor R considered as a quadratic form on $\wedge^2 TM$. We shall do this by first finding canonical forms for the selfadjoint operator $\bar{R}: \wedge^2 TM \rightarrow \wedge^2 TM$. It suffices to work pointwise, so consider \mathbb{R}^3 with the Minkowski inner product given by $\eta = \text{diag}[1, -1, -1]$ and let \bar{R} be a $\wedge^2 \eta$ -selfadjoint operator $\wedge^2 \mathbb{R}^3 \rightarrow \wedge^2 \mathbb{R}^3$. Note, $\wedge^2 \eta = \text{diag}[-1, -1, 1]$.

From O'Neill [10, pp. 261–262, ex. 19], we find that there are four canonical forms for selfadjoint operators such as \bar{R} . Let e_1, e_2, e_3 denote the usual basis for \mathbb{R}^3 . We choose $e_{12} = e_1 \wedge e_2$, $e_{13} = e_1 \wedge e_3$, $e_{23} = e_2 \wedge e_3$ as our associated basis for $\wedge^2 \mathbb{R}^3$. Changing bases from O'Neill's to ours, we have

LEMMA 2.1. *The selfadjoint operator \bar{R} appears, with respect to some η -orthonormal basis of \mathbb{R}^3 , in precisely one of the following forms on the associated basis of $\wedge^2 \mathbb{R}^3$:*

$$\begin{aligned} & \begin{bmatrix} -B & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & A \end{bmatrix}, & \begin{bmatrix} -B & 0 & 0 \\ 0 & A & -F \\ 0 & F & A \end{bmatrix}, \\ & \begin{bmatrix} -B & 0 & 0 \\ 0 & \lambda \mp \frac{1}{2} & \mp \frac{1}{2} \\ 0 & \pm \frac{1}{2} & \lambda \pm \frac{1}{2} \end{bmatrix}, & \begin{bmatrix} \lambda & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \lambda \end{bmatrix}, \end{aligned}$$

where A, B, C, F, λ are independent real constants.

Now let M be a 3-manifold with Lorentzian metric tensor g . We agree that whenever we choose local coordinates at $x \in M$ which give rise to a g -orthonormal basis (e_1, e_2, e_3) of $T_x M$ with $g(e_1, e_1) = 1$, we shall use (e_{12}, e_{13}, e_{23}) as the associated basis of $\wedge^2 T_x M$. With these choices, the Riemann tensor R_x appears as a 3×3 symmetric matrix

$$\begin{bmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1213} & R_{1313} & R_{1323} \\ R_{1223} & R_{1323} & R_{2323} \end{bmatrix},$$

which we regard as a quadratic form on $\wedge^2 T_x M$ as in [1].

Applying Lemma 2.1 pointwise and using $R = (\wedge^2 g)\bar{R}$, we have

THEOREM 2.2. *At each point x of a Lorentzian 3-manifold (M, g) , there exists a choice of g -orthonormal coordinates with respect to which the Riemann tensor R_x on $\wedge^2 T_x M$ takes on exactly one of these canonical forms:*

$$\begin{aligned} \text{CF1} \quad & \text{diag}[B, C, A]; \\ \text{CF2} \quad & \begin{bmatrix} B & 0 & 0 \\ 0 & -A & F \\ 0 & F & A \end{bmatrix}, \quad F \neq 0; \\ \text{CF3} \quad & \begin{bmatrix} B & 0 & 0 \\ 0 & -\lambda \pm \frac{1}{2} & \pm \frac{1}{2} \\ 0 & \pm \frac{1}{2} & \lambda \pm \frac{1}{2} \end{bmatrix}; \\ \text{CF4} \quad & \begin{bmatrix} -\lambda & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \lambda \end{bmatrix}. \end{aligned}$$

We note that these forms can also be characterized in terms of eigenvectors of \bar{R} : CF1 corresponds to a timelike eigenvector, CF2 to a spacelike eigenvector, CF3 to a double null eigenvector, and CF4 to a triple null eigenvector; compare [6, §4.3].

We now determine the pointwise symmetry group of each associated sectional curvature $K_x = R_x / \wedge^2 g_x$. Before we begin, let us note that these symmetry groups may also be regarded as parameterizing the choices of local orthonormal coordinates with respect to which R_x appears in its canonical form.

Observe that if $R_x = c \wedge^2 g_x$, then $K_x = c$ is a constant and is invariant under all automorphisms of the Grassmannian $G_2(T_x M)$. Therefore, in this case the symmetry group is PGL_3 . In terms of our canonical forms, constant sectional curvature at x is characterized by CF1 with $A = -B = -C$. Note that in all other cases, the symmetry group will be a subgroup of $PO_1^2(- - +)$; see Table 1.

The procedure for determining the remaining symmetry groups is as follows. First, the invariance group of R_x on $\wedge^2 T_x M$, expressed with respect to an appropriate orthonormal basis of $T_x M$, is $\{A \in GL_3; A^t R_x A = R_x\}$. Note that with respect to the same basis, the invariance group of $\wedge^2 g_x$ is $O_1^2(- - +) \leq GL_3$. The desired symmetry group of K_x is then the projection of the intersection of these two invariance groups.

Table 1. Lorentzian sectional curvature symmetry groups

Canonical form of R_x	Symmetry group of $K_x = R_x / \wedge^2 g_x$
CF1: $\text{diag}[B, C, A]$	$A = -B = -C$ PGL_3 $B = C \neq -A$ PO_2 $A = -B \neq -C$ PO_1^1 $A = -C \neq -B$ PO_1^1 generic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$
CF2	\mathbb{Z}_2
CF3	$B \neq -\lambda$ \mathbb{Z}_2 $B = -\lambda$ PHT
CF4	1

Now consider CF1. Letting $A = [a_{ij}]$ with $1 \leq i, j \leq 3$, it is easy to see that if $B = C$, then $a_{13} = a_{23} = a_{31} = a_{32} = 0$ and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in O_2 \leq O_1^2(- - +).$$

Hence $a_{33} = \pm 1$ and the symmetry group is isomorphic to PO_2 . Similarly, if $A = -B$ or $A = -C$, it follows that the symmetry group is isomorphic to PO_1^1 . Finally, if none of these hold, direct calculation from $A^t R_x A = R_x$ shows that A is diagonal with all diagonal entries ± 1 , independently of each other. Projectivized, this group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Next, consider CF2. Again, one begins with $A^t R_x A = R_x$ and calculates directly to obtain the invariance group, finding that A is diagonal with all three diagonal entries ± 1 . But this time, one also obtains $a_{22} = a_{33}$. Projectivizing, we obtain a symmetry group isomorphic to \mathbb{Z}_2 .

Next, consider CF3. Using the same procedure, we find two cases. When $B \neq -\lambda$, we obtain the group isomorphic to \mathbb{Z}_2 as for CF2. When $B = -\lambda$, we obtain the group HT of horocyclic translations (called “null rotations” in relativity because there is a fixed null direction). The identity component of this group consists of all the matrices

$$\exp \begin{bmatrix} 0 & -t & t \\ t & 0 & 0 \\ t & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t & t \\ t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\ t & -\frac{t^2}{2} & 1 + \frac{t^2}{2} \end{bmatrix}, \quad t \in \mathbb{R},$$

and appears naturally as the nilpotent group in an Iwasawa decomposition of SO_1^{2+} . Under the canonical double covering $SL_2 \twoheadrightarrow SO_1^{2+}$, it is the image

of the parabolic elements of the form

$$\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

See [5] for more about this subgroup. Each component of $O_1^2(- - +)$ contains one component of HT . Projectivizing, we obtain PHT .

Finally, consider CF4. Direct calculation from $A^t R_x A = R_x$ again yields a diagonal A , now with $a_{11} = a_{22} = a_{33} = \pm 1$. Projectivizing, we obtain the trivial group 1.

This completes the determination of all possible pointwise symmetry groups for the sectional curvature function K_x of a Lorentzian 3-manifold, as listed in Table 1.

We conclude this section with the Riemannian case. Here, selfadjoint transformations correspond to symmetric matrices, which are always orthogonally diagonalizable. Thus only CF1 occurs for Riemannian metrics, and we obtain Table 2.

Table 2. Riemannian sectional curvature symmetry groups

Canonical form of R_x	Symmetry group of $K_x = R_x / \bigwedge^2 g_x$
CF1: $\text{diag}[B, C, A]$	$A = B = C$ PGL_3
	$A = B \neq C$ $\left. \begin{array}{l} A = C \neq B \\ B = C \neq A \end{array} \right\} PO_2$
	$A = C \neq B$
	$B = C \neq A$
	generic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

3. Existence of naturally reductive models

In this section, we determine which CF types can be realized on naturally reductive spaces so that the CF type of K (and thus the symmetry group) is the same at every point. Thus we now consider naturally reductive homogeneous spaces $M = G/H$ of dimension 3 with a left-invariant Lorentzian metric tensor.

According to Corollary 2.11, p.198, Theorem 3.3 (2), p. 201, and the definition preceding Proposition 3.4, p. 202, of [8], naturally reductive Lorentzian homogeneous spaces are complete. From Proposition 3.3, p. 255, and Lemma 2.8, p. 253, of [8], it follows that $\dim G \leq 6$. For connected M , $\dim G = 6$ implies constant curvature. Thus we need to consider only G of dimensions 3, 4 and 5.

Let \mathfrak{g} and \mathfrak{h} denote the Lie algebras of G and H , respectively. In order that G/H be reductive, we must have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and

$$(3.1) \quad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

It then suffices to consider ad_H -invariant nondegenerate symmetric bilinear forms B on \mathfrak{m} such that

$$(3.2) \quad B(X, [Y, Z]) - B([Z, X], Y) = 0$$

for all $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{h}$. The naturally reductive condition is then

$$(3.3) \quad B(X, [Y, Z]_{\mathfrak{m}}) - B([Z, X]_{\mathfrak{m}}, Y) = 0$$

for all $X, Y, Z \in \mathfrak{m}$. (Cf. [8, pp. 200–201].)

We begin by showing that all these spaces with $\dim G = 3$ are flat or of constant positive curvature. We shall use the summation convention without further explicit notice.

THEOREM 3.1. *If $M = G/H$ is an irreducible, naturally reductive, Lorentzian homogeneous space of dimension 3, then either M is flat or of constant positive curvature. In the former case, M is Minkowskian 3-space or one of its quotients by a discrete group of translations. In the latter, M is $SO_1^{2+}(\mathbb{R})$ or one of its coverings or quotients by a discrete subgroup.*

Proof. Let \mathfrak{g} have basis (e_1, e_2, e_3) chosen so that $B(e_1, e_1) = -B(e_2, e_2) = -B(e_3, e_3) = 1$ and $B(e_i, e_j) = 0$ for $i \neq j$. Let $[e_i, e_j] = c_{ij}^k e_k$ define the structure constants. Now $\mathfrak{m} = \mathfrak{g}$ and $\mathfrak{h} = 0$, so (3.1) is trivial and (3.2) becomes $B(X, [Y, Z]) - B([Z, X], Y) = 0$ for $X, Y, Z \in \mathfrak{g}$. Considering two and three different arguments in this equation, we obtain the structure equations of \mathfrak{g} as

$$[e_1, e_2] = -ae_3,$$

$$[e_2, e_3] = ae_1,$$

$$[e_3, e_1] = -ae_2.$$

If $a = 0$, then the Lie algebra is abelian and we obtain the flat cases. If $a \neq 0$, then the Lie algebra is isomorphic to \mathfrak{so}_1^2 .

We compute the curvature of the latter. Using the formula from [8, Thm. 3.3], the Levi-Civita connection is given by $\nabla_X Y = \frac{1}{2}[X, Y]$. A direct computation then yields

$$R_{1212} = -\frac{1}{4}a^2,$$

$$R_{1313} = -\frac{1}{4}a^2,$$

$$R_{2323} = -\frac{1}{4}a^2,$$

$$R_{1213} = R_{1223} = R_{1323} = 0,$$

and we find an example of CF1 with group PGL_3 . ■

We observe that this is the situation for $SL_2(\mathbb{R})$ considered by Nomizu [9].

3.1. $\dim G = 4$

Let \mathfrak{g} have basis (e_1, e_2, e_3, e_4) with $B(e_i, e_i) = \varepsilon_i = \pm 1$ and $B(e_i, e_j) = 0$ for $i \neq j$. Again, $[e_i, e_j] = c_{ij}^k e_k$ defines the structure constants. We take $\mathfrak{h} = \langle e_4 \rangle$ so that $\mathfrak{m} = \llbracket e_1, e_2, e_3 \rrbracket$, where $\langle \bullet \rangle$ denotes the algebra generated by what is enclosed and $\llbracket \bullet \rrbracket$ denotes the linear span of what is enclosed. Thus exactly one $\varepsilon_i = 1$ for $1 \leq i \leq 3$, but the choice of ε_4 is free. We want G/H to be naturally reductive, (3.3) and (3.2), and B to be ad_H -invariant (3.1).

Working all these out, we find that the structure equations for \mathfrak{g} become

$$(3.4) \quad \begin{cases} [e_1, e_2] = c_{12}^3 e_3 + c_{12}^4 e_4, \\ [e_2, e_3] = \varepsilon_1 \varepsilon_3 c_{12}^3 e_1 + c_{23}^4 e_4, \\ [e_3, e_1] = \varepsilon_2 \varepsilon_3 c_{12}^3 e_2 + c_{31}^4 e_4, \\ [e_1, e_4] = \varepsilon_1 \varepsilon_2 c_{42}^1 e_2 + c_{14}^3 e_3, \\ [e_2, e_4] = c_{24}^1 e_1 + \varepsilon_2 \varepsilon_3 c_{43}^2 e_3, \\ [e_3, e_4] = \varepsilon_1 \varepsilon_3 c_{41}^3 e_1 + c_{34}^2 e_2. \end{cases}$$

From the Jacobi identity, we obtain

$$(3.5) \quad \begin{cases} \varepsilon_1 c_{24}^1 c_{31}^4 - \varepsilon_3 c_{14}^3 c_{12}^4 = 0, \\ \varepsilon_2 c_{34}^2 c_{12}^4 - \varepsilon_1 c_{24}^1 c_{23}^4 = 0, \\ \varepsilon_3 c_{14}^3 c_{23}^4 - \varepsilon_2 c_{34}^2 c_{31}^4 = 0. \end{cases}$$

Thus all the connected, naturally reductive, Lorentzian homogeneous spaces with $\dim G = 4$ are obtained from all simultaneous solutions of (3.4) and (3.5) and their quotients by discrete subgroups of G . We calculate their Riemann tensors using the formula

$$(3.6) \quad \begin{aligned} R(X, Y)Z &= \frac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad - \frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} - [[X, Y]_{\mathfrak{h}}, Z]_{\mathfrak{m}}, \quad X, Y, Z \in \mathfrak{m}, \end{aligned}$$

found in the proof of Proposition 3.4 of [8, p. 202], obtaining

$$(3.7) \quad \begin{cases} R_{1212} = \frac{1}{4}\varepsilon_3(c_{12}^3)^2 + \varepsilon_1 c_{24}^1 c_{12}^4, \\ R_{1313} = \frac{1}{4}\varepsilon_2(c_{12}^3)^2 + \varepsilon_3 c_{14}^3 c_{31}^4, \\ R_{2323} = \frac{1}{4}\varepsilon_1(c_{12}^3)^2 + \varepsilon_2 c_{34}^2 c_{23}^4, \\ R_{1213} = \varepsilon_3 c_{41}^3 c_{12}^4 = \varepsilon_1 c_{24}^1 c_{13}^4, \\ R_{1223} = \varepsilon_1 c_{24}^1 c_{23}^4 = \varepsilon_2 c_{34}^2 c_{12}^4, \\ R_{1323} = \varepsilon_2 c_{34}^2 c_{13}^4 = \varepsilon_3 c_{41}^3 c_{23}^4. \end{cases}$$

Examining (3.7) and the canonical forms CF1–CF4, we claim that only those cases of CF1 in which two of A, B, C have the same absolute value and those of CF3 in which $-\lambda = B \leq 0$ are possible when $\dim G = 4$, and that CF2 and CF4 do not occur. We proceed to show this.

In order to obtain CF1, suppose first that at least two of the c_{ij}^4 in (3.4) are nonzero, whence all $c_{j4}^4 = 0$, or that all c_{ij}^4 vanish, whence all c_{j4}^i are arbitrary. Calculating, we find the curvature matrix

$$\frac{1}{4}(c_{12}^3)^2 \operatorname{diag}[\varepsilon_3, \varepsilon_2, \varepsilon_1]$$

which represents constant nonnegative curvature and a symmetry group of PGL_3 .

If exactly one $c_{ij}^4 \neq 0$, then c_{j4}^i is arbitrary and the others vanish. If $c_{12}^4 \neq 0$, then the curvature matrix is

$$\begin{bmatrix} \frac{1}{4}\varepsilon_3(c_{12}^3)^2 + \varepsilon_1 c_{24}^1 c_{12}^4 & 0 & 0 \\ 0 & \frac{1}{4}\varepsilon_2(c_{12}^3)^2 & 0 \\ 0 & 0 & \frac{1}{4}\varepsilon_1(c_{12}^3)^2 \end{bmatrix}.$$

For $c_{24}^1 = 0$ we have a symmetry group of PGL_3 , and for $c_{24}^1 \neq 0$ we have a symmetry group of PO_1^1 when $\varepsilon_1 = -\varepsilon_2$ and PO_2 when $\varepsilon_1 = \varepsilon_2$.

If $c_{31}^4 \neq 0$, then we find the curvature matrix

$$\begin{bmatrix} \frac{1}{4}\varepsilon_3(c_{12}^3)^2 & 0 & 0 \\ 0 & \frac{1}{4}\varepsilon_2(c_{12}^3)^2 + \varepsilon_3 c_{14}^3 c_{31}^4 & 0 \\ 0 & 0 & \frac{1}{4}\varepsilon_1(c_{12}^3)^2 \end{bmatrix}.$$

For $c_{14}^3 = 0$ the symmetry is PGL_3 , and for $c_{14}^3 \neq 0$ it is PO_1^1 when $\varepsilon_1 = -\varepsilon_3$ and PO_2 when $\varepsilon_1 = \varepsilon_3$.

Finally, if $c_{23}^4 \neq 0$ we obtain the curvature matrix

$$\begin{bmatrix} \frac{1}{4}\varepsilon_3(c_{12}^3)^2 & 0 & 0 \\ 0 & \frac{1}{4}\varepsilon_2(c_{12}^3)^2 & 0 \\ 0 & 0 & \frac{1}{4}\varepsilon_1(c_{12}^3)^2 + \varepsilon_2 c_{34}^2 c_{23}^4 \end{bmatrix}.$$

For $c_{34}^2 = 0$ the symmetry is PGL_3 , and for $c_{34}^2 \neq 0$ it is PO_2 when $\varepsilon_2 = \varepsilon_3$ and PO_1^1 when $\varepsilon_2 = -\varepsilon_3$.

We observe that in all cases of constant curvature it is nonnegative; it is not possible to obtain constant negative curvature with $\dim \mathfrak{g} = 4$. Also, in all cases at least two of A, B, C have the same absolute value and the symmetry group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot be realized. There are no other constraints on the production of CF1.

Note that while the form of CF1 does not depend on the ordering of (e_1, e_2, e_3) , all of CF2–4 do. Thus we may as well assume from now on that (e_1, e_2, e_3) are ordered so that $1 = \varepsilon_1 = -\varepsilon_2 = -\varepsilon_3$.

In order to obtain CF2, it follows from the structure equations (3.4) and the canonical form of Theorem 2.2 that this can occur if and only if $c_{12}^3 \neq 0$ and $\varepsilon_1 + \varepsilon_2 \neq 0$. This contradicts our assumption, so CF2 cannot be produced with $\dim \mathfrak{g} = 4$.

In order to obtain CF3, we must have $R_{1213} = R_{1223} = 0$ whence $c_{24}^1 = c_{12}^4 = 0$. We must also have $R_{1323} = \pm 1/2$ whence $c_{34}^2 c_{31}^4 = c_{14}^3 c_{23}^4 = \pm 1/2$. Now $-\frac{1}{4}(c_{12}^3)^2 = B$ so we can produce only those CF3 in which $B \leq 0$, but this is the only constraint on B . If we regard B as given and substitute using the preceding relations, then we obtain

$$\begin{cases} \mp \frac{c_{14}^3}{2c_{34}^2} = -\lambda \pm \frac{1}{2} - B, \\ \mp \frac{c_{34}^2}{2c_{14}^3} = \lambda \pm \frac{1}{2} + B. \end{cases}$$

Using the reciprocity of the left-hand sides, this system has a unique solution and we can produce CF3 only for $-\lambda = B \leq 0$.

Finally, in order to have CF4 we must have $R_{1323} = 0$ whence $c_{34}^2 c_{31}^4 = c_{14}^3 c_{23}^4 = 0$. We must also have $R_{1213} = R_{1223} = -1/\sqrt{2}$ whence $c_{14}^3 c_{12}^4, c_{24}^1 c_{31}^4, c_{24}^1 c_{23}^4, c_{34}^2 c_{12}^4 \neq 0$. These conditions are contradictory so CF4 cannot be produced with $\dim \mathfrak{g} = 4$.

3.2. $\dim G = 5$

Let \mathfrak{g} have basis $(e_1, e_2, e_3, e_4, e_5)$ and again assume that it is B -orthonormal. This time we take $\mathfrak{h} = \langle e_4, e_5 \rangle$, so $\llbracket e_4, e_5 \rrbracket$ must be a subalgebra, and $\mathfrak{m} = \llbracket e_1, e_2, e_3 \rrbracket$ again. We continue with $1 = \varepsilon_1 = -\varepsilon_2 = -\varepsilon_3$, and now both ε_4 and ε_5 are free. Working again from the reductive condition (3.1), the

ad_H -invariance condition (3.2), and the naturally reductive condition (3.3), and requiring that \mathfrak{h} be a subalgebra, the structure equations become

$$(3.8) \quad \begin{cases} [e_1, e_2] = \alpha e_3 + a_4 e_4 + a_5 e_5, & [e_1, e_5] = \gamma_1 e_2 + \gamma_3 e_3, \\ [e_2, e_3] = -\alpha e_1 + b_4 e_4 + b_5 e_5, & [e_2, e_4] = \beta_1 e_1 - \beta_2 e_3, \\ [e_3, e_1] = \alpha e_2 + c_4 e_4 + c_5 e_5, & [e_2, e_5] = \gamma_1 e_1 - \gamma_2 e_3, \\ [e_4, e_5] = n_4 e_4 + n_5 e_5, & [e_3, e_4] = \beta_3 e_1 + \beta_2 e_2, \\ [e_1, e_4] = \beta_1 e_2 + \beta_3 e_3, & [e_3, e_5] = \gamma_3 e_1 + \gamma_2 e_2. \end{cases}$$

From the Jacobi identity, we obtain

$$(3.9) \quad \begin{cases} \beta_1 c_4 + \gamma_1 c_5 + \beta_3 a_4 + \gamma_3 a_5 = 0, & \gamma_1 b_4 + c_4 n_4 + \gamma_2 a_4 = 0, \\ \beta_2 a_4 + \gamma_2 a_5 + \beta_1 b_4 + \gamma_1 b_5 = 0, & \gamma_1 b_5 + c_4 n_5 + \gamma_2 a_5 = 0, \\ \beta_3 b_4 + \gamma_3 b_5 - \beta_2 c_4 - \gamma_2 c_5 = 0, & \beta_3 a_4 - b_5 n_4 + \beta_1 c_4 = 0, \\ \beta_2 c_4 + a_5 n_4 - \beta_3 b_4 = 0, & \beta_3 a_5 - b_5 n_5 + \beta_1 c_5 = 0, \\ \beta_2 c_5 + a_5 n_5 - \beta_3 b_5 = 0, & \gamma_3 a_4 + b_4 n_4 + \gamma_1 c_4 = 0, \\ \gamma_2 c_4 - a_4 n_4 - \gamma_3 b_4 = 0, & \gamma_3 a_5 + b_4 n_5 + \gamma_1 c_5 = 0, \\ \gamma_2 c_5 - a_4 n_5 - \gamma_3 b_5 = 0, & \beta_3 \gamma_2 - \beta_2 \gamma_3 - \beta_1 n_4 - \gamma_1 n_5 = 0, \\ \beta_1 b_4 - c_5 n_4 + \beta_2 a_4 = 0, & \beta_2 \gamma_1 - \beta_1 \gamma_2 - \beta_3 n_4 - \gamma_3 n_5 = 0, \\ \beta_1 b_5 - c_5 n_5 + \beta_2 a_5 = 0, & \beta_1 \gamma_3 - \beta_3 \gamma_1 + \beta_2 n_4 + \gamma_2 n_5 = 0. \end{cases}$$

Thus all the connected, naturally reductive, Lorentzian homogeneous spaces with $\dim G = 5$ are obtained from all simultaneous solutions of (3.8) and (3.9) and their quotients by discrete subgroups of G . We calculate their Riemann tensors using (3.6), obtaining

$$(3.10) \quad \begin{cases} R_{1212} = -\frac{1}{4}\alpha^2 + \beta_1 a_4 + \gamma_1 a_5, \\ R_{1213} = \beta_3 a_4 + \gamma_3 a_5 = -\beta_1 c_4 - \gamma_1 c_5, \\ R_{1223} = -\beta_2 a_4 - \gamma_2 a_5 = \beta_1 b_4 + \gamma_1 b_5, \\ R_{1313} = -\frac{1}{4}\alpha^2 - \beta_3 c_4 - \gamma_3 c_5, \\ R_{1323} = \beta_2 c_4 + \gamma_2 c_5 = \beta_3 b_4 + \gamma_3 b_5, \\ R_{2323} = \frac{1}{4}\alpha^2 - \beta_2 b_4 - \gamma_2 b_5. \end{cases}$$

As before, not all cases of all canonical forms occur when $\dim G = 5$. Only the continuous symmetry groups from CF1 and CF3 occur and CF2 and CF4 do not. First, observe that adding and subtracting certain equations

from (3.9) yields the additional conditions

$$\begin{cases} a_5 n_4 - a_4 n_5 = 0, \\ b_5 n_4 - b_4 n_5 = 0, \\ c_5 n_4 - c_4 n_5 = 0. \end{cases}$$

For example, the first of these is obtained by taking the third equation plus the fourth plus the seventh from (3.9). It follows that unless

$$(3.11) \quad \begin{cases} a_4 b_5 - a_5 b_4 = 0, \\ a_4 c_5 - a_5 c_4 = 0, \\ b_4 c_5 - b_5 c_4 = 0, \end{cases}$$

then $n_4 = n_5 = 0$. Thus we shall initially assume that (3.11) holds.

In order to obtain CF1, we must have $R_{1213} = R_{1223} = R_{1323} = 0$, which via (3.10) is equivalent to

$$\begin{cases} \beta_1 b_4 + \gamma_1 b_5 = 0, & \beta_2 c_4 + \gamma_2 c_5 = 0, \\ \beta_1 c_4 + \gamma_1 c_5 = 0, & \beta_3 a_4 + \gamma_3 a_5 = 0, \\ \beta_2 a_4 + \gamma_2 a_5 = 0, & \beta_3 b_4 + \gamma_3 b_5 = 0. \end{cases}$$

Note that our assumption of (3.11) implies that β_i and γ_i do not necessarily all vanish, so it is possible to obtain nonconstant curvature. If all a_i, b_i, c_i vanish, however, we would also have constant curvature. Thus we also need at least one of the following satisfied:

$$(3.12) \quad \beta_1 \gamma_2 - \beta_2 \gamma_1 = 0;$$

$$(3.13) \quad \beta_1 \gamma_3 - \beta_3 \gamma_1 = 0;$$

$$(3.14) \quad \beta_2 \gamma_3 - \beta_3 \gamma_2 = 0.$$

If all three hold, then we are again reduced to constant nonnegative curvature. Thus we shall consider only the cases where exactly one or two hold.

If one does not hold and the other two do, then in each case it now follows from two of the last three equations in (3.9) that $n_4 = n_5 = 0$, whence the one in fact holds. Therefore this case is not possible.

If (3.12) holds and the other two do not, then $a_i = b_i = 0$ so $R_{1212} = -R_{2323} \leq 0$ and the symmetry group is PO_1^1 when $\beta_3 c_4 + \gamma_3 c_5 \neq 0$, PGL_3 when it is. If (3.13) holds and the other two do not, then $a_i = c_i = 0$ so $R_{1212} = R_{1313} \leq 0$ and the symmetry group is PO_2 when $\beta_2 b_4 + \gamma_2 b_5 \neq 0$, PGL_3 when it is. If (3.14) holds and the other two do not, then $b_i = c_i = 0$ so $R_{1313} = -R_{2323} \leq 0$ and the symmetry group is PO_1^1 when $\beta_1 a_4 + \gamma_1 a_5 \neq 0$, PGL_3 when it is. Note that in all these cases of constant curvature, it is also nonnegative.

The analysis for CF2 begins similarly. We must have

$$\begin{cases} \beta_3 c_4 + \gamma_3 c_5 + \beta_2 b_4 + \gamma_2 b_5 = 0, \\ \beta_3 a_4 + \gamma_3 a_5 = \beta_1 c_4 + \gamma_1 c_5 = 0, \\ \beta_2 a_4 + \gamma_2 a_5 = \beta_1 b_4 + \gamma_1 b_5 = 0, \\ \beta_2 c_4 + \gamma_2 c_5 = \beta_3 b_4 + \gamma_3 b_5 \neq 0. \end{cases}$$

It follows that (3.11), (3.12), and (3.13) are satisfied. Whether (3.14) holds or not, it turns out that $\beta_2, \gamma_2, \beta_3, \gamma_3, b_4, b_5, c_4, c_5 \neq 0$ and $\beta_1 = \gamma_1 = a_4 = a_5 = 0$. Since $b_4 c_5 - b_5 c_4 = 0$ (else $\beta_3 = \gamma_3 = 0$), $b_4 e_4 + b_5 e_5$ and $c_4 e_4 + c_5 e_5$ are linearly dependent. It follows that (3.14) holds, whence $\beta_3 e_1 + \beta_2 e_2$ and $\gamma_3 e_1 + \gamma_2 e_2$ are linearly dependent. But this implies $\beta_2 c_4 + \gamma_2 c_5 = \beta_3 b_4 + \gamma_3 b_5 = 0$, contradicting the requirements for CF2. Therefore CF2 cannot be produced with $\dim G = 5$.

In order to obtain CF3, we must have

$$\begin{cases} \beta_3 c_4 + \gamma_3 c_5 + \beta_2 b_4 + \gamma_2 b_5 = \mp 1, \\ \beta_3 a_4 + \gamma_3 a_5 = \beta_1 c_4 + \gamma_1 c_5 = 0, \\ \beta_2 a_4 + \gamma_2 a_5 = \beta_1 b_4 + \gamma_1 b_5 = 0, \\ \beta_2 c_4 + \gamma_2 c_5 = \beta_3 b_4 + \gamma_3 b_5 = \pm \frac{1}{2}. \end{cases}$$

These are sufficiently different from those for CF2, and the analysis again proceeds along similar lines to that for CF1. Using the first of the preceding conditions (equivalent to $R_{1313} + R_{2323} = \pm 1$), we obtain $-\lambda = B \leq 0$ when (3.11) holds; when it fails, CF3 cannot be produced.

We come at last to CF4. Considering (3.11) both holding and not, it is easy to see that this form cannot be produced with $\dim G = 5$.

As noted at the end of Section 2, only CF1 occurs in the Riemannian case. Theorem 3.1 continues to hold if "Lorentzian" is changed to "Riemannian", "Minkowskian" to "Euclidean", and " SO_1^{2+} " to " SO_3 ". If $\dim G = 4$, we take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and now obtain only those CF1 in which two of A, B, C have the same positive value. Thus there is no naturally reductive Riemannian model with $\dim G = 4$ and symmetry group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. When $\dim G = 5$, we also take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and still obtain only those symmetry groups from CF1 except $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note that these actions are not effective. This concludes our summary of the Riemannian case.

4. Naturally reductive examples

Here we shall present a few selected examples with $\dim G = 4$ and 5. All the cases in which we were able to identify the Lie algebra can be realized as circle bundles over surfaces, but the surfaces may be open. We do not

know if this is a general feature; this should be investigated further.

The first examples will be those with $\dim G = 4$. Recall the general structure equations (3.4). We also recall that here only those CF1 in which two of A, B, C have the same absolute value and only those CF3 in which $-\lambda = B \leq 0$ are possible.

We begin with CF1. Referring to the discussion following (3.7), the case where all $c_{ij}^4 \neq 0$ is typical for at least two of them not vanishing. Signature $+-$ is also typical here, so we shall write only it down. Then \mathfrak{g} has structure equations

$$\begin{cases} [e_1, e_2] = c_{12}^3 e_3 + c_{12}^4 e_4, \\ [e_2, e_3] = -c_{12}^3 e_1 + c_{23}^4 e_4, \\ [e_3, e_1] = c_{12}^3 e_2 + c_{31}^4 e_4. \end{cases}$$

If $c_{12}^3 \neq 0$, then \mathfrak{g} is a central extension of \mathfrak{so}_1^2 by $\langle e_4 \rangle$ and we obtain compact models with constant positive curvature as circle bundles over closed surfaces Σ_g of genus $g \geq 2$. If $c_{12}^3 = 0$, then \mathfrak{g} is a 4-dimensional nilpotent Lie algebra with center $\langle e_4 \rangle$ and we obtain the compact model of the flat torus T^3 .

If all $c_{ij}^4 = 0$, then \mathfrak{g} has the structure equations

$$\begin{cases} [e_1, e_2] = c_{12}^3 e_3, & [e_1, e_4] = c_{24}^1 e_2 + c_{14}^3 e_3, \\ [e_2, e_3] = -c_{12}^3 e_1, & [e_2, e_4] = c_{24}^1 e_1 - c_{34}^2 e_3, \\ [e_3, e_1] = c_{12}^3 e_2, & [e_3, e_4] = c_{14}^3 e_1 + c_{34}^2 e_2. \end{cases}$$

Now $\mathfrak{g} = \mathfrak{g}_1 \times_{\theta} \langle e_4 \rangle$ is a semidirect product with $\theta: \langle e_4 \rangle \rightarrow \text{Der}(\mathfrak{g}_1)$ given by

$$\theta(e_4) = \begin{bmatrix} 0 & -c_{24}^1 & -c_{14}^3 \\ -c_{24}^1 & 0 & -c_{34}^2 \\ -c_{14}^3 & c_{34}^2 & 0 \end{bmatrix}.$$

If $c_{12}^3 = 0$, then \mathfrak{g}_1 is abelian and we obtain the compact model of a flat T^3 if and only if $\exp(\theta)$ preserves a lattice in \mathbb{R}^3 . If $c_{12}^3 \neq 0$, then $\mathfrak{g}_1 \cong \mathfrak{so}_1^2$ and we obtain compact models with constant positive curvature as circle bundles over closed surfaces Σ_g of genus $g \geq 2$ if and only if $\exp(\theta)$ preserves a suitable lattice in $SO_1^{2+} \cong PSL_2(\mathbb{R})$.

For exactly one $c_{ij}^4 \neq 0$, the case $c_{12}^4 \neq 0$ is typical. The structure equations of \mathfrak{g} are

$$\begin{cases} [e_1, e_2] = c_{12}^3 e_3 + c_{12}^4 e_4, & [e_1, e_4] = -\varepsilon_1 \varepsilon_2 c_{24}^1 e_2, \\ [e_2, e_3] = \varepsilon_1 \varepsilon_3 c_{12}^3 e_1, & [e_2, e_4] = c_{24}^1 e_1, \\ [e_3, e_1] = \varepsilon_2 \varepsilon_3 c_{12}^3 e_2, & [e_3, e_4] = 0. \end{cases}$$

If $c_{24}^1 = 0$ and $c_{12}^3 \neq 0$, then we have a central extension of \mathfrak{so}_1^2 again and constant positive curvature. If $c_{24}^1 \neq 0$ and $c_{12}^3 = 0$, then the result depends

on the signature. If $\varepsilon_1 = -\varepsilon_2$, then $\mathfrak{g} \cong \mathfrak{so}_1^2 \times \langle e_3 \rangle$ and the symmetry group is PO_1^1 . If $\varepsilon_1 = \varepsilon_2$, then $\mathfrak{g} \cong \mathfrak{g}_1 \times \langle e_3 \rangle$ with $\mathfrak{g}_1 \cong \mathfrak{so}_3$ if $c_{12}^4 c_{24}^1 > 0$ and $\mathfrak{g}_1 \cong \mathfrak{so}_1^2$ if $c_{12}^4 c_{24}^1 < 0$, with the symmetry group of PO_2 in both cases. In all these cases e_4 is not the compact generator of \mathfrak{so}_1^2 , so we can obtain a compact model $S^1 \times S^2$ only with \mathfrak{so}_3 . Finally, if $c_{24}^1, c_{12}^3 \neq 0$, then \mathfrak{g} is unidentified and the symmetry group is PO_1^1 or PO_2 according to the signature.

For CF3, consider the special case $c_{23}^4 = c_{31}^4 \neq 0$ and $c_{12}^3 = 0$ so $B = -\lambda = 0$. Replacing e_1 by $e_1 + e_2$ and leaving e_2, e_3, e_4 as is, the structure equations for \mathfrak{g} can be written as

$$\begin{cases} [e_1, e_2] = 0, & [e_1, e_4] = 0, \\ [e_2, e_3] = ae_4, & [e_2, e_4] = -be_3, \\ [e_3, e_1] = 0, & [e_3, e_4] = be_1, \end{cases}$$

where $a = c_{23}^4$ and $ab = \mp 1/2$. We obtain a semidirect product $\mathfrak{g} = \mathfrak{g}_1 \rtimes_{\theta} \langle e_2 \rangle$ with $\mathfrak{g}_1 = \langle e_3, e_4, e_1 \rangle$ isomorphic to the Heisenberg algebra and

$$\theta(e_2) = \begin{bmatrix} 0 & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can get compact models if and only if $\exp(\theta)$ preserves a suitable lattice in G .

Next, some examples with $\dim G = 5$. When $\beta_i = \gamma_i = 0$, we have constant nonnegative curvature with a central extension of \mathfrak{so}_1^2 by the abelian $\langle e_4, e_5 \rangle$. We omit the details.

As a typical example of the other CF1 cases, suppose that $\beta_i = \gamma_1 = \gamma_3 = a_5 = c_5 = 0$ and $\gamma_2, b_5 \neq 0$. Then (3.9) implies that $a_4 = c_4 = n_4 = n_5 = 0$. The structure equations of \mathfrak{g} are

$$\begin{cases} [e_1, e_2] = \alpha e_3, \\ [e_2, e_3] = -\alpha e_1 + b_4 e_4 + b_5 e_5, \\ [e_3, e_1] = \alpha e_2, \\ [e_2, e_5] = \gamma_2 e_3, \\ [e_3, e_5] = \gamma_2 e_2. \end{cases}$$

If $\alpha = 0$, then $\mathfrak{g} \cong \langle e_1 \rangle \times \mathfrak{g}_1$ where \mathfrak{g}_1 is a central extension (a product if $b_4 = 0$) of \mathfrak{so}_1^2 by $\langle e_4 \rangle$ and we obtain models with symmetry group PO_2 . Since e_5 is not the compact generator of \mathfrak{so}_1^2 , none of them are compact. If $\alpha \neq 0$, then \mathfrak{g} is a central extension (a product if $b_4 = 0$) by $\langle e_4 \rangle$ of an unidentified 4-dimensional Lie algebra.

All of the other Lie algebras remain unidentified. We hope to return to these later.

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Received November 3th, 1993.