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# ON THE FORM OF SOLUTIONS OF THE ITERATIVE FUNCTIONAL EQUATION OF THE $n$-TH ORDER

The paper contains two theorems on the shape of some continuous solutions of the iterative functional equation of the n-th order, with constant coefficients.

**1.** We consider the functional equation of the n-th order with constant coefficients:

$$\phi_1(f^n(x)) + a_{n-1}\phi_1(f^{n-1}(x)) + \ldots + a_0\phi_1(x) = 0 \tag{1}$$

where $\phi_1$ is an unknown function, $f^i$ denotes the i-th iterate of given function f. This equation has been studied in detail in [1] (chpt. XIII p. 259). The theorems on the shape of solutions of the equation (1), formulated in this paper complement some results from [1]. These theorems can be applied in the theory of iterative functional inequalities of the n-th order, which will be subject of the next paper.

**2.** Assume the following about the function $f$:

**(H1)** $f : I \to I$, $I = [0, a)$, $a > 0$, $f$ *is continuous and strictly increasing function and* $0 < f(x) < x$, $x \in I \setminus \{0\}$.

Polynomial:

$$w(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \tag{2}$$

is called the characteristic polynomial of equation (1). Denote by $\lambda_1, \ldots, \lambda_n$ the roots of this polynomial. One can consider, instead of (1), the equivalent system (cf. [1], chpt. XIII, p. 262):

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$$(3)\qquad \begin{cases} \phi_1(f(x)) - \lambda_1\phi_1(x) = \phi_2(x) \\ \phi_2(f(x)) - \lambda_2\phi_2(x) = \phi_3(x) \\ \vdots \\ \phi_{n-1}(f(x)) - \lambda_{n-1}\phi_{n-1}(x) = \phi_n(x) \\ \phi_n(f(x)) - \lambda_n\phi_n(x) = 0. \end{cases}$$

Denote by $\lambda_n$ the root of the characteristic polynomial, which has the smallest absolute value. We prove the following theorem for solutions of the equation (1) under the following assumption on roots of the characteristic polynomial

**(H2)** $|\lambda_i| < 1$, $i = 1, \ldots, n$ $\lambda_n$ *is the single root of the polynomial* (2).

THEOREM 1. *Assume* (**H1**) *and* (**H2**).

a) *If* $\phi_1 : I \to K$ *($K$ is the set of real or complex numbers) is a continuous solution of equation* (1) *such that* $\phi_n$ *from the system* (3) *satisfies:* $\phi_n(x) \neq 0$ *in* $I \setminus \{0\}$, *and the limit*

$$(4)\qquad p_n := \lim_{x\to 0} \frac{\phi_1(x)}{\phi_n(x)}$$

*exists, then* $\phi_1$ *is given by the formula*

$$(5)\qquad \phi_1(x) = \frac{\phi_n(x)}{\prod_{i=1}^{n-1}(\lambda_n - \lambda_i)}.$$

b) *If* $\phi_n : I \to K$, ($\phi_n(x) \neq 0$ *in* $I \setminus \{0\}$) *is a continuous solution of the equation*

$$(6)\qquad \phi_n(f(x)) - \lambda_n\phi_n(x) = 0$$

*then the function* (5) *is the only continuous solution of equation* (1), *satisfying condition* (4).

Proof. By induction with respect to the order of equation.

We prove at first a) and b) for the second order equation. Let $\phi_1$ be a continuous solution of equation (1), for n=2, i.e.

$$(7)\qquad \phi_1(f^2(x)) + a_1\phi_1(f(x)) + a_0\phi_1(x) = 0.$$

The roots of the characteristic polynomial

$$w(\lambda) = \lambda^2 + a_1\lambda + a_0$$

are $\lambda_1, \lambda_2$ and

$$\phi_2(x) := \phi_1(f(x)) - \lambda_1\phi_1(x)$$

is a solution of the equation

$$\phi_2(f(x)) - \lambda_2\phi_2(x) = 0.$$

It follows from the assumption, that the limit (4) exists (for n=2, it is equal to $p_2$), so the function

$$\widehat{\phi}_1(x) = \begin{cases} \frac{\phi_1(x)}{\phi_2(x)}, & x \in I \setminus \{0\} \\ p_2, & x = 0 \end{cases} \tag{8}$$

is a continuous solution of the equation:

$$\widehat{\phi}_1(f(x)) - \frac{\lambda_1}{\lambda_2}\widehat{\phi}_1(x) = \frac{1}{\lambda_2} \tag{9}$$

in case of $|\frac{\lambda_1}{\lambda_2}| > 1$, the equation (9) has exactly one continuous solution ([1], p. 53, Th. 2.7, [2], chpt. 3.1C) which is the following constant function

$$\widehat{\phi}_1(x) = \frac{1}{\lambda_2 - \lambda_1}, \quad x \in I.$$

In view of the form (8) of the function $\widehat{\phi}_1$ we have

$$\phi_1(x) = \frac{\phi_2(x)}{\lambda_2 - \lambda_1}, \quad x \in I. \tag{10}$$

When $\lambda_1 = -\lambda_2$, the equation (9) has also exactly one continuous solution, which is the constant function ([1], Th. 2.11, p. 58), ([2], chpt. 3.1C):

$$\widehat{\phi}_1(x) = \frac{1}{2\lambda_2}, \quad x \in I.$$

Again from the shape of $\widehat{\phi}_1$ we get

$$\phi_1(x) = \frac{\phi_2(x)}{2\lambda_2}.$$

The case remains when $|\frac{\lambda_1}{\lambda_2}| = 1$ and $\frac{\lambda_1}{\lambda_2} \in \mathbf{C}$, where $\mathbf{C}$ is the set of complex numbers. From [1] (Th. 2.6, p. 52 ), [2] (chpt. 3.1C) it follows that in this case equation (9) has either exactly one continuous solution, or none at all. Since the function

$$\widehat{\phi}_1(x) = \frac{1}{\lambda_2 - \lambda_1}, \quad x \in I$$

fulfils equation (9) in that case too, we have

$$\phi_1(x) = \frac{\phi_2(x)}{\lambda_2 - \lambda_1}, \quad x \in I.$$

Proof of the part b). Suppose that $\phi_2$, ($\phi_2(x) \neq 0$ in $I \setminus \{0\}$) is a continuous solution of the equation $\phi_2(f(x)) - \lambda_2\phi_2(x) = 0$. Then the function

$$\phi_1(x) = \frac{\phi_2(x)}{\lambda_2 - \lambda_1}, \quad x \in I$$

is the solution of the equation

$$\phi_1(f(x)) - \lambda_1\phi_1(x) = \phi_2(x). \tag{11}$$

So $\phi_1$ is also a solution of equation (7).

We prove now the uniqueness of $\phi_1$. Let's assume that there exist $\phi_1$ and $\phi_1'$ satisfying (11) and for which the limit (4) exists. The functions $\frac{\phi_1}{\phi_2}$ and $\frac{\phi_1'}{\phi_2}$ fulfil the equation (9). Since $|\frac{\lambda_1}{\lambda_2}| > 1$ or $\frac{\lambda_1}{\lambda_2} = -1$ or $\frac{\lambda_1}{\lambda_2} \in \mathbf{C}$ and $|\frac{\lambda_1}{\lambda_2}| = 1$, then the equation (9) has at most one continuous solution [1] (p. 52, p. 53), [2] (chpt. 3.1C). Therefore $\phi_1 = \phi_1'$.

We have proven the validity of the theorem for $n = 2$. Assume the theorem holds for the equation of the order $n-1$. Let $\phi_1$ be a continuous solution of the n-th order equation. Then the function

$$\phi_2(x) := \phi_1(f(x)) - \lambda_1\phi_1(x)$$

satisfies the equation of the order n-1. The characteristic polynomial of this equation has the roots $\lambda_2, \ldots, \lambda_n$. From the assumed existence of the limit (4) and from the definition of the function $\phi_2$ it also follows the existence of the limit

$$\lim_{x\to 0}\frac{\phi_2(x)}{\phi_n(x)}.$$

Therefore the function $\phi_2$ satisfies the assumptions of our theorem and it is given by the following formula

$$\phi_2(x) = \frac{\phi_n(x)}{\prod_{i=2}^{n-1}(\lambda_n - \lambda_i)}.$$

Thus the function

$$\tilde{\phi}_1(x) = \begin{cases} \dfrac{\phi_1(x)}{\phi_n(x)}, & x \in I\setminus\{0\} \\ p_n, & x = 0 \end{cases} \tag{12}$$

is a continuous solution of the equation

$$\tilde{\phi}_1(f(x)) - \frac{\lambda_1}{\lambda_n}\tilde{\phi}_1(x) = \frac{1}{\lambda_n\prod_{i=2}^{n-1}(\lambda_n - \lambda_i)}.$$

In each of the cases

$$|\lambda_1| > |\lambda_n|, \quad \lambda_1 = -\lambda_n, \quad \left|\frac{\lambda_1}{\lambda_n}\right| = 1 \text{ and } \frac{\lambda_1}{\lambda_n} \neq -1$$

there exists exactly one solution of this equation and it is the constant function:

$$\tilde{\phi}_1(x) = \prod_{i=1}^{n-1}(\lambda_n - \lambda_i)^{-1}.$$

Considering the form of $\tilde{\phi}_1$ (12) we get (5), which is the thesis a) of the theorem.

Proof of the thesis b). Assume $\phi_n$ is a continuous solution of equation (6). From the induction assumption, it follows that the function

$$\phi_2(x) = \frac{\phi_n(x)}{\prod_{i=2}^{n-1}(\lambda_n - \lambda_i)}$$

is the only continuous solution of the equation of the $n-1$ order, which satisfies (4). The characteristic polynomial of this equation has the roots $\lambda_2, \ldots, \lambda_n$. Thus each continuous solution of the equation

$$\phi_1(f(x)) - \lambda_1\phi_1(x) = \frac{\phi_n(x)}{\prod_{i=2}^{n-1}(\lambda_n - \lambda_i)} \tag{13}$$

is also a continuous solution of the equation (1). Substituting, we realize that the function

$$\phi_1(x) = \frac{\phi_n(x)}{\prod_{i=1}^{n-1}(\lambda_n - \lambda_i)}$$

is a solution of equation (13). The proof of the uniqueness of $\phi_1$ is analogous to that for the case n=2. This completes the proof.

The second theorem on the form of solution of the equation (1) consider the case where the roots of the characteristic polynomial satisfy the following assumption:

**(H3)** $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| > 1$
$|\lambda_{k+1}| = |\lambda_{k+2}| = \ldots = |\lambda_{k+l}| = 1$
$1 > |\lambda_{k+l+1}| \geq |\lambda_{k+l+2}| \geq \ldots \geq |\lambda_{k+l+m}|, \quad k+l+m = n$
$\lambda_n$ is the single root.

The system of equations equivalent to the equation (1) can be divided into three subsystems, corresponding to different sets of roots in **(H3)**:

$$\begin{cases} \phi_1(f(x)) - \lambda_1\phi_1(x) = \phi_2(x) \\ \vdots \\ \phi_k(f(x)) - \lambda_k\phi_k(x) = \phi_{k+1}(x) \end{cases} \tag{14}$$

$$\begin{cases} \phi_{k+1}(f(x)) - \lambda_{k+1}\phi_{k+1}(x) = \phi_{k+2}(x) \\ \vdots \\ \phi_{k+l}(f(x)) - \lambda_{k+l}\phi_{k+l}(x) = \phi_{k+l+1}(x) \end{cases} \tag{15}$$

$$\begin{cases} \phi_{k+l+1}(f(x)) - \lambda_{k+l+1}\phi_{k+l+1}(x) = \phi_{k+l+2}(x) \\ \vdots \\ \phi_n(f(x)) - \lambda_n\phi_n(x) = 0. \end{cases} \tag{16}$$

We prove a theorem on the shape of solution of the equation (1) in the case where the roots of the characteristic equation satisfy **(H3)**. We divide the roots with absolute value equal to one, into three groups

$$\lambda_{k+i} \notin \{1,-1\}, \quad \text{for } i = 1,\dots,p,$$
$$\lambda_{k+i} = -1, \quad \text{for } i = p+1,\dots,p+q,$$
$$\lambda_{k+i} = 1, \quad \text{for } i = p+q+1,\dots,p+q+r = l, \text{ where } p,q,r \in \{0,\dots,l\}.$$

THEOREM 2. *Assume* (**H1**) *and* (**H3**). *Let* $\phi_1 : I \to K$ *be a continuous solution of equation* (1) *such that the following limit exists*

$$\lim_{x\to 0} \frac{\phi_{k+l+1}(x)}{\phi_n(x)}, \quad \phi_n(x) \neq 0 \ \text{ in } \ I\setminus\{0\} \tag{17}$$

*where* $\phi_{k+l+1}$ *and* $\phi_n$ *are defined in system* (14)–(16).

*If* $r \neq 0$, *then there exists* $c \in \mathbf{K}$, *such that*

$$\phi_1(x) = \frac{c}{\prod_{i=1}^{k+p}(1-\lambda_i)} + \frac{\phi_n(x)}{\prod_{i=1}^{n-1}(\lambda_n - \lambda_i)}.$$

Proof. Let $\phi_1$ be a continuous solution of the equation (1). We construct the system (14)–(16). Consider at first the part (16) of the system. We assumed that the limit (17) exists, thus from the Theorem 1 we get the shape of the function $\phi_{k+l+1}$

$$\phi_{k+l+1}(x) = \frac{\phi_n(x)}{\prod_{i=k+l+1}^{n-1}(\lambda_n - \lambda_i)}. \tag{18}$$

Consider now the subsystem (15). We divide it into three groups, corresponding to different values of roots with the same absolute value equal to one.

Denoting

$$\alpha_i = \lambda_{k+i}, \quad i = 1,\dots l,$$
$$\psi_i(x) = \phi_{k+i}(x), \quad i = 1,\dots l$$

we can rewrite (15) according to this notation:

$$
\begin{aligned}
&\psi_1(f(x)) - \alpha_1\psi_1(x) = \psi_2(x)\\
&\vdots\\
&\psi_p(f(x)) - \alpha_p\psi_p(x) = \psi_{p+1}(x)\\
&\psi_{p+1}(f(x)) + \psi_{p+1}(x) = \psi_{p+2}(x)\\
&\vdots\\
&\psi_{p+q}(f(x)) + \psi_{p+q}(x) = \psi_{p+q+1}(x) \qquad (19)\\
&\psi_{p+q+1}(f(x)) - \psi_{p+q+1}(x) = \psi_{p+q+2}(x)
\end{aligned}
$$

$$\vdots$$

$$\psi_{l-1}(f(x)) - \psi_{l-1}(x) = \psi_l(x)$$

$$\psi_l(f(x)) - \psi_l(x) = \psi_{l+1}(x).$$

The function $\psi_{l+1} = \phi_{k+l+1}$ and is of the form (18). From [1] (Th. 2.11, p. 58), [2] (chpt. 3.1C) and from the form of the function $\psi_{l+1}$ (it fulfils the last equation in (16) too) it follows that the *l-th* (the last) equation of system (19) has a one parameter family of solutions. Thus there exists $b \in \mathbf{K}$, such that $\psi_l$ has the form:

$$\psi_l(x) = b - \sum_{i=0}^{\infty} \psi_{l+1}(f^i(x)) = b - \psi_{l+1}(x) \sum_{i=0}^{\infty} (\lambda_n)^i = b + \frac{\psi_{l+1}(x)}{\lambda_n - 1}.$$

The function $\psi_l$ is continuous and it is the right hand side of the *(l-1)-st* equation of system (19), therefore it must satisfy $\psi_l(0) = 0$, thus b=0. This implies that $\psi_l$ fulfils the last equation in (16), too. Thus the $(l-1)$-*st* equation has also a one parameter family of solutions and the function $\psi_{l-1}$ has the form

$$\psi_{l-1}(x) = \frac{\psi_{l+1}(x)}{(\lambda_n - 1)^2}, \quad x \in I.$$

Repeating the latter reasoning r times, for next equations up, to the equation no. $(p+q+1)$, we obtain $\psi_{p+q+1}$:

$$\psi_{p+q+1}(x) = g + \frac{\psi_{l+1}(x)}{(\lambda_n - 1)^r}, \quad x \in I \tag{20}$$

with some $g \in K$, since $\psi_{p+q+1}$ belongs to the one parameter family of solutions of the equation no. $(p+q+1)$.

Let's rewrite the equation no. $(p+q\ )$ of system (19) taking into account the form (20) of the function $\psi_{p+q+1}$:

$$\psi_{p+q}(f(x)) + \psi_{p+q}(x) = g + \frac{\psi_{l+1}(x)}{(\lambda_n - 1)^r}.$$

The above equation has exactly one continuous solution [1] (Th. 2.11, p. 58), [2] (chpt. 3.1C). It is given by

$$\psi_{p+q}(x) = \frac{1}{2}g + \sum_{i=0}^{\infty}(-1)^i \frac{\psi_{l+1}(f^i(x))}{(\lambda_n - 1)^r} = \frac{1}{2}g + \frac{\psi_{l+1}(x)}{(\lambda_n - 1)^r} \sum_{i=0}^{\infty} (-\lambda_n)^i, \quad x \in I.$$

Repeating the same q times for next equations of system (19) up to the equation no. $(p+1)$, we get for the function $\psi_{p+1}$:

$$\psi_{p+1}(x) = c + \frac{\psi_{l+1}(x)}{(\lambda_n - 1)^r(\lambda_n + 1)^q}, \quad x \in I,\ c = \frac{g}{2^q}. \tag{21}$$

The function given by (21) is the right hand side of the equation no. $p$ of the system (19). This equation and all which follow have a coefficient $\alpha_i$ different from 1 and $-1$. We rewrite once more the equation $p$:

(22) $$\psi_p(f(x)) - \alpha_p\psi_p(x) = \psi_{p+1}(x).$$

Considering the shape (21) of $\psi_{p+1}$ one can easily check that each solution of the equation (22) satisfies

(23) $$\psi_p(0) = \frac{c}{1-\alpha_p}.$$

To get the function $\psi_p$, we iterate equation (22)

$$\psi_p(x) = \frac{\psi_p(f^s(x))}{(\alpha_p)^s} - \sum_{i=0}^{s-1}\left[\frac{c}{\alpha_p^{i+1}} + \frac{\psi_{l+1}(f^i(x))}{\alpha_p^{i+1}(\lambda_n-1)^r(\lambda_n+1)^q}\right], \qquad s\in \mathrm{N},\ x\in I.$$

Reducing the right hand side of the last equation, we get

$$\psi_p(x) = \frac{\psi_p(f^s(x))}{(\alpha_p)^s} - \left(\frac{c}{\alpha_p}\right)\frac{1-\alpha_p^{-s}}{1-\alpha_p^{-1}} - \sum_{i=0}^{s-1}\frac{\psi_{l+1}(x){\lambda_n}^i}{\alpha_p^{i+1}(\lambda_n-1)^r(\lambda_n+1)^q}$$

$$\psi_p(x) = \frac{1}{\alpha_p^s}\left[\psi_p(f^s(x)) + \frac{c}{\alpha_p-1}\right]$$

$$-\frac{c}{\alpha_p-1} - \frac{\psi_{l+1}(x)}{(\lambda_n-1)^r(\lambda_n+1)^q\alpha_p}\sum_{i=0}^{s-1}\left(\frac{\lambda_n}{\alpha_p}\right)^i.$$

Since $\psi_p$ is continuous in 0 and (23) holds, then the sequence in square bracket tends to zero, when $s\to\infty$. From this and from the fact that the sequence $\{\frac{1}{\alpha_p^s}\}$ is bounded ($|\alpha_p| = 1$) it follows, that $\psi_p$ is of the form

$$\psi_p(x) = \frac{c}{1-\alpha_p} + \frac{\psi_{l+1}(x)}{(\lambda_n-1)^r(\lambda_n+1)^q(\lambda_n-\alpha_p)}.$$

The function $\psi_p$ is the right side of the $(p-1)$-st equation of system (19). Applying the same reasoning to all remaining equations up to the first one, we obtain

$$\psi_1(x) = \frac{c}{\prod_{i=1}^{p}(1-\alpha_i)} + \frac{\psi_{l+1}(x)}{(\lambda_n-1)^r(\lambda_n+1)^q\prod_{i=1}^{p}(\lambda_n-\alpha_i)}$$

thus

$$\psi_1(x) = \frac{c}{\prod_{i=1}^{p}(1-\alpha_i)} + \frac{\psi_{l+1}(x)}{\prod_{i=1}^{l}(\lambda_n-\alpha_i)}.$$

Coming back to the notation in system (14)–(16), we have

(24) $$\phi_{k+1}(x) = \psi_1(x) = \frac{c}{\prod_{i=1}^{p}(1-\lambda_{k+i})} + \frac{\phi_{k+l+1}(x)}{\prod_{i=1}^{l}(\lambda_n-\lambda_{k+i})}.$$

The function $\phi_{k+1}$ is the right hand side of the last equation of the subsystem (14). Each equation of this subsystem has ( right hand side being continuous) exactly one continuous solution ($|\lambda_i| > 1$). The solution of the $k$-th equation is given by the following formula

$$\phi_k(x) = -\sum_{i=0}^{\infty} \frac{\phi_{k+1}(f^i(x))}{\lambda_k^{i+1}}, \quad x \in I.$$

Substituting (24) for $\phi_{k+1}$, we get

$$\phi_k(x) = -\sum_{i=0}^{\infty} \left[ \frac{c}{\prod_{j=1}^{p}(1-\lambda_{k+j})\lambda_k^{i+1}} + \frac{\phi_{k+l+1}(f^i(x))}{\prod_{j=1}^{l}(\lambda_n - \lambda_{k+j})\lambda_k^{i+1}} \right].$$

Using (18) for $\phi_{k+l+1}$, and summing up the geometrical series we obtain

$$\phi_k(x) = \frac{c}{\prod_{j=1}^{p}(1-\lambda_{k+j})(1-\lambda_k)} + \frac{\phi_{k+l+1}(x)}{\prod_{j=1}^{l}(\lambda_n - \lambda_{k+j})(\lambda_n - \lambda_k)}.$$

The function $\phi_k$ appears as the right side of the $(k-1)$-st equation of the system (14)-(16) and has similar form (24), with different constants only. One can analogically determine $\phi_{k-1}$. After such k steps one gets the formula for $\phi_1$:

$$\phi_1(x) = \frac{c}{\prod_{j=1}^{p}(1-\lambda_{k+j})\prod_{j=1}^{k}(1-\lambda_j)} + \frac{\phi_{k+l+1}(x)}{\prod_{j=1}^{l}(\lambda_n - \lambda_{k+j})\prod_{j=1}^{k}(\lambda_n - \lambda_j)}.$$

This can be rewritten in the following way

$$\phi_1(x) = \frac{c}{\prod_{j=1}^{p+k}(1-\lambda_j)} + \frac{\phi_{k+l+1}(x)}{\prod_{j=1}^{l+k}(\lambda_n - \lambda_j)}.$$

Considering the form (18), of the function $\phi_{k+l+1}$, we get our thesis proven.

**Note.** If r=0, then the system (19) contains only first two subsystems. The function:

$$\psi_{p+q+1} = \phi_{k+l+1}$$

has then the form (18). Using this in formula (20) and repeating the proof from this place on we get for $\phi_1$:

$$\phi_1(x) = \frac{\phi_n(x)}{\prod_{i=1}^{n-1}(\lambda_n - \lambda_i)}.$$

**3.** We assumed in this paper that the root of the characteristic polynomial with the smallest absolute value is single. This assumption is crucial, because if it is a multiple root then the equation (1) does not have any solution satisfying the assumptions of Theorem 1. We prove that assuming that $\lambda_n$ is a double root of the polynomial (2). Assume for contradiction that $\phi_1$ is a solution of (1), fulfilling the assumptions of Theorem 1. The functions

$$\phi_i(x) = \phi_{i-1}(f(x)) - \lambda_{i-1}\phi_{i-1}(x), \quad i = 2, \dots n$$

satisfy the system

$$\begin{cases} \phi_1(f(x)) - \lambda_1\phi_1(x) = \phi_2(x) \\ \vdots \\ \phi_{n-1}(f(x)) - \lambda_n\phi_{n-1}(x) = \phi_n(x) \\ \phi_n(f(x)) - \lambda_n\phi_n(x) = 0. \end{cases}$$

From the existence of $\lim_{x\to 0} \frac{\phi_1(x)}{\phi_n(x)}$ and from the way the latter system was constructed, it follows that there exists $\lim_{x\to 0} \frac{\phi_{n-1}(x)}{\phi_n(x)}$. Thus the function

$$\tilde{\phi}(x) = \begin{cases} \frac{\phi_{n-1}(x)}{\phi_n(x)}, & \text{for } x \in I \setminus \{0\} \\ \lim_{x\to 0} \frac{\phi_{n-1}(x)}{\phi_n(x)}, & \text{for } x = 0 \end{cases}$$

is a continuous solution of the equation

$$\tilde{\phi}(f(x)) - \tilde{\phi}(x) = \frac{1}{\lambda_n}.$$

This is a contradiction, since the above equation does not possess continuous solutions in $I = [0, a)$.

## References

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