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CONVERGENCE WITH RESPECT TO THE σ-IDEAL OF MEAGER SETS IN SEPARABLE CATEGORY BASES

The aim of this paper is to prove that convergence with respect to the σ-ideal of meager sets in separable category bases does not yield the Fréchet topology in the space of all Baire functions.

Let us recall some basic definitions and concepts of category bases. All of them come from John Morgan II and are contained in [3].

A category base on a set X is a pair $(X,\mathcal{S})$ such that X is a set and $\mathcal{S}$ is a family of subsets of X, with nonempty subsets called regions, satisfying the following axioms:

(1) $\bigcup \mathcal{S} = \mathcal{X}$.

(2) Let A be a region and $\mathcal{D}$ a nonempty family of disjoint regions of cardinality less than that of $\mathcal{S}$,

then

(i) if $A \cap (\bigcup \mathcal{D})$ contains regions, then there is a region $B \in \mathcal{D}$ such that $A \cap B$ contains a region;

(ii) if $A \cap (\bigcup \mathcal{D})$ contains no regions, then there is a region $B \subset A$ which is disjoint from $\bigcup \mathcal{D}$.

Standard examples of category bases include topologies and sets of positive measure with respect to a σ-finite measure.

We shall say that a set $C \subset X$ is singular if, for every region A, there exists a region $B \subset A$ such that $B \cap C = \emptyset$. A set $M \subset X$ is meager if M is a countable union of singular sets. A set Y which is not meager is called an abundant set. A set Z is abundant everywhere in a region A if, for each region $B \subset A$, $B \cap Z$ is abundant. The class of meager sets for a category base $(X,\mathcal{S})$ will be denoted by $\mathcal{M}(\mathcal{S})$. A set C is a Baire set if, for every region A, there exists a subregion B such that one of the sets $B \cap C$

or $(X - B) \cap C$ is meager. In the case when the category base $(X, \mathcal{S})$ is a topology, the class of all Baire sets is identical with the family of all sets with the Baire property, and the class of meager sets is identical with the family of all sets of the first category.

A category base $(X, \mathcal{S})$ is separable if there exists a countable family $\mathcal{P}$ of regions such that every abundant set is abundant everywhere in some region of the family $\mathcal{P}$.

We also apply the property, called in [3] the fundamental theorem, that every abundant set is abundant everywhere in some region.

Let $\mathcal{B}(\mathcal{S})$ be the family of all Baire sets in a category base $(X, \mathcal{S})$. It constitutes the σ-field of subsets of X containing the family of all regions and meager sets (see Th. 1.6 in [3]).

We shall say that a real function $f : X \to R$ is Baire if it is measurable with respect to $\mathcal{B}(\mathcal{S})$. Let $\mathcal{L}^*(\mathcal{S})$ denote the family of all Baire functions in a category base $(X, \mathcal{S})$.

Similarly, as it was presented in [4] in the abstract sense, we define the convergence with respect to σ-ideal of meager sets.

A sequence $\{f_n\}_{n \in N} \subset \mathcal{L}^*(\mathcal{S})$ is convergent with respect to the σ-ideal $\mathcal{M}(\mathcal{S})$ to a function f (in abbr. $f_n \xrightarrow[n\to\infty]{\mathcal{M}(\mathcal{S})} f$) if every subsequence $\{f_{k_n}\}_{n \in N}$ contains a subsequence $\{f_{l_{k_n}}\}_{n \in N}$ convergent to f $\mathcal{M}(\mathcal{S})$-a.e. which means that the set $\{x : f_{l_{k_n}}(x) \not\to_{n\to\infty} f(x)\}$ is meager.

The space $\mathcal{L}^*(\mathcal{S})$ eqipped with the convergence with respect to $\mathcal{M}(\mathcal{S})$ is a Fréchet space. Hence it is possible to define the closure operation on $\mathcal{L}^*(\mathcal{S})$ by letting $f \in \bar{A}$ iff A contains a sequence convergent with respect to $\mathcal{M}(\mathcal{S})$ to the function f. This closure operation has the properties:

$$\bar{\emptyset} = \emptyset, \quad A \subset \bar{A}, \quad \overline{A \cup B} = \bar{A} \cup \bar{B} \quad \text{for any } A, B \subset \mathcal{L}^*(\mathcal{S});$$

however, the equality $\bar{\bar{A}} = \bar{A}$ holds for each $A \subset L^*(\mathcal{S})$ iff the following condition labelled (L4) is satisfied:

(L4) if $f_{j,n} \xrightarrow[j\to\infty]{\mathcal{M}(\mathcal{S})} f_j$ and $f_j \xrightarrow[j\to\infty]{\mathcal{M}(\mathcal{S})} f$ for any $j, n \in N$, then there exist sequences $\{j_p\}_{p \in N}$ and $\{n_p\}_{p \in N}$ of positive integers such that $f_{j_p,n_p} \xrightarrow[j\to\infty]{\mathcal{M}(\mathcal{S})} f$.

If (L4) is satisfied, then the topology introduced in $\mathcal{L}^*(\mathcal{S})$ in the sense described above is called the Fréchet topology.

The problem of topologizing the space of all measurable functions was studied in [1], [2], [4]. At this moment, we recall the criterion formulated by Wagner in [4].

THEOREM W. *Let $(X,\mathcal{S})$ be a measurable space and $\mathcal{I}$ an arbitrary σ-ideal of subsets of X. Suppose that $\mathcal{I} \subset \mathcal{S}$ and each family of disjoint sets in $\mathcal{S}\backslash\mathcal{I}$ is at most countable. Then the convergence with respect to $\mathcal{I}$ yields the Fréchet topology in the space of all real $\mathcal{S}$-measurable functions iff the pair $(\mathcal{S},\mathcal{I})$ satisfies the following condition:*

(E) *for each set $D \in \mathcal{S}\backslash\mathcal{I}$, and every double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets, such that*

(1^0) *$B_{j,n} \subset B_{j,n+1}$ for any $j,n \in N$,*
(2^0) *$\bigcup_{n=1}^{\infty} B_{j,n} = D$ for each $j \in N$,*

there exists a sequence $\{n_j\}_{j\in N}$ of positive integers such that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{I}$.

We adopt this theorem to prove the following

THEOREM 1. *Let $(X,\mathcal{S})$ be a separable category base such that every abundant region contains a decreasing sequence of abundant Baire sets with the meager intersection; then the convergence with respect to the σ-ideal $\mathcal{M}(\mathcal{S})$ does not yield the Fréchet topology in the space $L^*(\mathcal{S})$ of Baire functions.*

Proof. We observe that, in a separable category base, the family of pairwise disjoint abundant Baire sets is at most countable. Indeed, otherwise, by the separability condition, we would obtain that two disjoint Baire sets A and B are abundant everywhere in some region P. By Theorem 1.3 in [3], the sets $P-A$ and $P-B$ are meager but, at the same time, their union is the set P which is abundant. Further, we shall prove that condition (E) in Theorem W fails to hold. Namely, let $\{\mathcal{P}_j\}_{j\in N}$ be the sequence of abundant regions which assures the separability for the category base $(X,\mathcal{S})$. Let $\{B'_{j,n}\}_{j\in N}$ be a decreasing sequence of Baire subsets of $\mathcal{P}_j$ such that $\bigcap_{n=1}^{\infty} B'_{j,n} \in \mathcal{M}(\mathcal{S})$. Without loss of generality, let us suppose that $\bigcap_{n=1}^{\infty} B'_{j,n} = \emptyset$. Let $B_{j,n} = X - B'_{j,n}$ for any $j,n \in N$. It is clear that conditions (1^0) and (2^0) of (E) are satisfied. We prove that, for an arbitrary sequence $\{n_j\}_{j\in N}$ of positive integers, $\bigcap_{j=1}^{\infty} B_{j,n_j} \in \mathcal{M}(\mathcal{S})$. Let us suppose that, for some sequence $\{n_j\}_{j\in N}$ of positive integers, $X - \bigcup_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{M}(\mathcal{S})$. Then, by the fundamental theorem, there exists a region P such that the set $X - \bigcup_{j=1}^{\infty} B'_{j,n_j}$ is abundant everywhere in P. By the separability condition, there exists a region P_{j_0} such that the set $X - B'_{j_0,n_{j_0}}$ is abundant everywhere in P_{j_0}. Since $B'_{j_0,n_{j_0}}$ is an abundant set, there exists a region P such that $B'_{j_0,n_{j_0}}$ is abundant everywhere in the region P. Since $P \cap B'_{j_0,n_{j_0}} \subset P \cap P_{j_0}$, we conclude that the set $P \cap P_{j_0}$ is abundant, and thus, by Theorem 1.2 in [3], contains a region Q. Then, by Theorem 1.3 in [3], $Q - B'_{j_0,n_{j_0}}$ is a meager set. But, at the same time, if $Q \subset P_{j_0}$, then the set $Q - B'_{j_0,n_{j_0}} = Q \cap (X - B'_{j,n_{j_0}})$ is abundant, which is a contradiction.

In that way we have proved that, for an arbitrary sequence $\{n_j\}_{j \in N}$ of positive integers, the set $X - \bigcup_{j=1}^{\infty} B'_{j,n_j}$ is meager.

THEOREM 2. *Let $(X, \mathcal{T})$ be a topological T_1 space with a countable base. Then the convergence with respect to the σ-ideal of sets of the first category yields the Fréchet topology in the space $L^*(\mathcal{T})$ of functions with the Baire property iff $X = A \cup B$ where A is an open set of the first category and B is the countable set of all isolated points in the space X.*

Until we prove this theorem we propose the following lemma

LEMMA. *If $(X, \mathcal{T})$ is a topological space with a countable base, then there exists a representation $X = X_1 \cup X_2 \cup X_3$, where X_1 is a perfect Baire subspace, X_2 is an open set of the first category, X_3 is a countable set of all isolated points of X, and each of the second category set with the Baire property in X_1 is the second category with the Baire property in X.*

Proof. Let X_3 be the set of all isolated points. It is countable. Let X_2 be the union of all open sets of the first category. By the Banach theorem X_2 is open and the first category again. Putting $X_1 = X - (X_2 \cup X_3)$ we have a Baire subspace. It is clear that if $X_1 \neq \emptyset$, then every nonempty open set in X_1 is the second category in X. Thus, if $B \subset X_1$ is the second category with the Baire property, then in its representation $W \triangle P$, W is an open set of the second category in X_1 and P is the first category in X_1. We see that B is the second category in X and has the Baire property.

Proof of Theorem 2. To prove the sufficiency we can easily observe that condition (E) of Theorem W is satisfied. Namely, the case when $B = \emptyset$ is obvious. When $B \neq \emptyset$, we see that any set D of the second category intersects B. Let $x_0 \in B \cap D$. Having the sequence of sets $\{B_{j,n}\}_{j,n \in N}$ with the Baire property satisfying (1) and (2) of (E), for every j we choose a positive integer n_j such that $x_0 \in \bigcap_{j=1} B_{j,n_j}$. Then $\bigcap_{j=1} B_{j,n_j} \notin \mathcal{M}(\mathcal{S})$.

Necessity. Let us suppose that the convergence in space X yields the Fréchet topology in the space $L^*(\mathcal{T})$. We have by Lemma the representation $X = X_1 \cup X_2 \cup X_3$, where X_1 is a Baire perfect subspace, X_2 is an open set of the first category and X_3 is a countable set of all isolated points in X. Our goal is to show that $X_1 = \emptyset$. Let us suppose that $X_1 \neq \emptyset$. Then by the second part of Lemma and Theorem W we easily conclude that convergence with respect to the σ-ideal of sets of the first category in X_1 yields the Fréchet topology in $L^*(\mathcal{T}|X_1)$. From the other side, since X_1 is a perfect T_1 topological space with a countable base, then each singleton is a G_δ-set of the first category and each nonempty open set in the Baire space X_1

is the second category. Thus we conclude that every open set contains a decreasing sequence of open sets whose intersection is a first category set (e.g. a singleton). In such a way, by Theorem 1, we have that convergence with respect to the σ-ideal of sets of the first category in X_1 produce the Fréchet topology in the space of functions with the Baire property. This contradiction ends the proof.

COROLLARY (cf. Example 3 in [4]). *Convergence with respect to the σ-ideal of the sets of the first category in an uncountable Polish space does not yield the Fréchet topology in the space of all real functions with the Baire property.*

Remark. Let $(\mathbb{R}, \mathcal{S})$ denote the category base over the real line $\mathbb{R}$, where $\mathcal{S}$ is the family of all compact sets of positive Lebesgue measure. Then the family of Baire sets is identical with that of all Lebesgue measurable sets and the family of meager sets coincides with the family of all sets of Lebesgue measure zero. It is well known (see in [4]) that the convergence with respect to the σ-ideal of Lebesgue null sets yields the Fréchet topology in the space of all Lebesgue measurable functions. It is clear that each abundant region in the category base $(\mathbb{R}, \mathcal{S})$ possesses a decreasing sequence of abundant Baire sets whose intersection is a meager set. Then, by Theorem 1, we conclude that the category base $(\mathbb{R}, \mathcal{S})$ is not separable. And at the same time we see that the separability condition in Theorem 1 is necessary.

References

[1] J. Hejduk, *Some properties of topological σ-ideals*, Demonstratio Math. 22 (1989), 1183–1189.

[2] J. Hejduk, *Convergence with respect to some σ-ideals*, Review of Research Faculty of Science, Mathematics Series 21 (1991), 157–164.

[3] J. Morgan II, *Point Set Theory*, Marcel Dekler, New York & Basel, 1990.

[4] E. Wagner, *Sequences of measurable functions*, Fund. Math. 112 (1981), 89–102.

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