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ON CERTAIN SUBCLASS OF CONVEX FUNCTIONS

1. Introduction

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit disc $U = \{z : |z| < 1\}$ and Ω the class of bounded analytic functions $w(z)$ in U, satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. In [2] is introduced the class $S^*(\alpha, \beta, A, B)$ of functions (1.1) satisfying the inequality

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)\beta(\frac{zf'(z)}{f(z)} - \alpha) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < 1 \tag{1.2}$$

for some α, β, A, B such that

$$0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1. \tag{1.3}$$

In what follows the constants α, β, A, B verify (1.3).

Let $C(\alpha, A, B)$ denote the class of functions $f(z) \in S$ which satisfy the condition

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1 + Bw(z)}, \quad w(z) \in \Omega. \tag{1.4}$$

Observe that $C(\alpha, A, B) \subset C$ with $C(0, -1, 1) = C$, $C(\alpha, -1, 1) = C(\alpha)$, $C(0) = C$, where $C(\alpha)$ is the known class of convex functions of order α.

Motivated by a number of recent works ([3], [8], [1], [13], [2]), we introduce here the class $C^*(\alpha, \beta, A, B)$, defined as follows.

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DEFINITION 1. A function $f(z) \in S$ is in the class $C^*(\alpha, \beta, A, B)$ if and only if the inequality

$$(1.5) \qquad \left| \frac{\frac{zf''(z)}{f'(z)}}{(B-A)\beta(1+\frac{zf''(z)}{f'(z)}-\alpha)+A\frac{zf''(z)}{f'(z)}} \right| < 1$$

holds true for all $z \in U$.

It follows immediately from (1.2) and (1.5) that

$$(1.6) \qquad f(z) \in C^*(\alpha, \beta, A, B) \text{ if and only if } zf'(z) \in S^*(\alpha, \beta, A, B).$$

We note that, by specializing the parameters α, β, A, B, we obtain the following subclasses studied by various earlier authors:

(i) $C^*(\alpha, 1, -1, 1) = C(\alpha)$ (Robertson [11], Pinchuk [10]),
(ii) $C^*(0, 1, A, B) = C(A, B)$ (Mazur [6], Silverman and Silvia [12]),
(iii) $C^*(\alpha, 1, A, B) = C(\alpha, A, B)$,
(iv) $C^*(\alpha, \beta, -1, 1) = C(\alpha, \beta)$- the class of convex functions of order α and type β.

In the present paper, using the results proved in [2], we establish a representation formula, distortion properties and coefficient estimates for functions in the class $C^*(\alpha, \beta, A, B)$. A sufficient condition for a function to be in the class $C^*(\alpha, \beta, A, B)$ has been obtained. We also maximize $|a_3 - \mu a_2^2|$ over the class $C^*(\alpha, \beta, A, B)$. Finally, γ-spiral and γ-convex radius are obtained for the classes $C^*(\alpha, \beta, A, B)$ and $S^*(\alpha, \beta, A, B)$, respectively.

2. The representation formula

Let Q denote the class of functions $\psi(z)$ which are analytic in the unit disc U and satisfy $|\psi(z)| \leq 1$ for all $z \in U$. By (1.6), Theorem 1 of [2] implies the following one.

THEOREM 1. *A function $f(z) \in S$ is in the class $C^*(\alpha, \beta, A, B)$ if and only if*

$$(2.1) \qquad f'(z) = \exp\left[-b \int_0^z \frac{\psi(t)}{1+at\psi(t)} dt \right], \qquad z \in U,$$

for some $\psi(z) \in Q$, where

$$(2.2) \qquad a := A + \beta(B-A), \; b := (1-\alpha)(B-A)\beta.$$

3. A sufficient condition

We now establish a sufficient condition for a function to be in the class $C^*(\alpha, \beta, A, B)$.

THEOREM 2. *The function $f(z) \in S$ is in the class $C^*(\alpha, \beta, A, B)$ if*

$$\sum_{n=2}^{\infty} n\{n(1-a) - (1+A) + (A-B)\alpha\beta\}|a_n| \le b, \tag{3.1}$$

whenever $0 < \beta \le \frac{A}{A-B}$, *and*

$$\sum_{n=2}^{\infty} n\{n - 1 + |a(n-1) + b|\}|a_n| \le b, \tag{3.2}$$

whenever $\frac{A}{A-B} \le \beta \le 1$, *where a, b are defined by* (2.2).

Proof. The proof of the first part follows from Theorem 3 in [2] and by (1.6). For the proof of the second part, let $|z| = r < 1$. Noting that

$$|zf''(z)| < \sum_{n=2}^{\infty} n(n-1)|a_n|r, \tag{3.3}$$

we assume that (3.2) holds true for $\frac{A}{A-B} \le \beta \le 1$. In this case, we observe that

$$|(B-A)\beta(zf''(z) + (1-\alpha)f'(z))Azf''(z)| \ge \left\{b - \sum_{n=2}^{\infty} n|a(n-1) + b||a_n|\right\}r. \tag{3.4}$$

Making use of (3.3), (3.4) and (3.2), we complete the proof of Theorem 2.

4. Distortion theorem

Theorem 2 in [2] together with (1.6) yields the following distortion properties for the class $C^*(\alpha, \beta, A, B)$.

THEOREM 3. *If a function $f(z) \in C^*(\alpha, \beta, A, B)$, then for $|z| = r < 1$, by* (1.3), (2.2),

$$|f'(z)| \le (1 - ar)^{-\frac{b}{a}}, \ a \ne 0, \tag{4.1}$$

and

$$|f'(z)| \ge (1 + ar)^{-\frac{b}{a}}, \ a \ne 0 \tag{4.2}$$

whereas

$$|f'(z) \le \exp[-A(1-\alpha)r], \ a = 0, \tag{4.3}$$

and

$$|f'(z)| \ge \exp[A(1-\alpha)r], \ a = 0, \tag{4.4}$$

respectively. Equality in (4.1), (4.2) *holds for such a function $f(z)$ that*

$$f'(z) = (1 - az)^{-\frac{b}{a}}, \ a \ne 0, \tag{4.5}$$

whereas in (4.3), (4.4) *it holds for such a function* $f(z)$ *that*

$$f'(z) = \exp[-A(1-\alpha)z],\ a = 0. \tag{4.6}$$

5. Coefficient estimates

We shall require the following lemmas in our investigation.

LEMMA 1 [9]. *Let the function* $w(z)$ *be defined by*

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \tag{5.1}$$

and be in the class Ω. *Then*

(5.2) $|c_1| \le 1$,

(5.3) $|c_2| \le 1 - |c_1|^2$.

LEMMA 2 [4]. *Let the function* (5.1) *be in the class* Ω. *Then*

$$|c_2 - \nu c_1^2| \le \max\{1, |\nu|\}, \tag{5.4}$$

for any complex number ν. *Equality in (5.4) may be attained with the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\nu| < 1$ *and* $|\nu| \ge 1$, *respectively.*

THEOREM 4. *If a function* $f(z) \in C^*(\alpha, \beta, A, B)$, $\beta \ne \frac{A}{A-B}$, *then*

(a) *for any real number* μ *we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{b}{6}[b(1-\frac{3}{2}\mu)+a], & \text{if } \mu \le \frac{2}{3b}[A+(B-A)(2-\alpha)\beta-1] =: b^*, \\ \frac{b}{6}, & \text{if } b^* \le \mu \le b^{**} := \frac{2}{3b}[A+(B-A)(2-\alpha)\beta+1], \\ \frac{b}{6}[b(\frac{3}{2}\mu-1)-a], & \text{if } \mu \ge b^{**}, \end{cases} \tag{5.5}$$

(b) *for any complex number* μ, *we have*

$$|a_3 - \mu a_2^2| \le \frac{b}{6}\max\{1, |b(\frac{3}{2}\mu - 1) - a|\}. \tag{5.6}$$

The result is sharp for each μ *either real or complex.*

Proof. Since $f(z) \in C^*(\alpha, \beta, A, B)$, (2.1) gives

$$\frac{zf''(z)}{f'(z)} = -\frac{bw(z)}{1+aw(z)}, \tag{5.7}$$

where $w(z) \in \Omega$. From (5.7), we have

$$w(z) = \frac{-zf''(z)}{azf''(z) + bf'(z)} \tag{5.8}$$

and, by (1.1), it can be shown that

$$w(z) = -\frac{1}{b}\left\{2a_2 z + (6a_3 - \frac{4}{b}(a+b)a_2^2)z^2 + \dots\right\}. \tag{5.9}$$

Comparing the coefficients of z and z^2 on both sides of (5.9), by (5.1), we obtain the relations

$$(5.10) \quad c_1 = -\frac{2a_2}{b},$$

$$(5.11) \quad c_2 = -\frac{6a_3}{b} + \frac{4}{b^2}(a+b)a_2^2$$

implying

$$(5.12) \qquad |a_3 - \mu a_2^2| = \frac{b}{6}|c_2 - [a - b(\frac{3}{2}\mu - 1)]c_1^2|.$$

(a): For any real μ, by using Lemma 1 for $|c_1|$ and $|c_2|$, we can write (5.12) in the form

$$(5.13) \qquad |a_3 - \mu a_2^2| \leq \frac{b}{6}[|b(\frac{3}{2}\mu - 1) - a|].$$

Thus from (5.13), after simple computations, we obtain (5.5).

(b): For any complex number μ, applying Lemma 2 in (5.12), we get (5.6). Finally, the assertions (5.5), (5.6) of Theorem 4 are sharp, in view of the fact that the assertion (5.4) of Lemma 2 is sharp. Equality occurs for the functions obtained by letting $w(z) = z$ and $w(z) = z^2$ in (5.7).

COROLLARY 1. *If* $f(z) \in C^*(\alpha, \beta, A, B)$, $\beta \neq \frac{A}{A-B}$, *then*

$$(5.14) \quad |a_2| \leq \frac{b}{2}$$

$$(5.15) \quad |a_3| \leq \frac{b}{6}\max\{1, |a+b|\}.$$

The bounds in (5.14), (5.15) *are attained with the function* $f(z)$ *satisfying* (4.5).

Proof. The assertions (5.14) and (5.15) of Corollary 1 follow directly from (5.10) and (5.6), respectively.

THEOREM 5. *Let* $f(z) \in C^*(\alpha, \beta, A, B)$ *with* (1–3) *and let*

$$K := \frac{k-1}{(B-A)^2\beta}\{(k-1)(1-A^2) - (B-A)\beta[(B-A)\beta + 2A](k-\alpha)\},$$

$$N := [(1-\alpha)(k-\alpha)K^{-1}]$$

for $k = 2, 3, \ldots, n-1$.

(a) *If* $(1-\alpha)(k-\alpha)\beta > K$, *then*

$$(5.16) \quad |a_n| \leq \frac{1}{n!}\prod_{k=2}^{n}[a(k-2)+b], \; n = 2, 3, \ldots N+2,$$

$$(5.17) \qquad |a_n| \leq \frac{1}{(N+1)!n(n-1)} \prod_{k=2}^{N+3} [a(k-2)+b], \ n > N+2.$$

(b) If $1-\alpha)(k-\alpha)\beta \leq K$, then

$$(5.18) \qquad |a_n| \leq \frac{b}{n(n-1)}, \ n \geq 2.$$

The estimates in (5.16) are sharp for such a function $f(z)$ that

$$(5.19) \qquad 1 + \frac{zf''(z)}{f'(z)} = \frac{1-(a-b)z}{1-az}, \ a \neq 0,$$

while the estimates in (5.18) are sharp for such a function $f_n(z)$ that

$$(5.20) \qquad f'_n(z) = \begin{cases} (1-az^{n-1})^{-\frac{b}{a(n-1)}}, & a \neq 0, \ n \geq 2, \\ \exp\left[\frac{-A(1-\alpha)z^{n-1}}{n-1}\right], & a = 0, \ n \geq 2. \end{cases}$$

Proof. Since $zf'(z) = z + 2a_2z^2 + \dots$ is in the class $S^*(\alpha, \beta, A, B)$, this theorem is an immediate consequence of Theorem 5 in [2].

6. Radius of γ-spiral and γ-convex

Let S_1 be the family of all normalized functions which are analytic and univalent in U. In [5] Libera introduced as follows the concept of "γ-spiral radius" (γ-s.r.) for the classes of univalent functions.

DEFINITION 2. If $f(z) \in S_1$ and $|\gamma| < \frac{\pi}{2}$, then the γ-spiral radius of $f(z)$ is

$$(6.1) \qquad \gamma\text{-s.r}\{f(z)\} = \sup\left\{r : Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0, \ |z| < r\right\},$$

and if the set $F \subset S_1$, then γ-spiral radius of F is

$$(6.2) \qquad \gamma\text{-s.r.}\, F = \inf_{f \in F} [\gamma\text{-s.r.}\{f(z)\}].$$

Also in [7] Mogra introduced the concept of "γ-convex radius" (γ-c.r.) as follows.

DEFINITION 3. If $f(z) \in S$ and $|\gamma| < \frac{\pi}{2}$, then the γ-convex radius of $f(z)$ is

$$(6.3) \qquad \gamma\text{-c.r.}\{f(z)\} = \sup\left\{r : Re\left\{e^{i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0; \ |z| < r\right\}.$$

DEFINITION 4. If the set $G \subset S$ and $|\gamma| < \frac{\pi}{2}$, then γ-convex radius of G is

$$(6.4) \qquad \gamma\text{-c.r.}G = \inf_{f \in G} [\gamma\text{-c.r.}\{f(z)\}].$$

THEOREM 6. *γ-c.r. $S^*(\alpha,\beta,A,B)$, $\beta \neq \frac{A}{B-A}$, is the smallest positive root r of the equation.*

$$\cos\gamma - br - a(a-b)r^2\cos\gamma = 0. \tag{6.5}$$

The result is sharp.

Proof. Let $f(z) \in S^*(\alpha,\beta,A,B)$. Then, by (1.6) and (2.1), we have

$$\frac{zf'(z)}{f(z)} = \frac{1+(a-b)w(z)}{1+aw(z)}, \quad w(z) \in \Omega. \tag{6.6}$$

If $B(z) = e^{i\gamma}\frac{zf'(z)}{f(z)}$ and $|\gamma| < \frac{\pi}{2}$, then (6.6) may be written as

$$w(z) = \frac{e^{i\gamma} - B(z)}{aB(z) - e^{i\gamma}(a-b)}, \quad z \in U. \tag{6.7}$$

Now, by applying Schwarz's Lemma [9], it follows that $B(z)$ maps the disc $|z| < r$ onto a disc

$$|B(z) - \xi| < R, \tag{6.8}$$

where

$$\xi = \frac{e^{i\gamma}\{1-a(a-b)r^2\}}{1-ar^2}, \tag{6.9}$$

$$R = \frac{br}{1-a^2r^2}. \tag{6.10}$$

Hence $Re\big(e^{i\gamma}\frac{zf'(z)}{f(z)}\big) \geq 0$ if and only if

$$Re\left\{\frac{e^{i\gamma}\{1-a(a-b)r^2}{1-a^2r^2}\right\} \geq R, \tag{6.11}$$

which, after simplification and with the aid of (6.2), concludes the proof of Theorem 6. To show the sharpness we take

$$f(z) = \begin{cases} z(1-az)^{-\frac{b}{a}}, & a \neq 0, \\ z\exp[-A(1-\alpha)z], & a = 0, \end{cases} \tag{6.12}$$

and put

$$\zeta = \frac{r(ar - e^{-i\gamma})}{1 - are^{-i\gamma}}. \tag{6.13}$$

We thus obtain

$$e^{i\gamma}\frac{\zeta f'(\zeta)}{f(\zeta)} = e^{i\gamma} + be^{i\gamma}\frac{ar^2 - re^{-i\gamma}}{1-a^2r^2}$$

implying the equality

$$Re\left\{e^{i\gamma}\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} = \cos\gamma + b\frac{ar^2\cos\gamma - r}{1-a^2r^2} = \frac{\cos\gamma - br - a(a-b)r^2\cos\gamma}{1-a^2r^2} \tag{6.14}$$

which shows that the inequality (6.11) is sharp. Hence the bound obtained in Theorem 6 is sharp.

THEOREM 7. *γ-c.r.$C^*(\alpha,\beta,A,B)$ is the smallest positive root r of the equation* (6.5). *The result is sharp for such a function $f(z)$ that*

$$f'(z) = \begin{cases} (1-az)^{-\frac{b}{a}}, & a \neq 0, \\ \exp[-A(1-\alpha)z], & a = 0, \end{cases} \tag{6.15}$$

ζ being defined by (6.13).

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