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A NOTE ON CONTRACTORS IN RANDOM NORMED SPACES

1. Introduction

Fixed point theorems have been studied in the probabilistic metric spaces (or more briefly PM spaces) by many authors (we refer to [3]–[7] and [10]). In [10] Sehgal and Bharucha Reid obtained the contraction principle in the PM spaces and in [5] Istratescu generalized their result. Altman [1] obtained a new concept of an inverse derivative that can be used instead of the derivative and formulated the concepts of contractors and contractor directions.

Altman's concepts provide new techniques for solving nonlinear equations and also generalize the contraction principle in Banach spaces. Altman's theory of contractors has been further generalized by Balakrishna Reddy and Subrahmanyam [2]. Lee and Padgett [8] obtained the random analogues of Altman's existence theorem involving contractors.

In this note we extend the concept of contractors in random normed spaces, introduced by Serstnev [11] (see also [9]), and therein we obtain an existence theorem generalizing that of Altman. Our theorem also generalizes the contraction principle in random normed spaces.

2. Preliminaries

For the definition of a PM space and a Menger space we refer to [9]. We begin with the following definition of a T-norm.

DEFINITION 2.1. A mapping $t : [0,1] \times [0,1] \to [0,1]$ is a T-norm if it satisfies for $a, b, c, d \in [0,1]$ the following conditions:

(i) $t(a,1) = a, t(0,0) = 0$,
(ii) $t(a,b) = t(b,a)$,
(iii) $t(c,d) \geq t(a,b)$ for $c \geq a$ and $d \geq b$,
(iv) $t(t(a,b),c) = t(a,t(b,c))$.

DEFINITION 2.2. The (ε, λ)-topology in a probabilistic metric space is defined to be a topology on S determined by the family of neighbourhoods $\{U_v(\varepsilon, \lambda) : \varepsilon > 0,\ \lambda \in (0,1)\}$ for each $v \in S$, where

$$U_v(\varepsilon, \lambda) = \{u : F_{u,v}(\varepsilon) > 1 - \lambda\}.$$

In the (ε, λ)-topology a sequence $\{p_n\}$ in S converges to $p \in S$ if and only if for every $\varepsilon > 0$ and $\lambda > 0$ there exists an integer $M(\varepsilon, \lambda)$ such that $p_n \in U_p(\varepsilon, \lambda)$, i.e., $F_p, p_n(\varepsilon) > 1 - \lambda$, whenever $n \geq M(\varepsilon, \lambda)$. The sequence $\{p_n\}$ is Cauchy's one, if for each $\varepsilon > 0$ and $\lambda > 0$ there is an integer $M(\varepsilon, \lambda)$ such that $F_{p_n . p_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq M(\varepsilon, \lambda)$. A Menger space S is complete, if every Cauchy sequence in S converges to an element in S. The following notion of a random normed space has been introduced in [11]. Let Δ denote the collection of all distribution functions (i.e., the set of all $F : R \to R^+$ which is non-decreasing, left continuous with $\inf F = 0$ and $\sup F = 1$).

DEFINITION 2.3. (see [10]). A mapping T of a PM space $(S, \mathbb{F})$ into itself will be called a contraction mapping if and only if there exists a constant $k \in (0,1)$ such that $F_{Tp,Tq}(kx) \geq F_{p,q}(x)$ for all $x > 0$, and $p, q \in S$.

DEFINITION 2.4. A triplet $(S, \mathcal{F}, F)$ of a real or complex linear space S, a mapping $\mathcal{F} : S \to \Delta$ and a T-norm t is called a random normed space, if it satisfies the following conditions in which F_p denotes the distribution function $F(p)$ for $p \in S$:

(i) $F_p(0) = 0$ for all $p \in S$,

(ii) $F_p(u) = 1$ for all $u > 0$ if and only if $p = 0 \in S$,

(iii) If λ is a nonzero scalar, then $F_{\lambda p}(u) = F_p\left(\frac{u}{|\lambda|}\right)$ for all $p \in S$ and all $u \in R$,

(iv) $F_{p+2}(u + v) \geq t(F_p(u), F_q(v))$ for all $p, q \in S$, $u > 0$, $v > 0$, where $t(u, v) \geq \max(u + v - 1, 0) = T_m(u, v)$, $u, v \in [0,1]$.

A random normed space in a Menger space under the mapping F' defined by $F'(p, q) = F_{p-q}$ for all $p, q \in S$. If the T-norm t is continous, then S is a Hausdorff linear topological space under the (ε, λ)-topology.

DEFINITION 2.5 (see [11]). Let X, Y be Banach spaces, $P : X \to Y$ a nonlinear operator and $\Gamma : Y \to X$ a linear bounded operator associated with $x \in X$. If $\Gamma(x)$ has the property

$$\|y\|^{-1} \|P(x + \Gamma(x)y) - Px - y\| \to 0 \quad \text{as } y \to 0,$$

then $\Gamma(x)$ is called an inverse derivative at x of P.

Using Definition 2.5, Altman proved the following existence theorem, concerning the convergence of iteration scheme $x_{n+1} = x_n - \Gamma(x_n)Px_n$, $n = 1, 2, \ldots$, to a solution of the nonlinear operator equation $Px = \emptyset$.

We assume that the inverse derivative $\Gamma(x)$ of P exists in a neighbourhood $S(x_0, r) = (x : \|x - x_0\| \leq r)$.

THEOREM 2.1. *Let $P : X \to Y$ be a nonlinear operator, where X, Y are Banach spaces. Suppose that there exist positive numbers q, r, η, B with $0 < q < 1$ such that the inverse derivative satisfies the uniformity condition $\|y\|^{-1} \|P(x+\Gamma(x)y) - Px - y\| \leq q$ for $x \in S(x_0, r) = \{x \in X, \|x - x_0\| \leq r\}$ and $\|y\| \leq \eta$, $\|\Gamma(x)\| \leq B$, $x \in S(x_0, r)$, $\|Px_0\| \leq \eta$, $B\eta(1-q)^{-1} \leq 1$, where P is a closed operator on $S(x_0, r)$, i.e.,*

$$(x_n \in S, \{x_n\} \to x \quad \text{and} \quad \{Px_n\} \to y) \Rightarrow (x \in S \quad \text{and} \quad y = Px).$$

Then there exists a solution $x^* \in S(x_0, r)$ and the sequence $\{x_n\}$ converges towards x^*. Besides $Px^* = \emptyset$, $x^* \in S(x_0, r)$ and $\|x_n - x^*\| \leq B\eta q^n (1-q)^{-1}$.

For proving the global existence theorem Altman formulated the following definition.

DEFINITION 2.6. Let X, Y be Banach spaces, $P : X \to Y$ a nonlinear operator and $\Gamma : Y \to X$ a linear bounded operator associated with $x \in X$. The operator P has a contractor $\Gamma(x)$, if there is a positive number $q < 1$ such that $\|P(x + \Gamma(x)y) - Px - y\| \leq q\|y\|$, where $x \in X$ and $y \in Y$.

3. An existence theorem in random normed spaces

We prove below a theorem giving sufficient conditions for the nonlinear operator $P : X \to Y$ (where X, Y are complete random normed space) to have a zero in X. This result is an analogue of Altman's Theorem 2.1 [1] for random normed spaces.

THEOREM 3.1. *Let (X, F, t), (Y, F', t') be two complete random normed spaces, where t, t' are continuous T-norms and $t(x, x) \geq x$ for every $x \in [0, 1]$. Let $P : X \to Y$ be a nonlinear operator, $r > 0$ and $x_0 \in X$. Suppose that $\Gamma : Y \to X$ is a linear operator for $x \in X$ such that $F_{x-x_0}(r) > 0$. Consider the iteration $x_{n+1} = x_n - \Gamma(x_n)Px_n$. Suppose that there exist $q, \eta, B, \alpha > 0$ with $q < 1$ such that*

(i) $F'_{P(x+\Gamma(x)y)-Px-y}(a) \geq F'_y(\frac{a}{q})$ *for every $a > 0$ and for every x, y with $F_{x-x_0}(r) > \alpha$ and $F'_y(\eta) > \alpha$,*

(ii) $F_{\Gamma(x)y}(a) \geq F'_y(\frac{a}{B})$ *for every x with $F_{x-x_0}(r) > \alpha$ and for every $y \in Y$,*

(iii) $F'_{Px_0}(\eta) > \alpha$ *and* $B\eta(1-q)^{-1} \leq r$,

(iv) *P is a closed operator.*

Then there exists x^* with $Px^* = \emptyset$ and $\{x_n\} \to x^*$.

Proof. By induction on n we shall prove that $F_{x_n - x_0}(r) > \alpha$ and $F'_{Px_n}(\eta) > \alpha$ for every n. For $n = 0$ the result is obvious. Suppose that for every $k \leq n-1$ it holds. Then, by triangle inequality, we have

$$(1) \quad \begin{cases} F_{x_n - x_0}(r) \geq t(F_{x_n - x_1}(r - B\eta), \ F_{x_1 - x_0}(B\eta)), \\ F_{x_n - x_k}(r - B\eta \ldots - B\eta q^{k-1}) \\ \geq t(F_{x_n - x_{k+1}}(r - B\eta \ldots - B\eta q^k, \ F_{x_{k+1} - x_k}(B\eta q^k)), \quad k < n-1, \end{cases}$$

and, by induction hypothesis,

$$(2) \qquad F_{x_{k+1} - x_k}(B\eta q^k) = F_{\Gamma(x_k)Px_k}(B\eta q^k) \geq F'_{Px_k}(\eta q^k).$$

Taking $y = -Px_{k-1}$ and $x = x_{k-1}$, by induction hypothesis in (1), we get

$$(3) \qquad F'_{Px_k}(\eta q^k) \geq F'_{Px_{k-1}}(\eta q^{k-1}) \geq \ldots \geq F'_{Px_0}(\eta)$$

and (2) reduces to the inequality

$$F_{x_{k+1} - x_k}(B\eta q^k) \geq F'_{Px_0}(\eta)$$

implying, by the increasing nature of t, that (1) becomes

$$F_{x_n - x_k}(\beta) \geq t(F_{x_n - x_{k+1}}(\beta - B\eta q^k), F'_{Px_0}(\eta)) \quad \text{with } \beta = r - B\eta \ldots - B\eta q^{k-1}.$$

As t is increasing and associative, repeated use of (1), (3) together with the triangle inequality leads to

$$(4) \qquad F_{x_n - x_0}(r) \geq t(F_{x_n - x_{n-1}}(\gamma), F'_{Px_0}(\eta))$$

with $\gamma = r - B\eta - \ldots - B\eta q^{n-2}$ and, by (ii) and the induction hypothesis, we have

$$(5) \qquad F_{x_n - x_{n-1}}(\gamma) = F_{\Gamma(x_{n-1})P(x_{n-1})}(\gamma) \geq F'_{Px_{n-1}}(B^{-1}\gamma).$$

By an argument similar to that used to establish (3), we get

$$F_{x_n - x_{n-1}}(\gamma) \geq F'_{Px_0}(B^{-1}q^{1-n}\gamma).$$

By (iii), $B^{-1}q^{1-n}\gamma \geq \eta$. As F'_{Px_0} increases, $F_{x_n - x_{n-1}}(\gamma) \geq F'_{Px_0}(\eta)$. Since $t(x, x) \geq x$ for every $x \in [0, 1]$, t is increasing and $F'_{Px_0}(\eta) > \alpha$, we get

$$F_{x_n - x_0}(r) \geq t(F'_{Px_0}(\eta), F'_{Px_0}(\eta)) \geq F'_{Px_0}(\eta) > \alpha.$$

By induction it can be readily verified that $F'_{Px_0}(\eta) > \alpha$ for every n. Setting $x = x_n$ and $y = -Px_n$ in (i), we have

$$F_{Px_{n+1}}(a) \geq F'_{Px_n}\left(\frac{a}{q}\right) \quad \text{and} \quad F_{\Gamma(x_n)Px_n}(a) \geq F'_{Px_n}\left(\frac{a}{B}\right)$$

for every $n \in N$ and $a > 0$. Thus, we choose $y = -Px_n$ in (i) for all n and obtain

$$F'_{Px_{n+1}}(a) \geq F'_{Px_n}\left(\frac{a}{q}\right) \ldots \geq F'_{Px_0}(aq^{-n-1})$$

for every $a > 0$. As $q < 1$ and $\sup_{y \in R} F'_{Px_0}(y) = 1$, it follows that $F'_{Px_{n+1}}(a) \to 1$ for every $a > 0$. So $\{Px_n\} \to \emptyset$.

Now we prove that $\{x_n\}$ is Cauchy's sequence. For $m > n$ and $\varepsilon > 0$, using triangle inequality, we get

$$F_{x_n - x_m}(\varepsilon) \geq t(F_{x_n - x_{n+1}}(\varepsilon - q\varepsilon), F_{x_{n+1} - x_m}(q\varepsilon)).$$

From (5) we have

$$\begin{aligned} F_{x_n - x_{n+1}}(\varepsilon - q\varepsilon) &= F_{\Gamma(x_n)P(x_n)}(\varepsilon - q\varepsilon) \\ &\geq F'_{Px_n}(B^{-1}(\varepsilon - q\varepsilon)) \geq F'_{Px_0}(B^{-1}q^{-n}(\varepsilon - q\varepsilon)). \end{aligned}$$

In general, by triangle inequality,

$$(6) \quad F_{x_k - x_m}(q^{k-n}\varepsilon) \geq f\{F_{x_k - x_{k+1}}(q^{k-n}\varepsilon - q^{k-n+1}\varepsilon), F_{x_{k+1} - x_m}(q^{k-n+1}\varepsilon)\}.$$

For $m > k > n$, in view of (5), the first component in (6) becomes

$$\begin{aligned} (7) \quad F_{x_k - x_{k+1}}(q^{k-n}\varepsilon - q^{k-n+1}\varepsilon) &= F_{\Gamma(x_k)Px_k}(q^{k-n}\varepsilon - q^{k-n+1}\varepsilon) \\ &\geq F'_{Px_k}((q^{k-n}\varepsilon - q^{k-n+1}\varepsilon)B^{-1}) \geq F'_{Px_0}((\varepsilon - q\varepsilon)B^{-1}q^{-n}). \end{aligned}$$

Let $d = (\varepsilon - q\varepsilon)B^{-1}q^{-n}$. As t is increasing and associative, repeated use of (6) and (7) yields

$$\begin{aligned} F_{x_n - x_m}(\varepsilon) &\geq t(F'_{Px_0}(d), F_{x_{m-1} - x_m}(q^{m-n-1}\varepsilon)), \\ F_{x_{m-1} - x_m}(q^{m-n-1}\varepsilon) &= F_{\Gamma(x_m)Px_{m-1}}(q^{m-n-1}\varepsilon) \\ &\geq F'_{Px_{m-1}}((q^{m-n-1}\varepsilon)B^{-1}) \\ &\geq F'_{Px_0}((q^{m-n-1}\varepsilon)B^{-1}q^{1-m} \geq F'_{Px_0}(B^{-1}q^{-n}\varepsilon). \end{aligned}$$

Using (5), since $t(x,x) \geq x$ and $Bq^n d < \varepsilon$, we get

$$F_{x_n - x_m}(\varepsilon) \geq t(F'_{Px_0}(d), F'_{Px_0}(d)) \geq F'_{Px_0}(d).$$

For large N with $F'_{Px_0}(d) > 1 - \lambda$ we have $F_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for $n, m > N$. Thus $\{x_n\}$ is a Cauchy sequence in X. As X is complete, $\{x_n\} \to x^*$ in X. Since $\{Px_n\} \to \emptyset$ and P is closed, $Px^* = \emptyset$.

Remark 3.1. In Theorem 3.1 it suffices to assume that P is a closed operator on $S_r = \{x : F_{x - x_0}(r) > \alpha\}$.

Remark 3.2. Let x_0, r, x_n and x be as in the proof of Theorem 3.1. For every $\varepsilon > r$ there is

$$F_{x^* - x_0}(\varepsilon) \geq t(F_{x_0 - x_n}(r), F_{x_n - x^*}(\varepsilon - r) \geq t(\alpha, F_{x_n - x^*}(\varepsilon - r))$$

and in the limit $F_{x^* - x_0}(\varepsilon) \geq t(\alpha, 1) = \alpha$.

We introduce the following concept of a random contractor for operators on random normed spaces.

DEFINITION 3.1. Let $(X, \mathcal{F}, t)$ and (Y, F', t') be random normed spaces, $P : X \to Y$ a nonlinear operator and $\Gamma(x)$ a linear bounded operator (i.e., there exists $B > 0$ such that $F_{\Gamma(x)y}(a) \geq F'_y(\frac{a}{B})$ for every $a > 0$). The operator P is said to have a random contractor, if there exists $q < 1$ such that $F_{P(x+T(x)y)-Px-y}(a) \geq F'_y(\frac{a}{q})$ for every $a > 0$, where $x \in X$, $y \in Y$.

For mappings having random contractors we prove the following global existence theorem.

THEOREM 3.2. *Let $P : D(P) \subseteq (X, \mathcal{F}, t) \to (Y, \mathcal{F}', t')$ be a nonlinear operator mapping $D(P)$ (a probabilistic closed subset of a complete random normed space $(X, \mathcal{F}, t)$) into a subset of $(Y, \mathcal{F}', t')$, where t and t' are continuous T-norms with $t(x, x \geq x$ for every $x \in [0,1]$. Suppose that P has a random contractor $\Gamma(x)$ for every $x \in D(P)$ such that $x + \Gamma(x)y \in D(P)$ for all $y \in Y$. If P is closed operator on $D(P)$, then there exists x^* such that $Px^* = \emptyset$.*

Proof. Let x_0 be an arbitrary element of $D(P)$. Define the iteration scheme inductively by $x_{n+1} = x_n - \Gamma(x_n)P_{x_n}$. From the contractor inequality it follows that for each $a > 0$

$$F'_{Px_{n+1}}(a) \geq F'_{Px_0}\left(\frac{a}{q^{n+1}}\right)$$

and so Px_{n+1} converges to $\emptyset$. Using an argument similar to that given in the proof of Theorem 3.1, it can be shown that $\{x_n\}$ is a Cauchy sequence in the complete random normed space $(X, \mathcal{F}, t)$. So it converges to an element x^* of X and, P being a closed operator, it follows that $Px^* = \emptyset$.

Remark 3.3. We can deduce from Theorem 3.2 the contraction principle of Sehgal and Bharucha Reid in a complete random normed space as follows. Let $(X, \mathcal{F}, t)$ be a complete random normed space and t a continuous T-norm with $t(x, x) \geq x$ for every $x \in [0,1]$. If $P : X \to X$ is a probabilistic contraction map with contraction constant q, then it can be readily verified that, for the operator $I - P$ identity operator I is a random contractor and hence, by Theorem 3.2, there is $(I - P)x^* = \emptyset$ for some $x^* \in S$. Then $Px^* = x^*$.

References

[1] M. Altman, *Contractors and Contractors Directions, Theory and Applications*, Marcel Dekker, New York 1977.

[2] K. Balakrishna Reddy, V. Subrahmanyam, *Altman's contractors and Matkowski's fixed point theorem*, Nonlinear Analysis: Theory, Methods and Applications, 5 (1981), 1061–1075.

[3] Ch. Bocsan, *On some fixed point theorems in probabilistic metric spaces*, Math. Balkanica, 4 (1974) 67–70.

[4] O. Hadzic, *Some theorems on the fixed points in probabilistic metric and random normed spaces*, Boll. Uni. Mat. Ital., 6 (1982), 381–391.

[5] V. I. Istratescu, I. Saciu, *Fixed point theorems for contraction mappings on probabilistic metric space*, Rev. Roum. Math. Pures Appl., 18 (1973), 1375–1380.

[6] I. Istratescu, E. Roventa, *On fixed point theorems for mappings on probabilistic metric space*, Bull. Math. Soc. Sci. Math. Rep. Soc. Roum., 67 (1975), 67–70.

[7] V. I. Istratescu, *Fixed point theory. An introduction*, D. Reidel Publishing Co, 1981.

[8] A. C. H. Lee, W. J. Padgett, *Random contractors and the solution of random non-linear equations*, Nonlinear Analysis: Theory, Methods and Applications, 1 (1977), 175–185.

[9] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North Holland series in Probability and Applied Mathematics, 5 (1983).

[10] V. M. Sehgal, A. T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Syst. Theory, 6 (1972), 97–102.

[11] A. N. Serstnev, *The notion of random normed spaces*, Dokl. Akad. Nauk, SSR Soc. 49 (1963), 281–283.

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