DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 3 1995

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ON THE STABILITY OF THE DIFFERENCE SCHEMA APPROXIMATING CAUCHY'S PROBLEM FOR A PARABOLIC EQUATION

1. Introduction

In [7] was considered a class of the difference schema for solving the Cauchy problem for parabolic equations with constant coefficients. These schemas are open absolute-stability with order of the approximate $0(\tau+h^2)$.

This paper extends [7] to the case of the equation with the variable coefficients independing of the time. Furthermore, it is considered in R^M, $M \geq 1$.

At first, consider the Cauchy problem ($M = 1$)

$$(1.1)\quad \begin{cases} \frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + c(x)u + f(x,t), & (x,t) \in R \times (0,T), \\ u(x,0) = \varphi(x), \quad x \in R, \end{cases}$$

where $a(x)$, $b(x)$, $c(x) \in C(R)$; $a(x) \geq a_0 > 0$; $f(x) \in C(R,(0,T))$; $\varphi \in C(R)$,

On the rectangular grid

$$R^2_{h\tau} := \{(x,t) : x = mh, t = n\tau; m,n \in Z; N\tau = T; h,\tau > 0\},$$

we consider the difference schema

$$(1.2)\quad \begin{cases} \frac{2}{\tau}[U^{n+1/2}_{2m+1} - U^n_{2m+1}] = L_{h,2m+1}U^n_{2m+1} + f^n_{2m+1}, \\ \frac{2}{\tau}[U^{n+1/2}_{2m} - U^n_{2m}] = L_{h,2m}U^{n+1/2}_{2m} + \frac{1}{2}[f^n_{2m} + f^{n+1}_{2m}], \\ \frac{2}{\tau}[U^{n+1}_{2m} - U^{n+1/2}_{2m}] = L_{h,2m}U^{n+1/2}_{2n} + \frac{1}{2}[f^n_{2m} + f^{n+1}_{2m}], \\ \frac{2}{\tau}[U^{n+1}_{2m+1} - U^{n+1/2}_{2m+1}] = L_{h,2m+1}U^{n+1}_{2m+1} + f^{n+1}_{2m+1}, \\ U^0_m = \varphi_m, \ m,n \in Z, \ 0 \leq n \leq N-1, \end{cases}$$

where $U^n_m := u(mh, n\tau)$, the difference operator $(L_{h,m}y)_k$ is defined as follows

$$L_{h,m}y_k := \frac{a_m}{h^2}[y_{k-1} - 2y_k + y_{k+1}] + \frac{b_m}{2h}[y_{k+1} - y_{k-1}] + c_m y_k,$$

$m, k \in Z$ and Z is the set of integers, and we obtain the following results.

2. The stability of the schema (1.2) for the problem (1.1)

We prove that the schema (1.2) is equivalent to the Crank–Nicolson schema. We have the following theorem.

THEOREM 2.1 *The sequence* $\{U_m^n\}$, $m, n \in Z$, *satisfies the system of the difference equations* (1.2) *if and only if this sequence satisfies the relations defined by Crank-Nicolson schema*

$$\text{(2.1)} \quad \begin{cases} \frac{1}{\tau}(U_m^{n+1} - U_m^n) = \frac{1}{2}[L_{h,m}U_m^{n+1} + f_m^{n+1}] + \frac{1}{2}[L_{h,m}U_m^n + f_m^n], \\ U_m^0 = \varphi_m, \ m, n \in Z, \ 0 \le n \le N-1. \end{cases}$$

Proof. At first we show that, if the sequence $\{U_m^n\}$ satisfies (1.2), it satisfies (2.1). Summing up the second and third equations of (1.2) and dividing by 2 we obtain

$$\text{(2.2)} \qquad \frac{1}{\tau}[U_{2m}^{n+1/2} + U_{2m}^n] = L_{h,2m}U_{2m}^{n+1/2} + \frac{1}{2}[f_{2m}^{n+1} + f_{2m}^n],$$

and, subtracting side by side, we obtain

$$\text{(2.3)} \qquad U_{2m}^{n+1/2} = \frac{1}{2}(U_{2m}^{n+1} + U_{2m}^n), \qquad m \in Z.$$

Applying the operator $L_{h,2m}$ to the (2.3) and putting to the (2.2), we have

$$\text{(2.4)} \quad \frac{1}{\tau}(U_{2m}^{n+1} - U_{2m}^n) = \frac{1}{2}[L_{h,2m}U_{2m}^{n+1} + f_{2m}^{n+1}] + \frac{1}{2}[L_{h,2m}U_{2m}^n + f_{2m}^n].$$

Similarly, adding the first and fourth equations of the (1.2) and dividing by 2, we obtain

$$\text{(2.5)} \quad \frac{1}{\tau}(U_{2m+1}^{n+1} - U_{2m+1}^n) = \frac{1}{2}[L_{h,2m+1}U_{2m+1}^{n+1} + f_{2m+1}^{n+1}] + \frac{1}{2}[L_{h,2m+1}U_{2m+1}^n + f_{2m+1}^n].$$

From (2.4), (2.5) and the condition $U_m^0 = \varphi_m$ this schema gives (2.1).

To show that the sequence $\{U_m^{n+1/2}\}$ defined by the schema (2.1) satisfies (1.2), we introduce an auxiliary sequence $\{U_m^{n+1/2}\}$ as follows

$$\text{(2.6)} \qquad U_{2m+1}^{n+1/2} = U_{2m+1}^n + \frac{\tau}{2}[L_{h,2m}U_{2m+1}^n + f_{2m+1}^n],$$

$$U_{2m}^{n+1/2} = \frac{1}{2}(U_{2m}^{n+1} + U_{2m}^{n}) \qquad m, n \in Z. \tag{2.7}$$

Using (2.7) and (2.1), we obtain

$$\begin{aligned}\frac{2}{\tau}(U_{2m}^{n+1/2} - U_{2m}^{n}) &= \frac{1}{\tau}(U_{2m}^{n+1} + U_{2m}^{n}) \\ &= \frac{1}{2}[L_{h,2m}U_{2m}^{n} + f_{2m}^{n}] + \frac{1}{2}[L_{h,2m}U_{2m}^{n+1} + f_{2m}^{n+1}].\end{aligned}$$

From the linearity of $L_{h,m}$ and (2.7) we have

$$\frac{2}{\tau}(U_{2m}^{n+1/2} - U_{2m}^{n}) = L_{h,2m}U_{2m}^{n+1/2} + \frac{1}{2}[f_{2m}^{n+1} + f_{2m}^{n}]. \tag{2.8}$$

Similarly

$$\begin{aligned}\frac{2}{\tau}(U_{2m}^{n+1} - U_{2m}^{n+1/2}) &= \frac{1}{2}(U_{2m}^{n+1} - U_{2m}^{n}) \\ &= \frac{1}{2}[L_{h,2m}U_{2m}^{n} + f_{2m}^{n}] + \frac{1}{2}[L_{h,2m}U_{2m}^{n+1} + f_{2m}^{n+1}] \\ &= L_{h,2m}U_{2m}^{n+1/2} + \frac{1}{2}[f_{2m}^{n+1} + f_{2m}^{n}].\end{aligned}$$

That is

$$\frac{2}{\tau}(U_{2m}^{n+1} - U_{2m}^{n+1/2}) = L_{h,2m}U_{2m}^{n+1/2} + \frac{1}{2}[f_{2m}^{n+1} + f_{2m}^{n}]. \tag{2.9}$$

From (2.6) we have

$$\frac{2}{\tau}(U_{2m}^{n+1} - U_{2m}^{n+1/2}) = \frac{2}{\tau}(U_{2m+1}^{n+1} - U_{2m+1}^{n}) - L_{h,2m+1}U_{2m+1}^{n} - f_{2m+1}^{n}.$$

Using (2.1), we obtain

$$\begin{aligned}\frac{2}{\tau}(U_{2m}^{n+1} - U_{2m}^{n+1/2}) &= 2[\frac{1}{2}(L_{h,2m+1}U_{2m+1}^{n+1} + f_{2m+1}^{n+1}) \\ &\quad + \frac{1}{2}(L_{h,2m+1}U_{2m+1}^{n} + f_{2m+1}^{n})] \\ &\quad - L_{h,2m+1}U_{2m+1}^{n} - f_{2m+1}^{n} \\ &= L_{h,2m+1}U_{2m+1}^{n+1} + f_{2m+1}^{n+1}.\end{aligned}$$

That is

$$\frac{2}{\tau}(U_{2m+1}^{n+1/2} - U_{2m+1}^{n}) = L_{h,2m+1}U_{2m+1}^{n+1} + f_{2m+1}^{n+1}. \tag{2.10}$$

Analogously, from (2.6) we have

$$\frac{2}{\tau}(U_{2m+1}^{n+1/2} - U_{2m+1}^{n}) = L_{h,2m+1}U_{2m+1}^{n} + f_{2m+1}^{n}.$$

Theorem (2.1) have been proved.

We can prove the following theorem.

THEOREM 2.2. *If the coefficients of the equation* (1.1) *are independent of the time, continuous and bounded with respect to* x, *the function* f *is continuous and bounded with respect to* x, t, *then there exists* τ_* *depending only on the norm of the coefficients such that for all* $\tau \in (0, \tau_*)$ *the schema* (1.2) *is unconditionally stable with the error* $0(\tau + h^2)$.

Proof. By Theorem (2.1), the schema (1.2) is equivalent to the Crank-Nicholson schema, whose unconditional stability is proved in [6].

3. On the stability of difference schemas for the Cauchy problem in R^M

Consider the following initial problem ($M \geq 1$)

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f(x,t), \ (x,t) \in R^M \times (0,T), \\ u(x,0) = \varphi(x), \ x \in R^M, \end{cases} \tag{3.1}$$

where L is an elliptic operator defined by

$$Lu := -\sum_{i,j=1}^{M} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{M} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

and $L \in (C^2(R^M), C(R^M))$.

The difference schema approximating (3.1) will be constructed on the parallelepiped grid

$$R^M_{h\tau} := R^M_h \cdot \omega_\tau \text{ with } R^M_h := \{x \in R^M : \ x = (m_1h_1, \dots, m_M h_M), \ m_1, \dots, m_M \in Z\},$$
$$h := \max_i h_i \text{ with } h_1, h_2, \dots, h_M \geq h_0 > 0$$
$$\text{and } \omega_\tau := \{t = n\tau, \ n = 0, \dots N; \ N\tau = T\}.$$

Denote by $\partial_i, \overline{\partial}_i$, $i = 1, \dots, M$, the difference operators

$$\partial_i V(x) = \frac{1}{h_i}[V(x) - V(x - e_i h_i)], \ x \in R^M,$$
$$\overline{\partial}_i V(x) = \frac{1}{h_i}[V(x + e_i h_i) - V(x)], \ x \in R^M,$$

by $e_1, e_2, \dots, e_M$ vectors corresponding to axes $Os_1, \dots, Os_M$, respectively, by L^0_h, L^1_h, L_h operators defined as follows

$$L_h^0 V(x) := -\frac{1}{2}\sum_{i,j=1}^{M}[\partial_i(a_{ij}(x)\overline{\partial}_j V(x) + \overline{\partial}_i a_{ij}(x)\partial_j V(x)],$$

$$L_h^1 V(x) := \frac{1}{2}\sum_{i=1}^{M} b_i(x)[\partial_i V(x) + \overline{\partial}_i V(x)] + c(x)V(x),$$

$$L_h V(x) := L_h^0 V(x) + L_h^1 V(x)$$

and by Ω_{1h}, Ω_{2h} non-empty domains of R_h^M such that

$$R_h^M = \Omega_{1h} \cup \Omega_{2h}, \qquad \Omega_{1h} \cap \Omega_{2h} = \emptyset.$$

We determine a sequence $\{U^n(x)\}$, $x \in R_h^M$, $n \in Z$, such that $U^0(x) = \varphi(x)$, $x \in R_h^M$. Then the problem (3.1) will be approximated by the following schema

$$\begin{cases} \frac{2}{\tau}(U^{n+1/2}(x) - U^n(x)) + L_h U^n(x) = f^n(x), \ x \in \Omega_{1h}, \\ \frac{2}{\tau}(U^{n+1/2}(x) - U^n(x)) + L_h U^{n+1/2}(x) \\ \qquad\qquad = \frac{1}{2}[f^{n+1}(x) + f^n(x)], \ x \in \Omega_{2h}, \\ \frac{2}{\tau}(U^{n+1}(x) - U^{n+1/2}(x)) + L_h U^{n+1/2}(x) \\ \qquad\qquad = \frac{1}{2}[f^{n+1}(x) + f^n(x)], \ x \in \Omega_{2h}, \\ \frac{2}{\tau}(U^{n+1}(x) - U^{n+1/2}(x)) + L_h U^{n+1}(x) = f^{n+1}(x), \ x \in \Omega_{1h}, \\ U^0(x) = \varphi(x), \ x \in R_h^M, \ U^n(x) := u(x, n\tau). \end{cases}$$

THEOREM 3.1 *The sequence* $\{U^n(x)\}$, $x \in R_h^M$, $n \in Z$, *satisfies* (*3.2*) *if and only if it satisfies the following difference problem*

$$(3.3) \qquad \begin{cases} \frac{1}{\tau}(U^{n+1}(x) - U^{n+1/2}(x)) + \frac{1}{2}[L_h U^{n+1}(x) + L_h U^n(x)] \\ \qquad\qquad = \frac{1}{2}[f^{n+1}(x) + f^n(x)], \\ U^0(x) = \varphi(x), \ x \in R_h^M. \end{cases}$$

The proof is similar to that of Theorem 2.1.

THEOREM 3.2 *If the functions* a_{ij}, b_i, c, f, $i,j = 1,\dots,M$, *are continuous and bounded on* R^M, *then there exists a positive number* τ_* *dependent only on the norms of the coefficients and such that for all* $\tau \in (0, \tau_*)$ *the difference schema* (3.2) *is unconditionally stable and approximates the problem* (3.1) *with order* $0(\tau + h^2)$.

The proof is analogous to that of Theorem (3.3) in [6].

The schema (3.2) is absolute stable but it is not always open, as shown in the following theorem (see [6]).

THEOREM 3.3 *The schema (3.2) with the condition $\Omega_{1h} \cap \Omega_{2h} = \emptyset$ is the open schema, if $a_{ij} \equiv 0$ for $i \neq j$ and*

$$\Omega_{1h} := \Big\{ x \in R_h^M : \ x = (m_1 h_1, \dots, m_M h_M), \ \frac{1}{2}\Big(\sum_{j=1}^{M} m_j - 1 \Big) \in Z \Big\},$$

$$\Omega_{2h} := \Big\{ x \in R_h^M : \ x = (m_1 h_1, \dots, m_M h_M), \ \frac{1}{2}\Big(\sum_{j=1}^{M} m_j \Big) \in Z \Big\}.$$

CONSEQUENCE. The schema (3.2), with Ω_{1h}, Ω_{2h} such that $\Omega_{1h} \cap \Omega_{2h} = \emptyset$ is explicited, unconditionally stable and approximating the problem (3.1) with order $0(\tau + h^2)$.

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Received September 24, 1993.