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ON SOME CHARACTERIZATIONS OF δ-CONTINUOUS MULTIFUNCTIONS

1. Introduction

In 1980, Noiri introduced and investigated the concept of δ-continuous (single valued) functions. Some properties of δ-continuous functions were studied by I. L. Reilly and M. K. Vamanamurthy (1984). There is a vast literature that deals with other properties of δ-continuous functions and their connections with other types of application. In this paper several characterizations of upper (lower) δ-continuous multifunctions are obtained and their basic properties and their relationship to upper δ-continuous multifunctions and to their graph are investigated.

Throughout this paper spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. The topological spaces (X,T) and (Y,D) will be abbreviated as X and Y respectively. Let S be a subset of a space X. The closure of S and the interior of S are denoted by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$ respectively. A subset S of X is called a regular open [regular closed] iff $S = \mathrm{int}(\mathrm{cl}(S))$ [resp. $S = \mathrm{cl}(\mathrm{int}(S))$] [4]. It is easy to see that S is a regular open sets in X iff $X \setminus S$ is a regular closed set. The set of all regular open [regular closed] set in X are denoted by $RO(X)$ $[RC(X)]$.

A point x in X will be called an δ-closure point of a subset S of X iff $S \cap \mathrm{int}(\mathrm{cl}(U)) \neq \emptyset$ for each open neighborhood U of x. The set of all δ-closure points of S is called δ-closure of S and it is denoted by δ-$\mathrm{cl}(S)$. A subset S of X is called δ-closed if $S = \delta\text{-}\mathrm{cl}(S)$. The complement of a δ-closed set is called a δ-open. The family of all δ-open (δ-closed) sets of X are denoted by δ-$O(X)$ (δ-$C(X)$). [4].

Keywords: Upper (lower) δ-continuous multifunctions, δ-closed graphness of multifunctions.

A subset K of X is called nearly closed relative to X if every cover of K by regular open sets in X has a finite subcover. A space X is said to be nearly compact if X is nearly closed relative to X. [3].

The net $(x_\alpha)_{\alpha\in I}$ δ-convergent to x_0, if for each regular open set U containing x_0 there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in U$. [4].

A multifunction F of a set X into Y is a correspondence such that $F(x)$ is a nonempty subset of Y, for each $x \in X$, that is it is a function $F : X \to P(Y)\setminus\{\emptyset\}$, where $P(Y)$ is the power set of Y. We will denote such a multifunction by $F : X \to Y$. For a multifunction F, the upper and lower inverse of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. The multifunction F is point closed (nearly closed relative to Y) if $F(x)$ is closed (nearly closed relative to Y) in Y, for each $x \in X$. The graph $G(F)$ of a multifunction $F : X \to Y$ is said to be δ-closed if $G(F)$ is δ-closed set in the product space $X \times Y$. [2].

2. Upper and lower δ-continuous multifunctions

2.1. Definition. Let X, Y be topological spaces

a) A multifunction $F : X \to Y$ is upper-δ-continuous (in short, u.δ-c.) at a point $x_0 \in X$ if for any open set $V \subseteq Y$ such that $F(x_0) \subseteq V$ there exists an open set $U \subseteq X$ containing x_0 such that $F(\mathrm{int}(\mathrm{cl}(U)) \subseteq \mathrm{int}(\mathrm{cl}(V))$.

b) A multifunction $F : X \to Y$ is lower-δ-continuous (in short, l.δ-c.) at a point $x_0 \in X$ if for any open set $V \subseteq Y$ such that $F(x_0)\cap V \neq \emptyset$ there exists an open set $U \subseteq X$ containing x_0 such that $\mathrm{int}(\mathrm{cl}(U)) \subseteq F^-(\mathrm{int}(\mathrm{cl}(V)))$.

c) The multifunction $F : X \to Y$ is u.δ-c. (l.δ-c.) if it has this property at each point $x \in X$.

2.2. Theorem. *For a multifunction $F : X \to Y$ the following conditions are equivalent:*

1) *F is l.δ-c.*

2) *For any regular open set $V \subseteq Y$ and for each x of X with $F(x)\cap V \neq \emptyset$ there is a $U \in RO(X,x)$, where $RO(X,x) = \{U \in RO(X) \mid x \in U\}$ such that $U \subseteq F^-(V)$.*

3) *If $V \in RO(Y)$ then $F^-(V) \in \delta\text{-}O(X)$.*

4) *If $V \in \delta\text{-}O(Y)$ then $F^-(V) \in \delta\text{-}O(X)$.*

5) *If $K \in \delta\text{-}C(Y)$ then $F^+(K) \in \delta\text{-}C(X)$.*

6) *If $K \in RC(Y)$ then $F^+(K) \in \delta\text{-}C(X)$.*

7) *For each $B \subseteq Y$, $F^-(\delta\text{-int}(B)) \subseteq (\delta\text{-int}(F^-(B))$.*

8) *For each $A \subseteq X$, $F(\delta\text{-int}(A)) \subseteq \delta - \mathrm{int}(F(A))$.*

9) *For each* $y_0 \in F(x_0)$ *and for every net* $(x_\alpha)_{\alpha \in I}$ *δ-convergent to* x_0, *there exists a subnet* $(z_\beta)_{\beta \in J}$ *of the net* $(x_\alpha)_{\alpha \in I}$ *and a net* $(y_\beta)_{(\beta,V) \in J}$ *in* Y *so that* $(y_\beta)_{(\beta,V) \in J}$ *δ-convergent to* y *and* $y_\beta \in F(z_\beta)$.

Proof.

(1)⇒(2): Let $x \in X$ and let V be a regular open set in Y with $F(x) \cap V \neq \emptyset$. Then V is open set in Y. Since F is l.δ-c. at $x \in X$, there exists an open set W of X containing x such that $\mathrm{int}(\mathrm{cl}(W)) \subseteq F^-(V)$. We define $U = \mathrm{int}(\mathrm{cl}(W))$. Then $U \in RO(X,x)$. So we have $U \subseteq F^-(V)$.

(2)⇒(3): Let $V \subseteq Y$ be a regular open set and $x \in F^-(V)$, so $F(x) \cap V \neq \emptyset$. By (2), there exists a $U \in RO(X,x)$ such that $U \subseteq F^-(V)$. Consequently $F^-(V)$ is δ-open set in X.

(3)⇒(4): Let $V \subseteq Y$ be a δ-open set and $x \in F^-(V)$. So $F(x) \cap V \neq \emptyset$ there exists a $y \in Y$ such that $y \in F(x) \cap V$. Then $y \in F(x)$ and $y \in V$. Since V is a δ-open set there exists a regular open set $W \subseteq Y$ such that $y \in W \subseteq V$. Therefore we have $F(x) \cap W \neq \emptyset$. By (3), $F^-(W)$ is δ-open set of X. Because of $x \in F^-(W)$, there exists a regular open set $U \subseteq X$ such that $x \in U \subseteq F^-(V)$. Thus $F^-(V)$ is a δ-open set of X.

(4)⇒(5): Let $K \subseteq Y$ be any δ-closed set. Then $Y \setminus K$ is a δ-open set, by (4), $F^-(Y \setminus K)$ is a δ-open set. As we can write $F^+(K) = X \setminus F^-(Y \setminus K)$, $F^+(K)$ is a δ-closed set of X.

(5)⇒(6): Every regular closed sets are δ-closed sets. By (5), if $K \in$ δ-$RC(Y)$, then $F^+(K)$ is δ-closed set of X.

(6)⇒(3): Let $V \subseteq Y$ be a regular open set. Then $Y \setminus V$ is δ-closed set of Y, by (6), $F^+(Y \setminus V)$ is δ-closed set of X. As we can write $F^-(V) = X \setminus F^+(Y \setminus V)$, $F^-(V)$ is a δ-open set of X.

(3)⇒(1): Let $x \in X$ and $V \subseteq Y$ any open set such that $F(x) \cap V \neq \emptyset$. We know that $\mathrm{int}(\mathrm{cl}(V))$ is a regular open set with $V \subseteq \mathrm{int}(\mathrm{cl}(V))$ and $F(x) \cap (\mathrm{int}(\mathrm{cl}(V))) \neq \emptyset$. By (3), $F^-(\mathrm{int}(\mathrm{cl}(V))$ is a δ-open set containing x. There is a regular open set U such that $x \in U \subseteq F^-(\mathrm{int}(\mathrm{cl}(V)))$. Thus $U = \mathrm{int}(\mathrm{cl}(U))$ is an open set in X and $x \in \mathrm{int}(\mathrm{cl}(U)) \subseteq F^-(\mathrm{int}(\mathrm{cl}(V)))$.

(4)⇒(7): Let B be any subset of Y. Then we always have δ-$\mathrm{int}(B) \subseteq B$. δ-$\mathrm{int}(B)$ is a δ-open set of Y. By (4), F^-(δ-$\mathrm{int}(B)$) is δ-open set of X. On the other hand we have F^-(δ-$\mathrm{int}(B)) \subseteq F^-(B)$. So F^-(δ-$\mathrm{int}(B)$) $= \mathrm{int}(F^-(\mathrm{int}(B))) \subseteq \mathrm{int}(F^-(\mathrm{int}(B))$. Thus we obtain $F^-(\mathrm{int}(B)) \subseteq \mathrm{int}(F^-(B))$.

(7)⇒(4): Let V be any open set of Y. By (7), $F^-(V) = F^-(\mathrm{int}(V)) \subseteq \mathrm{int}(F^-(V))$. So we have $F^-(V) \subseteq$ δ-$\mathrm{int}(F^-(V))$. Thus $F^-(V)$ is δ-open set of X.

(4)⇒(8): Under the assumption (5), suppose (8) is not true i.e. for some $A \subseteq X$, F(δ-cl(A)) $\not\subseteq$ δ-cl($F(A)$). Then there exists a $y_0 \in Y$ such that

$y_0 \in F(\delta\text{-cl}(A))$, but $y_0 \notin \delta\text{-cl}(F(A))$. So $Y \setminus (\delta\text{-cl}(F(A)))$ is δ-open set containing y_0. By (4), we have $F^-(Y \setminus (\delta\text{-cl}(F(A)))$ is a *delta*-open set of X and $F^-(y_0) \subseteq F^-(Y \setminus (\delta\text{-cl}(F(A)))$. Since $(Y \setminus (\delta\text{-cl}(F(A))) \cap F(A) = \emptyset$ and $A \subseteq F^+(F(A))$ we have $F^-(Y \setminus (\delta\text{-cl}(F(A))) \cap F^+(F(A)) = \emptyset$ and $F^-(Y \setminus (\delta\text{-cl}(F(A))) \cap A = \emptyset$. Since $F^-(Y \setminus (\delta\text{-cl}(F(A)))$ is δ-open set clearly we will have $F^-(Y \setminus (\delta\text{-cl}(F(A))) \cap (\delta\text{-cl}(A)) = \emptyset$. On the other hand because of $y_0 \in F(\delta\text{-cl}(A))$, we have $F^-(y_0) \cap (\delta\text{-cl}(A)) \neq \emptyset$. But this contradicts with $F^-(Y \setminus (\delta\text{-cl}(F(A))) \cap (\delta\text{-cl}(A)) = \emptyset$. Thus $y \in F(\delta\text{-cl}(A))$ implies $y \in \delta\text{-cl}(F(A))$. Consequently $F(\delta\text{-cl}(A)) \subseteq \delta\text{-cl}(F(A))$.

(8)⇒(5): Let $K \subseteq Y$ be any δ-closed set. Since we always have $F(F^+(K)) \subseteq K$ we obtain $\delta\text{-cl}(F(F^+(K))) \subseteq \delta\text{-cl}(K) = K$. By (8), $F(\delta\text{-cl}(F^+(K))) \subseteq K$. Therefore $\delta\text{-cl}(F^+(K)) \subseteq F^+(F(\delta\text{-cl}(F^+(K))) \subseteq F^+(K)$ and so $F^+(K)$ is δ-closed set of X.

(1)⇒(9): Suppose F is l.δ-c. at x_0. Let $(x_\alpha)_{\alpha \in I}$ be a net δ-convergent to x_0. Let $y \in F(x_0)$ and V be an open set containing y. So we have $F(x_0) \cap V \neq \emptyset$. Since F is l.δ-c. at x_0. There exists an open set $U \subseteq X$ containing x_0 such that $\text{int}(\text{cl}(U)) \subseteq F^-(\text{int}(\text{cl}(V)))$. Since the net $(x_\alpha)_{\alpha \in I}$ is δ-convergent to x_0, for this U, there exists a $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in \text{int}(\text{cl}(U))$. Therefore we have the implication $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(\text{int}(\text{cl}(V)))$. For each open set $V \subseteq Y$ containing y, define the sets $I_v = \{\alpha_0 \in I \mid \alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(\text{int}(\text{cl}(V))\}$ and $J = \{(\alpha, V) \mid V \in D(y), \alpha \in I_v\}$ and order $\geq$ on J as follows: $(\alpha', V') \geq (\alpha, V)$ iff $\alpha' \geq \alpha$ and $V' \subseteq V$.

Define $\zeta : J \to I$ by $\zeta((\beta, V)) = \beta$. Then ζ increasing and cofinal in I, so ζ defines a subnet of $(x_\alpha)_{\alpha \in I}$. We denote the subnet $(z_\beta)_{(\beta,V) \in J}$. On the other hand for any $(\beta, V) \in J$, since $\beta \geq \beta_0 \Rightarrow x_\beta \in F^-(\text{int}(\text{cl}(V)))$ we have $F(z_\beta) \cap (\text{int}(\text{cl}(V))) = F(x_\beta) \cap (\text{int}(\text{cl})V))) \neq \emptyset$. Pick $y_\beta \in F(z_\beta) \cap (\text{int}(\text{cl}(V))) \neq \emptyset$. Then the net $(y_\beta)_{(\beta,V) \in J}$ is δ-convergent to y. To see this, let $V_0 \subseteq Y$ be an open set containing y. There exists $\beta_0 \in I$ such that $\beta_0 = \zeta((\beta_0, V_0)); (\beta_0, V_0) \in J$ and $y_{\beta 0} \in \text{int}(\text{cl}(V))$. If $(\beta, V) \geq (\beta_0, V_0)$ this means that $\beta \geq \beta_0$ and $V \subseteq V_0$. Therefore $y_\beta \in F(z_\beta) \cap (\text{int}(\text{cl}(V))) \subseteq F(x_\beta) \cap (\text{int}(\text{cl}(V))) \subseteq F(x_\beta) \cap (\text{int}(\text{cl}(V_0)))$ so $y_\beta \in \text{int}(\text{cl}(V_0))$. Thus $(y_\beta)_{(\beta,V) \in J}$ is δ-convergent to y.

(9)⇒(1): Under the supposing of (9), suppose (1) is not true i.e. F is not l.δ-c. at x_0. Then there exists an open set $V \subseteq Y$ so that $F(x_0) \cap V \neq \emptyset$ and for any neighbourhood $U \subseteq X$ of x_0, there is a point $x_U \in \text{int}(\text{cl}(U))$ for which $F(x_U) \cap (\text{int}(\text{cl}(V))) = \emptyset$. Let us consider the net $(x_U)_{U \in T(x_0)}$ where $T(x_0)$ is the system of T-neighbourhoods of x_0. Obviously $(x_U)_{U \in T(x_0)}$ is δ-convergent to x_0. Let $y_0 \in F(x_0) \cap V$. By (9) there is a subnet $(z_w)_{w \in W}$ of $(x_U)_{U \in T(x_0)}$ and $y_w \in F(z_w)$ like $(y_w)_{w \in W}$ δ-convergent to y_0. As $y_0 \in V$ and $V \subseteq Y$ is an open set there is $w_0' \in W$ so that $w \geq w_0'$ implies $y_w \in \text{int}(\text{cl}(V))$. On the other hand $(z_w)_{w \in W}$ is a subnet of the net $(x_U)_{U \in T(x_0)}$

and so there is a function $h : W \to T(x_0)$ such that $z_w = x_{h(w)}$ and for each $U \in T(x_0)$ there is $w_0'' \in W$ such that $h(w_0'') \geq U$. If $w \geq w_0''$ then $h(w) \geq h(w_0'') \geq U$. Let us consider now $w_0 \in W$ so that $w_0 \geq w_0'$ and $w_0 \geq w_0''$. Therefore $y_w \in \text{int}(\text{cl}(V))$. By the definition of the net $(x_U)_{U \in T(x_0)}$ we have $F(z_w) \cap (\text{int}(\text{cl}(V))) = F(x_{h(w)}) \cap (\text{int}(\text{cl}(V))) = \emptyset$ and this means that $y_w \notin \text{int}(\text{cl}(V))$. Thus is a contradiction and so F is l.δ-c. at x_0.

2.3. THEOREM. *For a multifunction $F : X \to Y$ the following conditions are equivalent:*

1) *F is u.δ-c.*

2) *For any regular open set $V \subseteq Y$ and for each x of X with $F(x) \subseteq V$ there is a regular open set $U \subseteq X$ containing x of X such that $F(U) \subseteq V$.*

3) *If $V \in RO(Y)$ then $F^+(V) \in \delta\text{-}O(X)$.*

4) *If $V \in \delta\text{-}O(Y)$ then $F^+(\text{int}(\text{cl}(V))) \in \delta\text{-}O(X)$.*

5) *If $K \in \delta\text{-}C(Y)$ then $F^-(\text{cl}(\text{int}(K))) \in \delta\text{-}C(X)$.*

6) *If $K \in (1,2)\text{-}RC(Y)$ then $F^-(K) \in \delta\text{-}C(X)$.*

7) *For each net $(x_\alpha)_{\alpha \in I}$ δ-convergent to x_0, and for each open set V of Y with $F(x_0) \subseteq V$ there is $\alpha_0 \in I$ such that $F(x_\alpha) \subseteq \text{int}(\text{cl}(V))$, for all $\alpha \geq \alpha_0$.*

Proof. The proofs of (1)⇔(2)⇔(3) ⇔(4)⇔(5)⇔(6) are quite similar to proofs in Theorem 2.2. and we skip them. We only prove (1)⇔(7).

(1)⇒(7): Let V be an open set of Y with $F(x_0) \subseteq V$. By (1), there is an open set $U \subseteq X$ containing x_0 of Xsuch that $F(\text{int}(\text{cl}(U))) \subseteq \text{int}(\text{cl}(V))$. Since $U \in T(x_0)$ and since $(x_\alpha)_{\alpha \in I}$ is δ-convergent to x_0, there is a $\alpha_0 \in I$ such that $x_\alpha \in \text{int}(\text{cl}(U))$, for all $\alpha \geq \alpha_0$ and then $F(x_\alpha) \subseteq \text{int}(\text{cl}(V))$, for all $\alpha \geq \alpha_0$ and hence (7) follows.

(7)⇒(1): Under the supposing of (7), suppose (1) is not true. Then there is an open set V in Y with $F(x_0) \subseteq V$ such that for each open set U of X, $F(\text{int}(\text{cl}(U))) \not\subseteq \text{int}(\text{cl}(V))$ i.e. there is a $x_U \in \text{int}(\text{cl}(U))$ such that $F(x_U) \not\subseteq \text{int}(\text{cl}(V))$. Then such x_U's will form a net in X with directed set $T(x_0)$ (under inclusion relation). Clearly this net is δ-convergent to x_0. But $F(x_U) \not\subseteq \text{int}(\text{cl}(V))$, for all $U \in T(x_0)$. This contradicts the hypothesis of (7).

2.4. THEOREM. *Let $F : X \to Y$ be a point-compact multifunction and assume that Y is an almost regular space. The following conditions are equivalent:*

1) *F is u.δ-c.*

2) *If $V \in \delta\text{-}O(Y)$ then $F^+(V) \in \delta\text{-}O(X)$.*

3) *If $K \in \delta\text{-}C(Y)$ then $F^-(K) \in \delta\text{-}C(X)$.*

4) *$\delta\text{-cl}(F^-(B)) \subseteq F^-(\delta\text{-cl}(B))$, for each $B \subseteq Y$.*

Proof.

(1)$\Rightarrow$(2): Let V be a δ-open set of Y, and $x \in F^+(V)$. So we have $F(x) \subseteq V$. Since V is a δ-open subset of Y, for each $y \in F(x)$, there is a regular open set W_y such that $y \in W_y$. Since Y is almost regular space there exists an open set T_y such that $y \in T_y \subseteq \mathrm{cl}(T_y) \subseteq \mathrm{int}(\mathrm{cl}(W_y)) = W_y$.

Therefore we have $F(x) \subseteq \bigcup\{T_y \mid y \in F(x)\} \subseteq \bigcup\{\mathrm{cl}(T_y) \mid y \in F(x)\} \subseteq \bigcup\{w_y \mid y \in F(x)\} \subseteq V\}$. Since $F(x)$ is a compact set, there exists points $y_1, y_2, y_3, \ldots, y_n \in F(x)$ such that $F(x) \subseteq \bigcup\{T_{yi} \mid y_i \in F(x),\ i = 1,2,3,\ldots \ldots, n\} \subseteq \bigcup\{\mathrm{cl}(T_{yi}) \mid y_i \in F(x),\ i = 1,2,3,\ldots,n\} \subseteq \bigcup\{W_{yi} \mid y_i \in F(x),\ i = 1,2,3,\ldots,n\} \subseteq V$.

Therefore we obtain $F(x) \subseteq \mathrm{int}(\bigcup\{T_{yi} \mid y_i \in F(x), i = 1,2,3,\ldots,n\}) = \bigcup\{T_{yi} \mid y_i \in F(x), i = 1,2,3,\ldots,n\} \subseteq \mathrm{int}(\mathrm{cl}(\bigcup\{T_{yi} \mid y_i \in F(x), i = 1,2,3,\ldots,n\})) \subset V$. Put $T_1 = \mathrm{int}(\mathrm{cl}(\bigcup\{T_{yi} \mid y_i \in F(x), i = 1,2,3,\ldots,n\}))$ and observe that T_1 is regular open set of Y. Since F is u.δ-c. there is a regular open set U containing x such that $F(U) \subseteq T_1 \subseteq V$. Therefore we have $x \in U \subseteq F^+(V)$ and this means that $F^+(V)$ is δ-open set of X.

(2)$\Rightarrow$(3): Let K be a δ-closed set of Y. Since $Y \setminus K$ is a δ-open set of Y, by (4) we conclude that $F^+(Y \setminus K)$ is a δ-open set of X. So $F^-(K) = X \setminus F^+(Y \setminus K)$ is a δ-open set of X.

(3)$\Rightarrow$(1): Let $x \in X$ and V be a regular-open set Y such that $F(x) \subseteq V$. So $Y \setminus V$ is a δ-closed set of Y and by (3), $F^-(Y \setminus V)$ is a δ-closed set of X. Therefore $F^+(V) = X \setminus F^-(Y \setminus V)$ is a δ-open set of X. Since $x \in F^+(V)$, there exists a regular-open set U of X such that $x \in U \in F^+(V)$ and this means that $F(U) \subseteq V$. F is u.δ-c.

(3)$\Rightarrow$(4): Let B be arbitrary subset of Y. Because of $B \subseteq \delta\text{-cl}(B)$, we always have $F^-(B) \subseteq F^-(\delta\text{-cl}(B))$. Since the set $\delta\text{-cl}(B)$ is a δ-closed set, by (3), $F^-(\delta\text{-cl}(B))$ is a δ-closed subset of X. Therefore we obtain $(\delta\text{-cl}(F^-(B)) \subseteq \delta\text{-cl}(F^-(\delta\text{-cl}(B))) = F^-(\delta\text{-cl}(B))$ and thus $\delta\text{-cl}(F(B)) \subseteq F^-(\delta\text{-cl}(B))$.

(4)$\Rightarrow$(3): Let B be a δ-closed set of Y. Because of $B = \delta\text{-cl}(B)$, we have $F^-(B) = F(\delta\text{-cl}(B))$. By (4), we obtain $\delta\text{-cl}(F^-(B)) \subseteq F^-(\delta\text{-cl}(B)) = F^-(B)$. So $\delta\text{-cl}(F^-(B) \subseteq F^-(B)$ and $F^-(B)$ is a δ-closed set of X.

2.5. THEOREM. *If $F : X \to Y$ and $G : Y \to Z$ are l.δ-c. multifunctions, then $G \circ F : X \to Z$ is l.δ-c. multifunction.*

Proof. Let V be a δ-open set of Z. Since $(G \circ F)^-(V) = F^-(G^-(V))$ and F, G are l.δ-c. multifunctions. $(G \circ F)^-(V)$ is a δ-open set of X. Thus $G \circ F$ is l.δ-c.

2.6. PROPOSITION. *Let (X,T) be a topological space, $A \subset Y$ an open set and $U \subset X$ a regular open set. Then $W = A \cap U$ is regular open set in A.*

Proof. Denote by $T|_A$ the topology T restricted to A. Since W is a $T|_A$-open set in $A, W \subseteq \text{int}_A(\text{cl}_A(W))$. Now suppose there exists a point $z \in A$ such that $z \in \text{int}_A(\text{cl}_A(W))$ but $z \notin W$. Then there exists a $T|_A$-open set V_A in A such that $z \in V_A \subseteq \text{cl}_A(W)$. The fact that $\text{cl}_A(W) = \text{cl}_X(W) \cap A$ allows us to write $z \in V_A \subseteq \text{cl}_A(W) \subseteq \text{cl}_X(W) \subseteq \text{cl}_X(U)$. Since $V_A \in T|_A$ and $A \in T$ then V_A is open set in X. But $z \in V_A \subseteq \text{cl}_X(U)$, so $z \in \text{int}_X(\text{cl}_X(U))$. Since U is a regular open set in X, $z \in U = \text{int}_X(\text{cl}_X(U))$. Because of $z \in A$ and $z \notin W$, $z \notin U$. Thus no such points z can exist, what shows that $W \subseteq \text{int}_A(\text{cl}_A(W))$ and $W = \text{int}_A(\text{cl}_A(W))$. Consequently W is a regular open set in A.

2.7. THEOREM. *For a multifunction $F : (X,T) \to (Y,D)$, the following statements are true*

1) *If F is l.δ-c. (u.δ-c.) and A is an open subset of X, then $F|_A : (A, T|_A) \to (Y,D)$ is l.δ-c.(u.δ-c).*

2) *Let $\Phi = \{U_\alpha \mid \alpha \in I\}$ be a regular open cover of X. Then a multifunction $F : (X,T) \to (Y,D)$ is l.δ-c. (uδ-c). iff the restrictions $F_\alpha = F|U_\alpha : U_\alpha \to Y$ are l.δ-c. (u.δ-c.) for each $\alpha \in I$.*

Proof. 1) The proof is an easy consequence of Prop. 2.6 and definitions of l.δ-c (u.δ-c).

2)($\Rightarrow$) Let F be l.δ-c. and $\alpha \in I$ be fix $x \in U_\alpha$ and let V be any regular open set of Y such that $F_\alpha(x) \cap V \neq \emptyset$. Since $F(x) = F_\alpha(x)$ and F is l.δ-c., there exists a regular open set U_0 containing x such that $U_0 \subseteq F^-(V)$. Put $U = U_\alpha \cap U_0$ and observe that U is regular open subset of U_α and $x \in U$. Therefore $U \subseteq F^-(V) \cap U_\alpha = F_\alpha^-(V)$. Thus F_α is l.δ-c. at x. Since $\alpha \in I$ and $x \in U_\alpha$ is arbitrary, this shows that F_α is l.δ-c.

($\Leftarrow$) Suppose that F_α is l.δ-c. for each $\alpha \in I$. Let $x \in X$ and fix a regular open set V such that $F(x) \cap V \neq \emptyset$. There exists $\alpha \in I$ such that $x \in U_\alpha$. Since $F(x) = F_\alpha(x)$, we have $F_\alpha(x) \cap V \neq \emptyset$. Since F_α is l.δ-c. there exists a regular open set U in U_α such that $x \subseteq F_\alpha^-(V) = F^-(V) \cap U_\alpha$. Take a regular open set $W \subset X$ such that $U = U_\alpha \cap W$. Because the intersection of any two regular open sets is regular open set, U is a regular open set in X. Thus F is l.δ-c. at x.

Note: As an application of the above theorems we see that the characterizing theorem of δ-continuous single valued functions as deduced in (Noiri, 1980) can be derived from the above theorems only treating F to be single valued.

3. δ-closed graphness and upper δ-continuousness of multifunctions

3.1. THEOREM. *Let $F : X \to Y$ be a point-closed multifunction. If F is u.δ-c. and assume that Y is a regular space, then $G(F)$ is δ-closed.*

Proof. Suppose $(x,y) \notin G(F)$. Then we have $y \notin F(x)$. Since Y is regular, there exist disjoint open sets V_1, V_2 of Y such that $y \in V_1$ and $F(x) \subseteq V_2$. Since F is u.δ-c. at x, there exists an open set U in X containing x such that $F(\mathrm{Int}(\mathrm{Cl}(U))) \subseteq \mathrm{int}(\mathrm{Cl}(V_2))$. As V_1 and V_2 are disjoint, we have $(\mathrm{Int}(\mathrm{Cl}(V_1))) \cap (\mathrm{Int}(\mathrm{Cl}(V_2))) \neq \emptyset$. Therefore we obtain $x \in \mathrm{Int}(\mathrm{Cl}(U))$, $y \in \mathrm{Int}(\mathrm{Cl}(V_1))$ and $(x,y) \in \mathrm{Int}(\mathrm{Cl}(U)) \times \mathrm{Int}(\mathrm{Cl}(V_1)) \subseteq X \times Y \setminus G(F)$ so $X \times Y \setminus G(F)$ is δ-open. Thus $G(F)$ is δ-closed in $X \times Y$.

3.2. THEOREM. *If $F : X \to Y$ is a multifunction with δ-closed graph and $K \subseteq Y$ is nearly closed set relative to Y. Then $F^-(K)$ is δ-closed set in X.*

Proof. If $x \in F^-(K)$, then we have $F(x) \cap K = \emptyset$. Therefore we have $(x,y) \notin G(F)$, for all $y \in K$. Since $G(F)$ is δ-closed in $X \times Y$, there are regular open sets $U_y \subseteq X$ containing x and $V_y \subseteq X$ containing y such that $F(U_y) \cap V_y = \emptyset$. $\Phi = \{V_y \mid y \in K\}$ is a regular open cover of K. Since K is nearly closed relative to Y, there are points $y_1, y_2, y_3, \ldots, y_n$ in K such that $K \subseteq \bigcup\{V_{yi} \mid i = 1,2,3,\ldots,n\}$. Set $U = \cap\{U_{yi} \mid i = 1,2,3,\ldots,n\}$ and observe that U is a regular open set containing x with $U \cap F^-(K) = \emptyset$. This shows that $x \notin \delta\text{-}\mathrm{Cl}(F^-(K))$. Thus $F^-(K)$ is δ-closed set.

3.3. THEOREM. *If Y nearly compact space and a multifunction $F : X \to Y$ has a δ-closed graph, then F is u.δ-c.*

Proof. Let K be a regular closed set in Y. Since regular closed subset of nearly compact space is nearly closed and F is a δ-closed graph multifunction, by 3.2. Theorem, $F^-(K)$ is a δ-closed set in X. Thus F is u.δ-c.

3.4. THEOREM. *Let Y be an almost regular space, $F : X \to Y$ be a point compact and u.δ-c. multifunction. If A is a nearly closed set relative to X, then $F(A)$ is a nearly closed set relative to Y.*

Proof. Let A be a nearly closed set relative to X and Φ be a regular open cover of $F(A)$. If $a \in A$ then we have $F(a) \subseteq \bigcup \Phi$. Thus Φ is a regular open cover of $F(a)$. Since $F(a)$ is compact, there exists a finite subfamily $\Phi_{n(a)}$ of Φ such that $F(a) \subseteq \bigcup \Phi_{n(a)} = V_a$. $V_a$is a δ-open set in Y. Since F is u.δ-c. at a, $F^+(V_a)$ is a δ-open set in X. So there exists a regular open set U_a of X such that $a \in U_a \subseteq F^+(V_a)$. Therefore $\zeta = \{U_a \mid a \in A\}$ is a regular open covering of A. Since A is nearly closed set relative to X, there exist points $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup\{U_{ai} \mid a_i \in A,\ i = 1,2,\ldots,n\}$. So we obtain $F(A) \subseteq F(\bigcup\{U_{ai} \mid a_i \in A,\ i = 1,2,\ldots,n\}) \subseteq \bigcup\{V_{ai} \mid a_i \in$

$A,\ i=1,2,\ldots,n\} \subseteq \bigcup\{U\Phi_{n(ai)} \mid ai \in A,\ i=1,2,\ldots,n\}$. Thus $F(A)$ is nearly closed set relative to Y.

3.5. COROLLARY. *Let Y be a almost regular space and $F: X \to Y$ be a point compact and u.δ-c. multifunction. If X is nearly compact, then Y is a nearly compact.*

Proof. The proof is clear.

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