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# ON THE OSCILLATION OF SOLUTIONS OF DIFFERENCE EQUATIONS

In several recent papers (see e.g. [1-6]) oscillatory behavior of solutions of difference equations and inequalities have been investigated. In this paper we study this property for solutions of the equation

$$(E) \qquad \Delta^2 y_n + p_{n-k} y_{n-k}^q = 0, n = k, k+1, \dots$$

where $q$ is a quotient of odd positive integers, $k$ any fixed nonnegative integer. Some results will be presented for suitable difference inequalities.

Equation (E) is a discrete analogue of Emden-Fowler differential equation with retarded arguments. Equations of type (E) with $k = -1$ were exhaustively considered in [2]. Some of the results contained therein were next generalized in [6]. Sufficient conditions for oscillation of first order difference equation with delay were studied in [3] (see also [5]).

In the paper we shall use the following notions: $R, R_+, N, N_m$ for the sets of reals, nonnegative reals, nonnegative integers, and integers no less than $m$. For any function $x : N \to R$ the forward difference operator $\Delta$ is defined by the equality $\Delta x_n = x_{n+1} - x_n$, $n \in N$, and for $k \geq 1$: $\Delta^k x_n = \Delta(\Delta^{k-1} x_n)$, $n \in N$. Furthermore we suppose

$$\sum_{j=k}^{k-m} x_j := 0 \quad \text{for any } k, m \in N, m \geq 1.$$

We call the function $x : N \to R$ oscillatory if there exists any infinite, increasing sequence $\{n_r\}_{r=1}^{\infty}$ of positive integers such that $x_{n_r} x_{n_r+1} \leq 0$ for all $r \geq 1$, $r \in N$.

By a solution (ordinary solution), generalized $N_\nu$ solution, generalized solution of (E) we mean real sequence $y := \{y_n\}_{n=0}^{\infty}$ satisfying (E) for all $n \in N$, for all $n \in N_\nu$, for all sufficiently large argument without specification the initial value for which the equation is satisfied, respectively.

Throughout the paper we shall suppose that $q$ is a quotient of odd positive integers, however in some statements it suffices $q$ be only positive real.

We start our investigations with some useful lemma, given here without proof, which is very simple.

LEMMA 1. *Let* $p : N \to R_+$ *and let for every* $m \in N$ *there is* $\sup_{n \geq m} p_n > 0$. *Assume that* $\{x_n\}_{n \in N}$ *is a solution of the inequality*

$$(I1) \qquad \Delta^2 x_n + p_{n-k} x_{n-k}^q \leq 0, n \in N_k$$

*such that*

$$(1) \qquad x_n > 0$$

*for all* $n \geq \nu$ *and some* $\nu \in N$. *Then*

$$\Delta x_n > 0 \quad \textit{for all } n \geq \nu + k.$$

Remark 1. Similar result holds for the reversed inequality

$$(I2) \qquad \Delta^2 x_n + p_{n-k} x_{n-k}^q \geq 0, n \in N_k.$$

THEOREM 1. *Let* $p : N \to R_+$ *be such that* $\sup_{n \geq m} p_n > 0$ *for every* $m \in N$. *If* $\{x_n\}_{n \in N}$ *is a solution of the inequality* (I1) *such that* (1) *holds, then equation* (E) *has the generalized* $N_\mu$ *solution* $y$, ($\mu := \nu + k$) *such that*

(2i) $0 < y_n \leq x_n$, $n \in N_\nu$,
(2ii) $0 < \Delta y_n \leq \Delta x_n$, $n \in N_\mu$,
(2iii) $\lim_{n \to \infty} \Delta y_n = 0$,
(2iv) $\sum_{j=\nu}^{\infty} p_j y_j^q$ *converges.*

Proof. By Lemma 1 we get

$$(3) \qquad \Delta x_n > 0 \quad \text{for } n \geq \mu.$$

Let us take some $m, n \in N_\mu$, $m \geq n$. Then from (I1) we obtain

$$\Delta x_{m+1} - \Delta x_n + \sum_{j=n}^{m} p_{j-k} x_{j-k}^q \leq 0.$$

Hence, by (3), we get

$$(4) \qquad \sum_{j=n}^{m} p_{j-k} x_{j-k}^q < \Delta x_n$$

and from there

$$(5) \qquad \sum_{j=n}^{\infty} p_{j-k} x_{j-k}^q \leq \Delta x_n \quad \text{for } n \in N_\mu.$$

Let us consider the set $S$ of all sequences $\{u_n\}_{n\in N}$ such that

$$(6)\qquad \begin{cases} u_n = x_n & \text{for } n = 0, 1, \dots, \mu, \\ u_n \geq 0 & \text{for } n > \mu, \\ \Delta u_n \leq \Delta x_n & \text{for } n \geq \mu. \end{cases}$$

For any $u \in S$ we have

$$(7)\qquad u_n \leq x_n \quad \text{for all } n \in N.$$

By (6) and (7) we get from (5) that

$$(8)\qquad \sum_{j=\mu}^{n-1} \sum_{i=j}^{\infty} p_{i-k} u_{i-k}^q \leq \sum_{j=\mu}^{n-1} \sum_{i=j}^{\infty} p_{i-k} x_{i-k}^q \leq x_n - x_\mu \quad \text{for all } n \in N_\mu.$$

Define an operator $T$ on $S$ by the formula $Tu = z = \{z_n\}_{n\in N}$ where

$$(9)\qquad \begin{aligned} z_n &= x_n \quad \text{for } n = 0, \dots, \mu \\ z_n &= x_\mu + \sum_{j=\mu}^{n-1} \sum_{i=j}^{\infty} p_{i-k} u_{i-k}^q \quad \text{for } n > \mu. \end{aligned}$$

Observe that by (8) the operator $T$ is well defined. Furthermore

$$z_n \geq x_\mu > 0, n \geq \mu$$

and

$$\Delta z_n = \sum_{i=n}^{\infty} p_{i-k} u_{i-k}^q \leq \sum_{i=n}^{\infty} p_{i-k} x_{i-k}^q \leq \Delta x_n, n \in N_\mu.$$

So $T : S \to S$.

Let now $u = \{u_n\}_{n\in N}$, $v = \{v_n\}_{n\in N}$ be any two elements of $S$. We shall say $u < v$ if

$$(10)\qquad u_n \leq v_n \quad \text{and} \quad \Delta u_n \leq \Delta v_n, n \in N.$$

One can easily check that operator $T$ is monotonic on $S$ i.e. $Tu < Tv$ whenever $u < v$. Fix $y^0 = \{x_n\}_{n\in N} \in S$ and consider the sequence $\{y^r\}_{r\in N}$ of consequtive iterations of $y^0$, i.e.

$$y^{r+1} = Ty^r \quad \text{for } r \geq 0.$$

We shall prove that $\{y^r\}_{r\in N}$ is monotonic i.e. $y^{r+1} < y^r$, $r = 0, 1, \dots$. By (9) for $r = 0$ we get $y_n^1 = y_n^0$, $n = 0, \dots, \mu$, and hence $\Delta y_n^1 = \Delta y_n^0$, $n = 0, \dots, \mu^{-1}$. Furthermore, by (5)

$$\Delta y_n^1 = \sum_{j=n}^{\infty} p_{j-k} (y_{j-k}^0)^q = \sum_{j=n}^{\infty} p_{j-k} (x_{j-k})^q \leq \Delta x_n = \Delta y_n^0$$

for $n \geq \mu$, from there it follows $y_n^1 \leq y_n^0$ for $n > \mu$. Hence we get $y^1 < y^0$. Since $y^1 \in S$ and $T$ is monotonic on $S$ we have $y^2 = Ty^1 < Ty^0 = y^1$. By the induction argument we obtain $y^{i+1} < y^i$ for all $i \in N$. For arbitrary $n \in N$ the sequence $\{y_n^r\}_{r\in N}$ is nonincreasing and bounded from below by $x_n$ for $n = 0, \ldots, \mu$ and by $x_\mu$ for $n > \mu$. So there exist finite limits

$$y_n = \lim_{r\to\infty} y_n^r \quad \text{for every } n \in N.$$

Hence, by (9) we have

$$y_n = \lim_{r\to\infty} y_n^{r+1} = x_\mu + \lim_{r\to\infty} \sum_{j=\mu}^{n-1} \sum_{i=j}^{\infty} p_{i-k}(y_{i-k}^r)^q, n > \mu.$$

From the above, taking into account that $y_\mu = x_\mu$, we have

$$\Delta y_n = \lim_{r\to\infty} \sum_{j=n}^{\infty} p_{j-k}(y_{j-k}^r)^q \quad \text{for } n \geq \mu, \tag{11}$$

and consequently

$$\begin{aligned}\Delta^2 y_n &= \lim_{r\to\infty} \sum_{j=n+1}^{\infty} p_{j-k}(y_{j-k}^r)^q - \lim_{r\to\infty} \sum_{j=n}^{\infty} p_{j-k}(y_{j-k}^r)^q \\ &= \lim_{r\to\infty} [-p_{n-k}(y_{n-k}^r)^q] = -p_{n-k} y_{n-k}^q, n \geq \mu.\end{aligned}$$

So the sequence $\{y_n\}_{n\in N}$ is a generalized $N_\mu$ solution of (E). Since for every $r \in N$ there is $y^r \in S$, then from $0 \leq y_n^r \leq x_n$, $n \geq \nu$ it follows

$$0 \leq y_n \leq x_n \quad \text{for all } n \in N_\nu. \tag{12}$$

Moreover $y_n = x_n$ for $n = 0, 1, \ldots, \mu$. By (5), (11), and $0 \leq y_n^r \leq x_n$ we get

$$0 \leq \Delta y_n \leq \sum_{j=n}^{\infty} p_{j-k} x_{j-k}^q \leq \Delta x_n, n \in N_\mu. \tag{13}$$

This means that the sequence $\{y_n\}_{n\in N}$ is nondecreasing at least for $n \geq \mu$. But $y_\mu = x_\mu > 0$ so $y_n \geq x_\mu > 0$ for $n \geq \mu$. Hence (2i) holds. Because the definition of the set $S$ yields $y_n = x_n > 0$ for $n = \nu, \nu+1, \ldots, \mu$, therefore by (E) $\Delta^2 y_n \leq 0$ for $n \geq \mu$. The sequence $\{\Delta y_n\}_{n\geq\mu}$ is nonnegative and nondecreasing, furthermore by $\sup p_n > 0$ this sequence contains some strictly decreasing subsequence so $\Delta y_n > 0$ for $n \in N_\mu$.

Moreover (2iii) and (2iv) follow from (13) and (12). Q.E.D.

Remark 2. Similar result for inequality (I2) can be expressed as follows

Let $p : N \to R_+$ be such that $\sup_{n\geq m} p_n > 0$ for every $m \in N$. If $\{x_n\}_{n\in N}$ is a solution of the inequality (I2) such that (1) holds, then equation (E) has the generalized $N_\mu$ $(\mu := \nu + k)$ solution $y$ such that

$$\begin{aligned}
&0 > y_n \geq x_n, n \in N_\nu,\\
&0 > \Delta y_n \geq \Delta x_n, n \in N_\mu,\\
&\lim_{n\to\infty} \Delta y_n = 0,\\
&\sum_{j=\nu}^{\infty} p_j y_j^q \quad \text{converges.}
\end{aligned}$$

Remark 3. Relation (2iv) describes the fact that the solution of (E) is $q$ summable with the weight function $p$.

Remark 4. To get similar result for ordinary solution we should have in the assumption (1) $\nu = 0$. The problem is: can we extended generalized solution up to $n = 0$? or in the other words does there exist ordinary solution of (E) which coincides for $n \geq \nu$ with the generalized solution obtained by Theorem 1? For the answer let us rewrite the equation (E) in the form

$$p_n y_n^q = -\Delta^2 y_{n+k}.$$

Considering the case $k > 0$ under the assumption $p_n \neq 0$ we can observe that having done $y_{n+k}, y_{n+k+1}, y_{n+k+2}$ one can find $y_n$ from the formula

$$y_n = [-(1/p_n)\Delta^2 y_{n+k}]^{1/q}$$

and following this way (if $p_n \neq 0$ for $n = 0, 1, \ldots, \nu - 1$) build the solution step by step on the left of $\nu$ up to $y_0$. In the case $k = 0$ we obtain from (E)

$$p_n y_n^q + y_n = -y_{n+2} + 2y_{n+1}.$$

Now, we can proceed in a similar way provided the equations

$$p_n z^q + z = -y_{n+2} + 2y_{n+1}, n < v$$

have solutions.

Some of researches reason that if they have any generalized solution they can obtain ordinary solution by a simple procedure of renumbering the terms or in other words by translation of the independent variable $n$ only in the solution. But it is not always allowed as we shall show in the example below.

EXAMPLE 1. Let us consider the equation

$$(E1) \qquad \Delta^2 y_n + p_n y_n^2 = 0, n \in N,$$

where

$$\begin{aligned}
&p_0 = p_1 = 0,\\
&p_n = \frac{2(n+2)}{(n+3)(n+4)(2n+3)^2} \quad \text{for } n \geq 2.
\end{aligned}$$

We can check that the sequence

$$y = \left\{ \frac{2n+3}{n+2} \right\}_{n=2}^{\infty} \tag{14}$$

fulfills (E1) for $n \geq 2$. Since $y_2 = \frac{7}{4}$, $y_3 = \frac{9}{5}$, $y_4 = \frac{11}{6}$ then $\Delta^2 y_2 = -p_2 y_2^2 = -\frac{1}{60}$. Supposing we translate the independent variable taking $y_0 = \frac{7}{4}, y_1 = \frac{9}{5}, y_2 = \frac{11}{6}$ then $\Delta^2 y_0 = -\frac{1}{60}$ but it is imposible because for $p_0 = 0$ we have $\Delta^2 y_0 = 0$.

Of course in the considered equation changing independent variable both in the sequence of coefficients $p$ and the solution $y$ we obtain the sequence which fulfills (E1) for all $n \geq 0$ but the new equation is not the same relation as before, because for $p_0 = 0$ we should have in the equation $\Delta^2 y_0 = 0$.

Let us follow the procedure of coming back, now, on the above example changing the values of $p_0$ and $p_1$ and studying different types of efekts we shall obtain during our activity. The equation (E1) can be transformed to the form

$$p_n y_n^2 + y_n + (y_{n+2} - 2y_{n+1}) = 0. \tag{15}$$

Supposing we have generalized solution of (E1) given by (14) we try to find ordinary solution.

Case 1. If $p_0 = p_1 = 0$ then from (15) we get for $n = 1$ and $n = 0$ the unique solution on $N$ with $y_1 = 17/10$, $y_0 = 33/20$.

Case 2. If in (E1) $p_1 = 0$ and $p_0 > 0$ then in this case equation (15) possesses always two ordinary solutions starting with two different $y_0$ which coincide with (14) for $n \geq 2$ and which are glued at $y_1 = 17/10$.

Case 3. Let $p_1 = 21/680$, $p_0 \in (0, 1/279)$. In this case the generalized solution can be extended on the left twofold $y_1' = -34$ or $y_1'' = 34/21$. Moreover each of these solutions can be further extended also twofold. So we get four different ordinary solutions which coincide with (14) for $n \geq 2$.

Case 4. Let $p_1 = 21/680, p_0 > 1/279$. In this case we obtain two $y_1$. For $y_1 = 34/21$ the solution can be continued to the left twofold like previously, but if we take $y_1 = -34$ then this solution can not be extended to the left.

Case 5. Let now $p_1 = 20/17$, $p_0 > 5$. This time we get two values of $y_1$ namely $y_1' = -17/10$ and $y_1'' = 17/20$, but none of these generalized solutions can be further extended to the left, that means we get two generalized solutions which coincide with the sequence (14) for $n \geq 2$ and differ in the point $y_1$ and which can not be extended to the left up to the $y_0$, or in other words there do not exist initial values $y_0, y_1$ such that suitable solution coincides with (14) for $n \geq 2$.

The above examples do not exhaustively describe all possible relations between ordinary and generalized solutions. We shall study the problem elsewhere.

THEOREM 2. *Let $p : N \to R_+$ be such that one of the following conditions holds*

(16i) $\lim_{n\to\infty} \sup p_n > 0$,

(16ii) *there exist $K \in N$ and $\alpha > 0$ such that $np_{nK} \geq \alpha$ for all $n \in N$,*

(16iii) *there exists positive, increasing sequence $\{\varphi_n\}$ such that $\lim_{n\to\infty} \varphi_n = \infty$ and $\limsup_{n\to\infty}(1/\varphi_n) \sum_{j=m}^{n} \varphi_j p_j > 0$ for $m \in N$.*

*Let $q \in R_+$ be a quotient of positive odd integers. Then inequality* (I1) *does not possess any solution which is eventually positive.*

Proof. Suppose contrary that the inequality (I1) possesses any solution $x$ which is eventually positive, say $x_n > 0$ for all $n \geq \nu$ and some $\nu \in N$. Since the assumptions of Theorem 1 are satisfied, so it follows that equation (E) possesses a positive solution with the properties

$$\Delta y_n \to 0, \Delta y_n > 0 \quad \text{for } n \geq \mu := \nu + k. \tag{17}$$

Suppose at first that the sequence $p$ satisfies condition (16i). Then by (17) we get

$$\limsup_{n\to\infty} p_n y_n^q > 0.$$

On the other hand

$$\lim_{n\to\infty} \Delta^2 y_{n+k} = \lim_{n\to\infty} \{\Delta y_{n+k-1} - \Delta y_{n+k}\} = 0.$$

So we obtain a contradiction, because

$$0 = \limsup_{n\to\infty}\{-\Delta^2 y_{n+k}\} = \limsup_{n\to\infty} p_n y_n^q > 0.$$

Considering conditions (16ii) we shall show that our assumptions also lead to contradiction. Rewrite equation (E) in the form

$$\Delta^2 y_{j+k} = -p_j y_j^q.$$

Multiplying the above equation by $j$ and summing next from $\mu$ to $n$ we obtain

$$\sum_{j=\mu}^{n} j\Delta^2 y_{j+k} = -\sum_{j=\mu}^{n} jp_j y_j^q. \tag{18}$$

Applying the method of summation by parts the left hand side of (18) is equal to

$$n\Delta y_{n+k+1} - \mu\Delta y_{\mu+k} - \sum_{j=\mu}^{n-1} \Delta y_{j+k+1}$$

while by monotonicity of the solution $y$ the sum on the right hand side can be estimated from above by $-y_\mu^q \sum_{j=\mu}^{n} jp_j$. Then using (16ii), we find

$$n\Delta y_{n+k+1} - \mu\Delta y_{\mu+k} - y_{n+k+1} + y_{\mu+k+1} \le -y_\mu^q \sum_{i=E(\mu/K)+1}^{E(n/K)} iKp_{iK} \le -\alpha K[E(n/K) - E(\mu/K)]y_\alpha^q,$$

where $E(m)$ denotes the entire part of $m$. Dividing by $n$ we get

$$\Delta y_{n+k+1} - \frac{\mu\Delta y_{\mu+k}}{n} - \frac{y_{n+k+1}}{n} + \frac{y_{\mu+k+1}}{n} \le -\alpha K\frac{E(n/K) - E(\mu/K)}{n}y_\alpha^q, \tag{19}$$

Suppose that $y_n \to a > 0$ for some $a \in R$, then of course $\lim_{n\to\infty}(y_{n+k+1}/n) = 0$. If $y_n \to \infty$, then by Stolz theorem we obtain $\lim_{n\to\infty}(y_{n+k+1}/n) = \lim_{n\to\infty} \Delta y_{n+k+1} = 0$. Therefore in both possible cases $y_{n+k+1}/n$ tends to zero. Since the same behavior characterize first, second, and fourth terms on the left hand side of (19), thus all the sum of this side tends to zero. On the other hand we have

$$\lim_{n\to\infty}\left[-\alpha K\frac{E(n/K) - E(\mu/K)}{n}y_\alpha^q\right] = -\alpha y_\mu^q < 0.$$

Therefore we have got

$$0 < -\alpha y_\mu^q < 0.$$

This contradiction completes the proof in the case (16ii). To see that conditions (16iii) also gives us a contradiction we proceed in a similar way like in the previous case, obtaining

$$\Delta y_{n+k+1} - \frac{\varphi_\mu\Delta y_{\mu+k}}{\varphi_n} - (1/\varphi_n)\sum_{j=\mu}^{n-1}(\Delta y_{j+k+1})(\Delta\varphi_j) \le -y_\mu^q(1/\varphi_n)\sum_{j=\mu}^{n}\varphi_j p_j. \tag{20}$$

Of course $\lim_{n\to\infty} \Delta y_{n+k+1} = 0$ and $\lim_{n\to\infty}(\varphi_\mu\Delta y_{\mu+k})/\varphi_n = 0$.

To determine the limit at infinity of the third term on the left hand side, let us observe that $\Delta y_j > 0$, $\Delta\varphi_j > 0$. Therefore we have two possible cases

(i) $$\lim_{n\to\infty}\sum_{j=\mu}^{n-1}(\Delta y_{j+k+1})(\Delta\varphi_j) = a > 0$$

for any constant $a$, or

(ii) $$\lim_{n\to\infty}\sum_{j=\mu}^{n-1}(\Delta y_{j+k+1})(\Delta\varphi_j) = \infty.$$

In the case (i), the considered term tends to zero. In the case (ii), by Stolz theorem we have: the limit of the sequence $\{[\sum_{j=\mu}^{n-1}(\Delta y_{j+k+1})(\Delta\varphi_j)]/\varphi_n\}_{n\in N_\mu}$ exists if exists the limit of quotient of differences of numerator and denominator and then these limits are the same. But

$$\lim_{n\to\infty}\Big(\Delta\Big[\sum_{j=\mu}^{n-1}(\Delta y_{j+k+1})(\Delta\varphi_j)\Big]/\Delta\varphi_n\Big)$$
$$= \lim_{n\to\infty}([(\Delta y_{n+k+1})(\Delta\varphi_n)]/\Delta\varphi_n) = \lim_{n\to\infty}\Delta y_{n+k+1} = 0.$$

This means that all the terms on the left hand side tend to zero. On the other hand, by (16iii)

$$\liminf_{n\to\infty}\Big(-y_\mu^q\Big[\sum_{j=\mu}^{n}\varphi_j p_j\Big]/\varphi_n\Big) \le -ay_\mu^q < 0$$

for some $a$. Hence in this case we get contradiction. So our supposition that (I1) possesses eventually positive solution was false. ∎

Remark 5. As a sequence $\{\varphi_n\}$ defined in condition (16iii) we can take for example sequences $\ldots,\{n^3\},\{n^2\},\{n\},\{n^{1/2}\},\{n^{1/3}\},\ldots,\{\log(n)\}$, $\{\log(\log(n))\},\ldots$ each of them give in turn larger class of sequences $p$ for which conditions (16iii) are satisfied. In fact, if we have two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n \le \beta_n$ and $\Delta(\beta_n/\alpha_n) \ge 0$ for sufficiently large $n$ (say for $n \ge m$) then

$$0 < \limsup_{n\to\infty}(1/\beta_n)\sum_{j=m}^{n}\beta_j p_j = \limsup_{n\to\infty}(1/\beta_n)\sum_{j=m}^{n}\alpha_j\frac{\beta_j}{\alpha_j}p_j$$
$$\le \limsup_{n\to\infty}(1/\alpha_n)\sum_{j=m}^{n}\alpha_j p_j.$$

Hence if condition (16iii) is satisfied with the bigger sequence $\beta$ then it also holds with the smaller one $\alpha$.

Remark 6. Similar result for the inequality (I2) can be expressed as follows

Let $p : N \to R_+$ be such that one of the following conditions hold

(i) $\limsup_{n\to\infty} p_n > 0$,

(ii) there exist $K \in N$ and $\alpha > 0$ such that $np_{nK} \ge \alpha$ for all $n \in N$,

(iii) there exists positive, increasing sequence $\{\varphi_n\}$ such that $\lim_{n\to\infty}\varphi_n = \infty$ and $\limsup_{n\to\infty}(1/\varphi_n)\sum_{j=m}^{n}\varphi_j p_j > 0$ for $m \in N$.

Let $q \in R_+$ be a quotient of positive odd integers. Then inequality (I2) does not possess any solution which is eventually negative.

EXAMPLE 2. Consider the equation

$$\Delta^2 y_n + p_n y_n = 0 \tag{E2}$$

where $p : N \to R_+ \setminus \{0\}$. If

$$p_n \le p_{n-1}/[2 + p_{n-1}] \quad \text{for } n \ge 2 \tag{21}$$

and $p_1 < 1/3$ then the equation (E2) possesses positive increasing solution.

To prove the above assertion we show that there exists positive solution of the inequality

$$\Delta^2 x_n + p_n x_n \le 0. \tag{I3}$$

From $p_1 < 1/3$ it follows $(1+p_0)/2 < (1+p_0)(1-p_1)/(1+p_1)$. Since the function $f(x) = (1+p_0)(1+x)/2$ is continuous and increasing for $x \ge 0$, we can find two positive real numbers $x_0$, $x_1$ such that $x_1 > x_0$ and $x_1 \in [f(x_0), (1+p_0)(1-p_1)/(1+p_1)]$. For $n \ge 2$ define the sequence $\{x_n\}_{n \ge 2}$ by

$$x_n = \prod_{j=0}^{n-2}(1+p_j).$$

Of course $x_n > 0$ for all $n \ge 0$. We check that this sequence is a solution of (I3). In fact

$$\Delta^2 x_0 = (1+p_0) - 2x_1 + x_0$$

but

$$x_1 \ge (1+p_0)(1+x_0)/2$$

from there by simple calculation we get

$$(1+p_0) - 2x_1 + x_0 \le -p_0 x_0$$

that is $\Delta^2 x_0 \le -p_0 x_0$. Similarly from

$$x_1 \le (1+p_0)(1-p_1)/(1+p_1)$$

we can easily deduce that

$$(1+p_0)(1+p_1) - 2(1+p_0) + x_1 \le -p_1 x_1$$

but this yields

$$\Delta^2 x_1 = (1+p_0)(1+p_1) - 2(1+p_0) + x_1 \le -p_1 x_1.$$

Furthermore it one can easily check, applying (21), that also for $n \ge 2$ we have $\Delta^2 x_n \le -p_n x_n$. Therefore the sequence $\{x_n\}_{n \in N}$ fulfills inequality (I3).

Hence by Theorem 2 we obtain the desired result about the solution of equation (E2). Condition (21) is satisfied for example for the sequence $p$

defined by $p_n = 3^{-n}$. Of course we can formulate similar result for (E). In the case $k = 0$ suitable assertion holds if for example

$$p_n \le \Big[p_{n-1} \prod_{j=0}^{n-2} (1 + p_j)\Big] \Big[ \prod_{j=0}^{n-1} (1 + p_j) + \prod_{j=0}^{n-2} (1 + p_j)^q \Big]^{-1}.$$

Other results can be obtained using different definitions of the sequence $\{x_n\}$.

Since every solution of (E) is the solution of (I1) and (I2) then as a consequence of Theorem 2, Remark 5, and Remark 6 we have

THEOREM 3. *Let $p : N \to R_+$ be such that one of the following conditions hold*

(i) $\limsup_{n\to\infty} p_n > 0$,

(ii) *there exist $K \in N$ and $\alpha > 0$ such that $np_{nK} \ge \alpha$ for all $n \in N$,*

(iii) *there exists positive, increasing sequence $\{\varphi_n\}$ such that $\lim_{n\to\infty} \varphi_n = \infty$ and $\limsup_{n\to\infty}(1/\varphi_n) \sum_{j=m}^{n} \varphi_j p_j > 0$ for $m \in N$.*

*Then every solution of* (E) *is oscillatory.*

This work is motivated by the paper of Ladas, Philos and Sficas [2], where these authors considered the equation

$$\text{(E3)} \qquad \Delta y_n + p_n y_{n-k} = 0$$

and where they have given sufficient conditions for all solutions of (E2) to be oscillatory, in the form

$$\text{(22)} \qquad \liminf_{n\to\infty} \Big[(1/k) \sum_{j=n-k}^{n-1} p_j\Big] > \frac{k^k}{(k+1)^{k+1}}$$

and stated that this result is sharp (in some sense). Here we want to say that difference equation like (E3) can be treated as the first order equation with constant delay but on the other hand, and sometimes better, it should be treated as the ordinary difference equation of order $k + 1$, and then the methods of higher order equation are more adequate and can give better results. As an example illustrative of our statement let us consider the equation

$$\text{(E4)} \qquad \Delta y_n + p_n y_{n-1} = 0.$$

The condition (22) reduces here to

$$\text{(23)} \qquad \liminf_{n\to\infty} p_n > 1/4.$$

Now, we treat (E4) as the second order difference equation

$$\text{(24)} \qquad y_{n+1} - y_n + p_n y_{n-1} = 0.$$

We shall transform (24) to the form (E). Dividing (24) by $2^{n+1}$ and substituting $y_n = z_n/2^n$ we get after simple transformations

$$\Delta^2 z_{n-1} + (4p_n - 1)z_{n-1} = 0. \tag{25}$$

This equation is of the type (E) and therefore by Theorem 3 we obtain conditions on oscillation of all solutions of (E4) (because solutions of (E4) and (25) are of the same behavior in relation to oscillation)

$$\limsup_{n\to\infty} p_n > 1/4,$$

which is essentially better than (23).

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