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A NOTE ON SUBMAXIMAL SPACES AND *SMPC* FUNCTIONS

1. Introduction

In 1977, Cameron [1] gives one of the fundamental properties of spaces which called submaximality. Also, several weaker types than openness of sets have been defined throughout the last twenty years ago, by a great staff of topologiests. Depending on these new sets the corresponding classes of functions on topological spaces are also obtained and many of their properties are studied. Therefore, in the last year, the concept of strongly M-precontinuous (SMPC) property of functions has been established by Abd El-Monsef et al. [2] depending on the preopenness notion due to Mashhour et al. [3] in 1982. So, we devote this note to study both of submaximal spaces and SMPC functions. So, all useful definitions and preliminaries which will be used throughout our present work contained in the second section, nextly. New studies about submaximality given throughout the part No.(3) of this note. Whenever, in the forth one, we charactrize SMPC using the filterbase concept. The relationships between submaximal space, SMPC function, hereditarily irresolvable subspace, preopen space, weakly precontinuous function and continiuous one are established in the last article which contains an important results throughout the fundamental theorem of our work.

2. Definitions and preliminaries

In this note, given a topological space (X, τ) consisting of a non-empty set X with topology τ of subsets of X. All topological spaces here without any separation properties, whenever such ones are needed its will be explicitly assumed. For any subset A of X, its closure and interior with respect to

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τ will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A is called preopen [3] (semi-open [4]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$ ($A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$). The collection of all preopen, semi-open sets in (X,τ) will be denoted by $PO(X,\tau)$ and $SO(X,\tau)$, respectively. For (X,τ), τ_P means the smallest topology on X containing $PO(X,\tau)$ [5], also, in [6] $\tau^\alpha = PO(X,\tau) \cap SO(X,\tau)$ is a topology on X.

A space (X,τ) is resolvable [7] if there is a dense subset D of X for which its complement is also dense. A space which is not resolvable called irresolvable. Any subset of X is resolvable (irresolvable) if it is resolvable (irresolvable) as a subspace. Also (X,τ) is hereditarily irresolvable [7] if each of its non-empty subsets is irresolvable. (X,τ) is said to be a submaximal space [1] if each of its dense subsets are open.

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be preirresolute [8] (strongly M-precontinuous (SPMC) [2]) if the inverse image of each preopen in (Y,σ) is preopen (open) in (X,τ).

A property P which is preserved under semi-homeomorphism [9] (α-homeomorphism [10]) is called semi-topological [9] (α-topological [11]) property, whenever semi and α-homeomorphisms are defined likewise ordinary one except continuity and openness are replaced by semi and α-corresponding ones.

3. Properties of submaximal spaces

Throughout this note, we give the obviously implication $\tau \subseteq \tau^\alpha \subseteq PO(X,\tau) \subseteq \tau_P$, which is useful in the next result.

PROPOSITION 1. *For any space (X,τ), the space (X,τ_P) is submaximal.*

Proof. Let D be τ_P-dense. Then it is τ-dense and hence $D \in PO(X,\tau)$. Thus $D \in \tau_P$ and therefore (X,τ_P) is submaximal.

THEOREM 1. *Submaximality is preserved by open surjections.*

Proof. If $f : (X,\tau) \to (Y,\sigma)$ is an open surjection and (X,τ) is submaximal and if $D \subseteq Y$ is dense, $f^{-1}(D)$ is dense and hence open in X so that $D = f(f^{-1}(D))$ is open.

COROLLARY 1. *If ΠX_α is submaximal then each X_α is submaximal.*

Proof. The projections are open surjections.

The converse of the result in Corollary 1 need not be held in general as the next example shows.

EXAMPLE 1. Let $X = \{0\} \cup \{\frac{1}{n} : n = 1,2,\ldots\}$ have the usual real subspace topology. then the only proper dense subset of X is $D = \{\frac{1}{n} : n = 1,2,\ldots\}$ which is open so that X is submaximal. But the product space X^2

is not submaximal. For $\{(0,0)\} \cup (D \times D)$ is dense and hence preopen in X^2 but not open.

LEMMA 1 [7]. *(X, τ^α) is submaximal iff (X, τ) contains an open, dense and hereditarily irresolvable subspace.*

Submaximality of (X, τ) satisfies in its finer space (X, τ^α) as a corollary of Proposition 1. While its converse does not hold as the following example illustrates.

EXAMPLE 2. The non-submaximal space (X^2, τ) of Example 1 had an open, dense, discrete and hence hereditarily irresolvable subspace so that (X^2, τ^α) is submaximal. So, in general, (Y, σ^α) submaximal does not imply that (Y, σ) is submaximal.

The following example shows that even though (X, τ^α) is submaximal, (X, τ) may fail to be hereditarily irresolvable.

EXAMPLE 3. Let $X = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\}$ have topology τ. Let each $A \subseteq \{\frac{1}{n} : n = 2, 3, \ldots\}$ be open and if U is open and $0 \in U$ then also $1 \in U$ and $\{\frac{1}{n} : n = 2, \ldots\}$-$U$ is finite. Clearly $\{\frac{1}{n} : n = 2, \ldots\}$ is an open, dense, hereditarily irresolvable subspace so that (X, τ^α) is submaximal whereas the subspace $\{0, 1\}$ is indiscrete and hence resolvable.

THEOREM 2. *Submaximality is a topological property which is not semi-topological.*

Proof. It has been shown recently in [12] that the semitopological properties are precisely the α-topological properties. Since (Y, σ^α) submaximal while (Y, σ) is not submaximal for some space (Y, σ), submaximality is not an α-topological property.

We conclude with an example showing that in general the intersection of two submaximal topologies is not submaximal.

EXAMPLE 4. Let $X = \{0, 0'\} \cup \{1, 2, \ldots\}$ have topologies τ and τ' defined by

$$\begin{aligned} A \in \tau \quad & \text{iff } 0 \in A \to A = X \quad \text{and} \\ A \in \tau' \quad & \text{iff } 0' \in A \to A = X. \end{aligned}$$

Then (X, τ) and (X, τ') are submaximal. In particular, if D is τ-dense in X, either $D = \{0', 1, 2, \ldots\}$ or $D = X$ and in either case $D \in \tau$. Similarly each τ'-dense D is an element of τ'. Finally, as a subspace of $(X, \tau \cap \tau')$, $\{0, 0'\}$ is indiscrete and hence resolvable so that $(X, \tau \cap \tau')$ is not even hereditarily irresolvable and hence not submaximal.

4. Characterizations of SMPC function

DEFINITION 1. A filterbase $\mathbb{F}$ P-converges to x and we write $\mathbb{F} \xrightarrow{P} x$ if for each preopen U with $x \in U$, there is an $F \in \mathbb{F}$ with $F \subseteq U$.

PROPOSITION 2. *If $\mathbb{F}$ is a filterbase in (X,τ), $\mathbb{F} \xrightarrow{P} x$ iff $\mathbb{F} \xrightarrow{\tau_P} x$. i.e. $\mathbb{F}$ P-converges to x iff $\mathbb{F}$ converges to x in (X,τ_P).*

Proof. If $\mathbb{F} \xrightarrow{P} x$ and $W \in \tau_P$ with $x \in W$, there exists $V = \bigcap_{k=1}^{n} A_k$ with each $A_k \in PO(X,\tau)$ and $x \in V \subseteq W$. There exists $F_k \in \mathbb{F}$ with $F_k \subseteq A_k$ for each $k = 1,2,\dots,n$. Since $\mathbb{F}$ is a filterbase, there is an $F \in \mathbb{F}$ with $F \subseteq \bigcap_{k=1}^{n} F_k \subseteq V \subseteq W$. Thus, $\mathbb{F} \xrightarrow{\tau_P} x$. The converse is clear since $PO(X,\tau) \subseteq \tau_P$.

THEOREM 3. *The following are equivalent:*

(i) *$f : (X,\tau) \to (Y,\sigma)$ is SMPC.*

(ii) *$f : (X,\tau) \to (Y,\sigma_P)$ is continuous.*

(iii) *For each filterbase $\mathbb{F}$ on X, $f(\mathbb{F}) \xrightarrow{P} f(x)$ if $\mathbb{F} \to x$.*

(iv) *$f : (X,\tau) \to (Y,\sigma)$ is continuous and $f^{-1}(D) \in \tau$ for each dense $D \subseteq Y$.*

(v) *$f : (X,\tau) \to (Y,\sigma)$ is continuous and $f^{-1}(E)$ is closed if $\operatorname{Int} E = \emptyset$.*

Proof. It has been shown that (i) and (ii) are equivalent and it is clear that (ii) and (iii) are equivalent in view of the previous proposition. Clearly (iv) and (v) are equivalent. Also (i)→(iv) since dense sets are preopen. It remains only to show that (iv) implies (i). Let $B \in PO(Y < \sigma)$. Then $B = U \cap D$ for some $U \in \sigma$ and dense $D \subseteq Y$. Since $f^{-1}(D) \in \tau$, and f is continuous, $f^{-1}(B) = f^{-1}(U) \cap f^{-1}(D) \in \tau$ showing that f is SMPC if (iv) holds.

Since $\tau \subseteq \tau^\alpha \subseteq PO(X,\tau) \subseteq \tau_P$, we have the next result.

PROPOSITION 3.

(1) *$f : (X,\tau) \to (Y,\sigma)$ is SMPC*

→ (2) *$f : (X,\tau) \to (Y,\sigma^\alpha)$ is continuous*

→ (3) *$f : (X,\tau^\alpha) \to (Y,\sigma^\alpha)$ is continuous.*

Functions $f : (X,\tau) \to (Y,\sigma)$ satisfying (2) above are called strongly α-irresolute in [11]. Functions satisfying (3) above are called α-iresolute functions (or α-functions in [13]).

The following examples show that in general the implications of the above proposition are irreversible.

EXAMPLES 5. If $\tau \neq \tau^\alpha$, the identity function $l_X : (X,\tau^\alpha) \to (X,\tau^\alpha)$ is continuous whereas $l_X : (X,\tau) \to (X,\tau^\alpha)$ is not continuous showing that strong α-irresoluteness is not implied by α-irresoluteness. Now let (R,σ) be the usual space of real numbers. Then (R, σ^α) is connected since for any space (X, τ), (X, τ) and (X, τ^α) share the same clopen (closed and open) subsets. Further, (R, σ) is resolvable so that no non-constant function from (R, σ^α) to (R, σ) can be SMPC. However, the identify $l_R : (R, \sigma^\alpha) \to (R, \sigma)$ is strongly α-irresolute and not SMPC.

5. Connections between SMPC and submaximality

We now turn our attention toward characterizing submaximality and finding some basic properties concerning submaximal spaces.

DEFINITION 2. A space (X, τ) is a preopen space if $\tau = \tau_P$.

PROPOSITION 3. *(X, τ) is a preopen space iff $\tau = PO(X, \tau)$.*

Proof. $\tau = \tau_P \to \tau \subseteq PO(X, \tau) \subseteq \tau_P \subseteq \tau \to \tau = PO(X, \tau)$. Conversely, if $\tau = PO(X, \tau)$, then $PO(X, \tau)$ is a topology so that $\tau_P = PO(X, \tau) = \tau$.

DEFINITION 3. A function $f : (X, \tau) \to (Y, \sigma)$ is weakly precontinuous if $f : (X, \tau_P) \to (Y, \sigma)$ is continuous.

Clearly because $PO(X, \tau) \subseteq \tau_P$, precontinuity implies weak precontinuity. The next example shows that the converse does not hold.

EXAMPLE 6. Let (R, σ) be the usual space of real numbers and let $l_R : (R, \sigma) \to (R, \sigma_P)$ be the identity function. Because (R, σ) is resolvable, σ_P is the discrete topology. Also $PO(R, \sigma) \neq \sigma_P$ since nonempty nowhere dense sets exist in (R, σ) which are not preopen. Thus, l_R is weakly precontinuous but not precontinuous.

Several of submaximality equivalents with SMPC and many of previous topological concepts are established throughout the following fundamental theorem in this note.

THEOREM 4. *The following are equivalent:*

(i) *(Y, σ) is submaximal.*
(ii) *For every space (X, τ), each continuous $f : (X, \tau) \to (Y, \sigma)$ is SMPC.*
(iii) *The identity function $l_Y : (Y, \sigma) \to (Y, \sigma)$ is SMPC.*
(iv) *$\sigma = PO(Y, \sigma)$.*
(v) *For all $B \subseteq Y$, $B - \operatorname{Int} B \neq \emptyset \to B - \operatorname{Int} \operatorname{Cl} B \neq \emptyset$.*
(vi) *(Y, σ) is a preopen space.*
(vii) *For every space (X, τ), $h : (X, \tau) \cong (Y, \sigma)$ iff $h : (X, \tau) \to (Y, \sigma)$ and $h^{-1} : (Y, \sigma) \to (X, \tau)$ are SMPC.*
(viii) *Every weakly precontinuous $g : (Y, \sigma) \to (Z, \rho)$ is continuous.*
(ix) *For every continuous $f : (X, \tau) \to (Y, \sigma)$, $f^{-1}(E)$ is closed whenever $\operatorname{Int} E = \emptyset$.*

(x) $\sigma = \{U - E : U \in \sigma$ *and* $\operatorname{Int} E = \emptyset\}$.
$= \{U \cap D : U \in \sigma$ *and* $\operatorname{Cl} D = Y\}$.

(xi) *There exists an open, dense, hereditarily irresolvable subspace* $D \subseteq Y$ *and* $\sigma = \sigma^\alpha$.

(xii) $PO(Y,\sigma) \subseteq SO(Y,\sigma)$ *and* $\sigma = \sigma^\alpha$.

(xiii) $A \subseteq Y$ *is nowhere dense iff* $\operatorname{Int} A = \emptyset$, *and* $\sigma = \sigma^\alpha$.

(xiv) (Y,σ^α) *is submaximal and* $\sigma = \sigma^\alpha$.

Proof. If (Y,σ) is submaximal and $f : (X,\tau) \to (Y,\sigma)$ is continuous then f is SMPC by Theorem 3.1 in [2]. So (i)→(ii). Clearly (ii)→(iii) and (iii)→(iv) and since (iv) is equivalent to $PO(Y,\sigma) \subseteq \sigma$, (iv)↔(v). We now show that (iv)→(i). Suppose that $\sigma = PO(Y,\sigma)$ and let $D \subseteq Y$ be dense. Since every dense set is preopen, $D \in \sigma$ so that (Y,σ) is submaximal. By Proposition 3 above, (iv)↔(vi). Now (i)↔(vii) since if (Y,σ) is submaximal and $h : (X,\tau) \cong (Y,\sigma)$ then (X,τ) is submaximal and by (ii), h and h^{-1} are each SMPC. Also, if (Y,σ) is submaximal and h and h^{-1} are each SMPC, then clearly h is a homeomorphism so that (i)→(vii). For the converse, (vii)→(iii)→(i). To see that (i)↔(viii), let (Y,σ) be submaximal and let $g : (Y,\sigma) \to (Z,\rho)$ be weakly precontinuous. Then $W \in \rho \to g^{-1}(W) \in \sigma_P = \sigma$ so that g is continuous. Conversely, if (viii) holds, then the identity $l_Y : (Y,\sigma) \to (Y,\sigma_P)$ is weakly precontinuous and hence continuous so that $\sigma_P \subseteq \sigma$ and (Y,σ) is a preopen space and hence submaximal. By Theorem 3 previously (ix)↔(ii) and clearly (x)↔(iv) by Proposition 1 of [7]. Since (i)→ $\sigma = \sigma^\alpha$ and by Proposition 1 by [14] (Y,σ^α) is submaximal and thus, $\sigma^\alpha = PO(Y,\sigma^\alpha) = PO(Y,\sigma) = \sigma$. Therefore, the equivalences of (i) with each of (xi), (xii), (xiii), and (xiv) follow from Theorems 2 and 4 of [7].

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