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ON PRINCIPAL MAPS OF THE PLANE

The concept of a principal map of the plane is strictly connected with that of a principal line. They were introduced in paper [1] which is devoted to the study of product final differential structures on the plane. These concepts are useful in the formulating of some properties of such structures (see [1], Corollaries 4.2 and 4.8), however, they are directly defined. In the present paper we treat principal maps of the plane $\mathbb{R}^2$ in a way independent of product final differential structures, that is, we carry out our considerations in the sense of classical geometry. To be precise, we also introduce more general quasi principal maps which, however, are not considered in detail. Give attention that the major part of this paper is devoted to the investigation of principal maps of the plane without any assumption concerning differentiability or even continuity. In this paper we present some characteristic properties of such maps (Theorems 2.9, 2.16 and 2.19) and give a full description of them (Theorems 3.1 and 3.9). Moreover, we also obtain results (Propositions 3.5 and 3.6) which are close to those of papers [1] (Proposition 4.9) and [2] (Propositions 2.24 and 2.25). Our language for principal maps is a continuation of that used in the abovementioned papers and it is fully presented here unless it concerns adduced results.

1. Preliminaries

First, if necessary, we shall regard the plane $\mathbb{R}^2$ and any of its subsets as topological spaces under the Euclidean topology. A *vertical* (*horizontal*) line in $\mathbb{R}^2$ is a straight line of the form $V_a = \{a\} \times \mathbb{R}$ ($H_b = \mathbb{R} \times \{b\}$) for some $a \in \mathbb{R}$ ($b \in \mathbb{R}$). By a *principal* line L in $\mathbb{R}^2$ we shall mean a vertical or horizontal one. For $X \in \{V, H, P\}$ we say that L is an X-*principal* line provided that it is vertical if $X = V$, horizontal if $X = H$, and principal if $X = P$. By an X-*principal* segment in $\mathbb{R}^2$ we shall mean a segment of an X-principal line, that is, a bounded connected subspace of it. In particular, every closed X-principal segment is compact. A P-principal segment will also be called *principal*. By a *principal K-graph* we shall mean a compact connected subspace of $\mathbb{R}^2$ which can be expressed as a finite union of closed principal segments. In turn, by a *principal cross* we shall mean a subset K of $\mathbb{R}^2$ of the form $\mathbb{K}_p = V_a \cup H_b$ where $p = (a, b)$ is called the *origin* of K. The principal cross $\mathbb{K} = \mathbb{K}_o$ with origin $o = (0,0)$ will also be called the *central principal* one. A subset A of $\mathbb{R}^2$ is said to be *locally K-subordinate* if for each $p \in A$ there are a neighbourhood U of p in $\mathbb{R}^2$ and a principal cross K such that $A \cap U \subseteq K$. It is easily seen that a compact connected subspace G of $\mathbb{R}^2$ is a principal K-graph if and only if G is locally K-subordinate (compare [2], the corresponding equivalent definition).

Let $X, Y \in \{V, H, P\}$ and $p \in \mathbb{R}^2$. A map f of $\mathbb{R}^2$ is said to be *locally (X, Y)-principal* at p in case there is an open neighbourhood U of p in $\mathbb{R}^2$ such that $f(U \cap L)$ is contained in a Y-principal line for each X-principal line L passing through p. In turn, we say that f is (X,Y)-*principal* if it transforms any X-principal line to a Y-principal one. Clearly, every such f is locally (X, Y)-principal at each point of $\mathbb{R}^2$. If f is (X, X)-principal (locally (X, X)-principal at p), we call shortly it X-*principal* (*locally X-principal at* p). Moreover, a P-principal (locally P-principal at p) map of $\mathbb{R}^2$ will also be called *principal* (*locally principal at* p). Clearly, every principal map of $\mathbb{R}^2$ transforms any principal line to a principal line. In turn, we say that a map f of $\mathbb{R}^2$ is *quasi principal* if for each principal segment S in $\mathbb{R}^2$ the image $f(S)$ is locally K-subordinate. Note that if in addition f is continuous, then it transforms any principal K-graph to a principal K-graph. Obviously, every principal map of $\mathbb{R}^2$ is quasi principal.

1.1. EXAMPLE. Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a smooth map such that $\vartheta(x) = 0$ for $x \leq 0$, $0 < \vartheta(x) < 1$ for $0 < x < 1$ and $\vartheta(x) = 1$ for $x \geq 1$. Define the map f of $\mathbb{R}^2$ by $f(x, y) = (\vartheta(x), y + \vartheta(-x))$. It is seen that $f(V_x) = V_{\vartheta(x)}$ for each $x \in \mathbb{R}$, and so, f is V-principal. On the other hand, $f(H_y) = \{0\} \times [y, y + 1] \cup [0, 1] \times \{y\}$ for each $y \in \mathbb{R}$, which means that f is not (H, P)-principal, and so, not principal. However, note that for any horizontal line H_y the map f is locally principal at each point of H_y except the point $(0, y)$. In turn, observe that f is smooth and quasi principal, which implies that f transforms any principal K-graph to a principal K-graph. Moreover, it turns out that f can be regarded as a smooth map from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_l \mathbb{R}$ where $k, l \in \{1, 2\}$ and $(k, l) \neq (2, 1)$ (see [1] and [2]). Indeed, since f is a smooth map of $\mathbb{R}^2$, we have $f \in \mathcal{F}^{k,l}$. Then by condition (d) of Proposition 2.24 from paper [2], we conclude that $f \in \mathcal{S}^{k,l}$, that is, f is a smooth map from $\mathbb{R} \times_k \mathbb{R}$ to $\mathbb{R} \times_l \mathbb{R}$. ∎

In paper [2] is given a characterization of a smooth map $f : \mathbb{R} \times_k \mathbb{R} \to \mathbb{R} \times_l \mathbb{R}$ where $k, l \in \{1,2\}$ as a continuous one for the corresponding Sikorski topologies such that $f|A : A \to f(A)$ is a smooth map of C^∞ subsets of $\mathbb{R} \times_k \mathbb{R}$ and $\mathbb{R} \times_l \mathbb{R}$, respectively (see [2], Proposition 2.25). Observe that such a map f has to be quasi principal because it transforms any principal K-graph to a principal K-graph (see [2], Lemma 2.23 and Proposition 2.25), however, it need not be principal (Example 1.1). Thus it is of interest to know more about quasi principal maps of $\mathbb{R}^2$ and, as the first step in this direction, we shall study here principal maps of $\mathbb{R}^2$. This is also justified by the fact that quasi principal maps can be of very complicated structure in contrast to much simpler principal ones.

2. Characteristic properties

We say that points $p, q \in \mathbb{R}^2$ are *coprincipal* if they are contained in some principal line in $\mathbb{R}^2$. It is easy to verify

2.1. LEMMA. *A subset A of $\mathbb{R}^2$ is contained in some principal line if and only if any two points of A are coprincipal.* ■

2.2. PROPOSITION. *A map f of $\mathbb{R}^2$ is principal if and only if $f(\mathbb{K}_p) \subseteq \mathbb{K}_{f(p)}$ for each $p \in \mathbb{R}^2$.*

Proof. The necessity is obvious. To prove the sufficiency, suppose to the contrary that there is a principal line L such that $f(L)$ is not contained in any principal line. By Lemma 2.1 there are points $p, q \in L$ such that $f(p)$ and $f(q)$ are not coprincipal, which implies that neither $f(p)$ nor $f(q)$ belong to $\mathbb{K}_{f(p)} \cap \mathbb{K}_{f(q)}$. On the other hand, we have $f(p), f(q) \in f(L) \subseteq \mathbb{K}_{f(p)} \cap \mathbb{K}_{f(q)}$ because $L \subseteq \mathbb{K}_p \cap \mathbb{K}_q$, a contradiction. ■

Denote by $\mathfrak{P}$ the class of all principal lines in $\mathbb{R}^2$. We have the relation $\|$ of *parallelism* in $\mathfrak{P}$ for which the notation $P \| Q$ means that the principal lines P and Q are parallel. This is an equivalence relation which divides $\mathfrak{P}$ for two subclasses $\mathfrak{P}_V$ and $\mathfrak{P}_H$ consisting of all vertical and horizontal lines in $\mathbb{R}^2$, respectively. By a *principal class* we shall further mean a nonempty subset $\mathfrak{A}$ of $\mathfrak{P}$. We say that $\mathfrak{A}$ is a *principal class* of *parallelism* if for any $P, Q \in \mathfrak{A}$ we have $P \| Q$. A principal class $\mathfrak{A}$ of parallelism is called *maximal* if it is maximal under the inclusion relation for all such classes, or equivalently, if $\bigcup \mathfrak{A} = \bigcup \{P : P \in \mathfrak{A}\} = \mathbb{R}^2$. It is seen that $\mathfrak{P}_V$ and $\mathfrak{P}_H$ are unique maximal principal classes of parallelism. For any principal class $\mathfrak{A}$ we set

$$\mathfrak{A}^{\|} = \{Q \in \mathfrak{P} : \forall P \in \mathfrak{A}\ Q \| P\}.$$

Clearly, $\mathfrak{A}^{\|}$ is a unique maximal principal class of parallelism containing $\mathfrak{A}$ in the case when $\mathfrak{A}$ is a principal class of parallelism and $\mathfrak{A}^{\|}$ is the empty set otherwise. In particular, for every $P \in \mathfrak{P}$ the set $P^{\|} = \{Q \in \mathfrak{P} : Q \,\|\, P\}$ is a unique maximal principal class of parallelism containing P, i.e. $P^{\|} = \mathfrak{P}_V$ if P is vertical and $P^{\|} = \mathfrak{P}_H$ if P is horizontal.

Let P and Q be principal lines in $\mathbb{R}^2$. It is seen that if P and Q are parallel, then $P = Q$ or $P \cap Q = \emptyset$. In turn, if P and Q are not parallel, then they are orthogonal, which is written as $P \perp$ Q. Moreover, in the latter case P and Q belong to distinct maximal principal classes of parallelism and $P \cap Q$ is a one-point set. For any principal class $\mathfrak{A}$ we set

$$\mathfrak{A}^{\perp} = \{Q \in \mathfrak{P} : \forall P \in \mathfrak{A}\; Q \perp P\}.$$

Clearly, $\mathfrak{A}^{\perp}$ is a maximal principal class of parallelism in the case when $\mathfrak{A}$ is a principal class of parallelism and $\mathfrak{A}^{\perp}$ is the empty set otherwise. We say that principal classes $\mathfrak{A}$ and $\mathfrak{B}$ are *orthogonal* if $P \perp Q$ for any $P \in \mathfrak{A}$ and $Q \in \mathfrak{B}$, or equivalently, $\mathfrak{A} \subseteq \mathfrak{B}^{\perp}$ ($\mathfrak{B} \subseteq \mathfrak{A}^{\perp}$). Clearly, in this case $\mathfrak{A}$ and $\mathfrak{B}$ are principal classes of parallelism. Note that if $\mathfrak{A}$ and $\mathfrak{B}$ are orthogonal principal classes, then so are $\mathfrak{A}^{\|}$ and $\mathfrak{B}^{\|}$ as well as $\mathfrak{A}^{\perp}$ and $\mathfrak{B}^{\perp}$, moreover, we have $\mathfrak{A}^{\|} = \mathfrak{B}^{\perp}$ and $\mathfrak{B}^{\|} = \mathfrak{A}^{\perp}$. It is seen that for any principal class $\mathfrak{A}$ of parallelism the set $\mathfrak{A}^{\perp}$ is a unique maximal one orthogonal to $\mathfrak{A}^{\|}$, i.e. $\mathfrak{A}^{\perp} = (\mathfrak{A}^{\|})^{\perp}$ ($(\mathfrak{A}^{\perp})^{\perp} = \mathfrak{A}^{\|}$).

Let f be a principal map of $\mathbb{R}^2$. We say that f *preserves parallelism* (*orthogonality*) in case for any parallel (orthogonal) $P, Q \in \mathfrak{P}$ there are parallel (orthogonal) $P', Q' \in \mathfrak{P}$ such that $f(P) \subseteq P'$ and $f(Q) \subseteq Q'$. Moreover, we say that f *strictly preserves parallelism* (*orthogonality*) in case for any parallel (orthogonal) $P, Q \in \mathfrak{P}$ the conditions $f(P) \subseteq P'$ and $f(Q) \subseteq Q'$ where $P', Q' \in \mathfrak{P}$ involve that P' and Q' are parallel (orthogonal). Give attention that the notions of the preserving of parallelism (orthogonality) introduced above, are formulated only for principal maps and only for $\mathfrak{P}$. This means that if f is a principal map of $\mathbb{R}^2$ preserving parallelism (orthogonality) in our "principal" sense, then it need not preserve parallelism (orthogonality) in the usual one, that is, for arbitrary parallel (orthogonal) straight lines, however, the converse assertion is true.

A principal map f of $\mathbb{R}^2$ is called *contracting* for $P \in \mathfrak{P}$ (for $\mathfrak{A} \subseteq \mathfrak{P}$) if $\operatorname{card} f(P) = 1$ ($\operatorname{card} f(P) = 1$ for each $P \in \mathfrak{A}$). For example, note that the projections $p_1 : (x, y) \mapsto (x, 0)$ and $p_2 : (x, y) \mapsto (0, y)$ are principal maps of $\mathbb{R}^2$ preserving parallelism (orthogonality) but not strictly. Moreover, it is seen that p_1 and p_2 are contracting for all vertical and horizontal lines, respectively. If f is not contracting for all $P \in \mathfrak{P}$, that is, card $f(P) \geq 2$ for each $P \in \mathfrak{P}$, we call it *totally P-noncontracting*. For example, note that every injective principal map of $\mathbb{R}^2$ is totally P-noncontracting. If f is a totally P-noncontracting principal map of $\mathbb{R}^2$, then for any $P \in \mathfrak{P}$ there exists a unique principal line P_f in $\mathbb{R}^2$, denoted sometimes by $(\mathrm{P})_f$, such that $f(P) \subseteq P_f$. In this case we have thus defined the assignment $P \mapsto P_f$ from $\mathfrak{P}$ to itself, called the *principal* one of f. Of course, such f strictly preserves parallelism (orthogonality) if and only if for any $P, Q \in \mathfrak{P}$ the condition $P \parallel Q$ ($P \perp Q$) involves $P_f \parallel Q_f$ ($P_f \perp Q_f$). It is easy to verify

2.3. Lemma. *Every principal map of $\mathbb{R}^2$ strictly preserving parallelism (orthogonality) is totally P-noncontracting.* ∎

The following example shows that a totally P-noncontracting principal map of $\mathbb{R}^2$ need not preserve parallelism or orthogonality.

2.4. Example. Let f be a map of $\mathbb{R}^2$ given by $f(x,y) = (x,y)$ if $(x,y) \in \mathbb{K}$ and $f(x,y) = (0,0)$ otherwise. It is seen that f is a totally P-noncontracting principal map. Moreover, observe that f does not preserve parallelism (orthogonality). Indeed, note that for horizontal lines H_0 and H_1 we have $(H_0)_f = H_0$ and $(H_1)_f = V_0$, so f does not preserve parallelism. Similarly, for orthogonal principal lines H_1 and V_0 we have $(H_1)_f = V_0$ and $(V_0)_f = V_0$, so f does not preserve orthogonality. ∎

2.5. Proposition. *Let f be a principal map of $\mathbb{R}^2$. Then the following conditions are equivalent*:

(a) *f is contracting, i.e. constant*;

(b) *f is contracting for all $P \in \mathfrak{P}$*;

(c) *f is contracting for a maximal principal class $\mathfrak{A}$ of parallelism and for at least one $P \in \mathfrak{A}^{\perp}$.*

Proof. Implications (a)⇒(b) and (b)⇒(c) are obvious. To prove implication (c)⇒(a), observe first that for an arbitrary $x \in \mathbb{R}^2$ there is $Q_x \in \mathfrak{A}$ such that $x \in Q_x$ and $Q_x \cap P = \{x'\}$. Since f is contracting for Q_x and P, we get $f(x) = f(x') = f(p)$ where p is a fixed point of P, so f is constant. ∎

The following example shows that condition (c) of this proposition is minimal in a certain sense.

2.6. Example. Let f be a principal map of $\mathbb{R}^2$ given as follows: $f(x,y) = (0,0)$ if $y \neq 0$ and $f(x,0) = (x,0)$ otherwise. It is seen that f is contracting for each horizontal line H_y with $y \neq 0$ and for the vertical line V_0. On the other hand, f is noncontracting for each vertical line V_x with $x \neq 0$ and for the horizontal line H_0. ∎

By an easy verification we get

2.7. LEMMA. *Every principal map of* $\mathbb{R}^2$ (*strictly*) *preserving orthogonality* (*strictly*) *preserves parallelism.* ∎

The next example shows that a principal map of $\mathbb{R}^2$ strictly preserving parallelism need not preserve orthogonality.

2.8. EXAMPLE. Since the sets $\mathbb{R}^2$ and $\mathbb{R}$ are equipotent, one has a bijective map f from $\mathbb{R}^2$ onto the vertical line V_0. Obviously, f is a principal map of $\mathbb{R}^2$ strictly preserving parallelism but not preserving orthogonality. ∎

2.9. THEOREM. *Let f be a principal map of $\mathbb{R}^2$. If $f(\mathbb{R}^2)$ is not contained in any principal cross, then f strictly preserves parallelism* (*orthogonality*).

Proof. First, we prove that f strictly preserves parallelism. Indeed, suppose to the contrary that there are principal lines P, Q, P' and Q' in $\mathbb{R}^2$ such that $P \| Q$, $f(P) \subseteq P'$, $f(Q) \subseteq Q'$ and $P' \perp Q'$. Since P' and Q' are orthogonal, we infer that $P' \cap Q' = \{a\}$ and the set $K = P' \cup Q'$ is a principal cross in $\mathbb{R}^2$ with origin a. Let x be an arbitrary point of $\mathbb{R}^2$. Since P and Q are parallel principal lines in $\mathbb{R}^2$, there is a unique principal one R such that $x \in R$ and R is orthogonal to both P and Q. Clearly, we have $R \cap P = \{p\}$ and $R \cap Q = \{q\}$, and so, $f(p) \in f(R) \cap f(P) \subseteq R' \cap P'$ and $f(q) \in f(R) \cap f(Q) \subseteq R' \cap Q'$ where R' is a principal line in $\mathbb{R}^2$ such that $f(R) \subseteq R'$. Therefore since $P' \perp Q'$, it follows that R' has to be parallel to either P' or Q'. Next, since $R' \cap P' \neq \emptyset$ and $R' \cap Q' \neq \emptyset$, we conclude that either $R' = P'$ or $R' = Q'$, so $R' \subseteq$ K. In particular, this means that $f(x) \in K$, which implies that $f(\mathbb{R}^2) \subseteq K$ because x can be an arbitrary point of $\mathbb{R}^2$, a contradiction.

Next, we prove that f strictly preserves orthogonality. By the first part of our proof f is strictly preserving parallelism. Therefore from Lemma 2.3 it follows that it is totally P-noncontracting, and so, f determines the principal assignment $P \mapsto P_f$. Let $\mathfrak{A}$ be a maximal principal class of parallelism. Since f strictly preserves parallelism, we conclude that $\mathfrak{A}_f = \{P_f : P \in \mathfrak{P}\}$ is a principal class of parallelism not maximal in general. Moreover, since $f(\mathbb{R}^2)$ is not contained in any principal line, one can find $P, P' \in \mathfrak{A}$ such that $P_f \neq P'_f$, for otherwise the image $f(\mathbb{R}^2) = f(\bigcup \mathfrak{A}) \subseteq \bigcup\{P_f : P \in \mathfrak{A}$ would be contained in a principal line. Let us take $Q \in \mathfrak{A}^\perp$. Of course, we have $Q \cap P \neq \emptyset$ and $Q \cap P' \neq \emptyset$, which implies $Q_f \cap P_f \neq \emptyset$ and $Q_f \cap P'_f \neq \emptyset$. Observe further that $Q_f \notin \mathfrak{A}^{\|}_f$, for otherwise we have

$Q \parallel P_f \parallel P'_f$, which together with the previous conclusion imply $Q_f = P = P'_f$, a contradiction. This means that $Q \in \mathfrak{A}_f^{\perp}$, and so, $Q_f \perp P_f$. Let now M and N be orthogonal principal lines in $\mathbb{R}^2$. We can thus assume that $M \parallel P$ and $N \parallel Q$ because $P \perp Q$. Since f strictly preserves parallelism, we have $M_f \parallel P_f$ and $N_f \parallel Q_f$. Furthermore, since $P_f \perp Q_f$, we conclude that $M_f \perp N_f$. Consequently, f strictly preserves orthogonality. ∎

Let f be a principal map of $\mathbb{R}^2$. We call f *epiprincipal* for $P \in \mathfrak{P}$, if it is noncontracting for P and such that $f(P) = P_f$. By an *epiprincipal* map of $\mathbb{R}^2$ we shall mean a principal one which is totally P-noncontracting and epiprincipal for each $P \in \mathfrak{P}$. Clearly, the composition of epiprincipal maps of $\mathbb{R}^2$ is again an epiprincipal one. The following example shows that if f is an epiprincipal map of $\mathbb{R}^2$, then the image $f(\mathbb{R}^2)$ can be contained in a principal line.

2.10. EXAMPLE. Let f be the map of $\mathbb{R}^2$ given by the assignment $(x, y) \mapsto (x + y, 0)$. It is seen that f is a principal map of $\mathbb{R}^2$ such that $f(\mathbb{R}^2) = H_0$. Moreover, for any $P \in \mathfrak{P}$ we have $f(P) = H_0$, so f is epiprincipal. ∎

By definition and Theorem 2.9 we obviously get the following corollaries:

2.11. COROLLARY. *If f is an epiprincipal map of $\mathbb{R}^2$ strictly preserving orthogonality, then $f(\mathbb{K}_p) = \mathbb{K}_{f(p)}$ for each $p \in \mathbb{R}^2$. In particular, this is satisfied if f is an epiprincipal map such that $f(\mathbb{R}^2)$ is not contained in any principal cross.* ∎

2.12. COROLLARY. *Let f be a principal map of $\mathbb{R}^2$ strictly preserving orthogonality. Then f is epiprincipal if and only if $f(\mathbb{K}_p) = \mathbb{K}_{f(p)}$ for each $p \in \mathbb{R}^2$.* ∎

2.13. PROPOSITION. *Every surjective principal map of $\mathbb{R}^2$ is epiprincipal.*

Proof. Let f be a principal map of $\mathbb{R}^2$ which is surjective, i.e. $f(\mathbb{R}^2) = \mathbb{R}^2$. Suppose to the contrary that there is a principal line P in $\mathbb{R}^2$ such that $f(P) \subset P_f$. Let us take a point $y \in P_f \setminus f(P)$. Since f is surjective, it follows that $f(x) = y$ for some $x \in \mathbb{R}^2$. Let V_x and H_x denote the vertical and horizontal lines passing through x in $\mathbb{R}^2$, respectively. Since f preserves orthogonality by Theorem 2.9, we infer that either $(V_x)_f$ or $(H_x)_f$ is parallel to P_f, or equivalently, that there is a principal line Q in $\mathbb{R}^2$ such that $x \in Q$ and $Q_f \parallel P_f$. Hence we get $P_f = Q_f$ because $y \in P_f \cap Q_f$. Observe further that $P \neq Q$ because $x \in Q \setminus P$. Moreover, note that $P \cap Q = \emptyset$, for otherwise $P \perp Q$, and so, $P_f \perp Q_f$ by Theorem 2.9, which contradicts $P_f = Q_f$. In particular, we have $P \parallel Q$. It follows that there is a principal line R in $\mathbb{R}^2$ passing through x and orthogonal to both P and Q. Clearly, we have $R \cap Q = \{x\}$ and $R \cap P = \{\overline{x}\}$ where $\overline{x}$ is a point of $\mathbb{R}^2$ distinct from x. Moreover, note that $y = f(x)$ and $\overline{y} = f(\overline{x})$ are distinct points of $\mathbb{R}^2$ because $y \in P_f \setminus f(P)$. Consequently, we get $y, \overline{y} \in R_f \cap P_f = R_f \cap Q_f$. Thus, we conclude that $R_f = Q_f = P_f$ because every straight line is defined by any two distinct points of it. On the other hand, $R \perp P$ and $R \perp Q$, and so, $R_f \perp P_f$ and $R_f \perp Q_f$ because f strictly preserves orthogonality, a contradiction. ∎

The following example shows that if f is a principal map of $\mathbb{R}^2$ such that $f(\mathbb{R}^2)$ is not contained in any principal cross, then f need not be epiprincipal.

2.14. EXAMPLE. Let $\varepsilon > 0$. Consider the map f of $\mathbb{R}^2$ defined by $f(x,y) = (x', y)$ where x' is unique such that $0 \leq x' < \varepsilon$ and $x = n\varepsilon + x'$ for some $n \in \mathbb{Z}$. Clearly, f is a principal map of $\mathbb{R}^2$. Moreover, note that $f(V_x) = V_{x'}$ for $x \in \mathbb{R}$, and so, f is epiprincipal for vertical lines. On the other hand, we have $f(H_y) = [0, \varepsilon) \times \{y\}$ for $y \in \mathbb{R}$, which means that f is not epiprincipal for horizontal lines. ∎

By applying Proposition 2.13 we obviously get

2.15. COROLLARY. *If f is a bijective principal map of $\mathbb{R}^2$, then so is the inverse map of it. In particular, f transforms every principal line onto a principal one.* ∎

2.16. THEOREM. *Let f be an injective principal map of $\mathbb{R}^2$. If $f(\mathbb{R}^2)$ is not contained in any principal line, then the principal assignment $P \mapsto P_f$ is one-to-one and f strictly preserves parallelism (orthogonality).*

Proof. First, we prove that the following statement holds:

(1) *If $P, Q \in \mathfrak{P}$ and $P \cap Q = \emptyset$, then $P_f \neq Q_f$.*

Indeed, suppose to the contrary that there are principal lines P and Q in $\mathbb{R}^2$ such that $P \cap Q = \emptyset$ and $P_f = Q_f$. Clearly, in this case P and Q are parallel. Moreover, for an arbitrary point $x \in \mathbb{R}^2$ there is a principal line R in $\mathbb{R}^2$ such that $x \in R$ and R is orthogonal to both P and Q. This means that $R \cap P = \{p\}$ and $R \cap Q = \{q\}$ where p and q are distinct points of $\mathbb{R}^2$. Thus we have $f(p) \in R_f \cap P_f$ and $f(q) \in R_f \cap Q_f$, which implies $f(p), f(q) \in R_f$ and $f(p), f(q) \in P_f = Q_f$ where $f(p) \neq f(q)$ because f is injective. Since every straight line is defined by any two distinct points of it, we conclude that $R_f = P_f = Q_f$. Thus we get $f(x) \in f(R) \subseteq R_f = P_f$, and so, $f(\mathbb{R}^2) \subseteq P_f$ because x can be an arbitrary point of $\mathbb{R}^2$, a contradiction. This completes the proof of statement (1).

Next, we prove that f strictly preserves orthogonality, that is, the following statement holds:

(2) *If $P, Q \in \mathfrak{P}$ and $P \perp Q$, then $P_f \perp Q_f$.*

Indeed, suppose to the contrary that there are principal lines P and Q such that $P \perp Q$ but P_f and Q_f are not orthogonal, i.e. $P_f \parallel Q_f$. Moreover, we have $P \cap Q = \{a\}$, and so, $f(a) \in P_f \cap Q_f$, which implies that $P_f = Q_f$. Let us take any principal line P' in $\mathbb{R}^2$ such that $P' \cap P = \emptyset$. Clearly, P' is parallel to P, and so, it is orthogonal to Q, which implies that $P' \cap Q = \{q\}$ for $q \neq a$. Hence we get $f(q) \in P'_f \cap Q_f = P'_f \cap P_f$. Since $P' \cap P = \emptyset$, it follows from statement (1) that $P'_f \neq P_f$, and so, $P'_f \cap P_f = \{f(q)\}$, i.e. $P'_f \perp P_f$. Similarly, we can take a principal line Q' in $\mathbb{R}^2$ such that $Q' \cap Q = \emptyset$, $Q' \cap P = \{p\}$ for $p \neq a$ and $Q'_f \cap Q_f = \{f(p)\}$, i.e. $Q'_f \perp Q_f$. Since $P_f = Q_f$, we obviously have $P'_f, Q'_f \in \{P_f, Q_f\}^{\perp}$, and so, $P'_f \parallel Q'_f$. Clearly, P' and Q' are orthogonal, and so, $P' \cap Q' = \{a'\}$ where $a' \in \mathbb{R}^2 \setminus (P \cup Q)$. Therefore $f(a') \in P'_f \cap Q'_f$ and since $P'_f \parallel Q'_f$, it follows that $P'_f = Q'_f$. Next, since $P_f = Q_f$, we get $P'_f \cap P_f = Q'_f \cap Q_f$. On the other hand, $f(p) \neq f(q)$ because $p \in P \setminus Q$, $q \in Q \setminus P$ and f is injective. Moreover, we have $f(p) \in P'_f \cap P_f$ and $f(q) \in Q'_f \cap Q_f$. Thus since $P'_f \cap P_f = Q'_f \cap Q_f$, we conclude that each principal line P'_f, P_f, Q'_f and Q_f contains the same distinct points $f(p)$ and $f(q)$, which implies $P'_f = P_f = Q'_f = Q_f$. But this contradicts the fact that $P'_f \perp P_f$ and $Q'_f \perp Q_f$. Consequently, we have proved statement (2).

Now, observe that if $P, Q \in \mathfrak{P}$ and $P \neq Q$, then either $P \cap Q = \emptyset$ or $P \perp Q$. Next, if $P \cap Q = \emptyset$, then $P_f \neq Q_f$ by statement (1); and if $P \perp Q$, then $P_f \neq Q_f$ by statement (2). Consequently, the assignment $P \mapsto P_f$ is one-to-one.

Finally, observe that from statement (2) and Lemma 2.7 it follows that f strictly preserves parallelism. ∎

From this theorem we obviously get

2.17. COROLLARY. *Let f be an injective principal map of $\mathbb{R}^2$ such that $f(\mathbb{R}^2)$ is not contained in a principal line. Then for any $P, Q \in \mathfrak{P}$ such that $P \cap Q = \emptyset$, we have $P_f \cap Q_f = \emptyset$.* ∎

2.18. PROPOSITION. *Let f be an injective principal map of $\mathbb{R}^2$. Then f is epiprincipal if and only if it is bijective.*

Proof. The sufficiency follows from Proposition 2.13.

To prove the necessity, we suppose that f is an injective epiprincipal map of $\mathbb{R}^2$. If now M is a fixed principal line in $\mathbb{R}^2$, then so is $M_f = f(M)$. Clearly, for an arbitrary point y of $\mathbb{R}^2$ there is a unique $Q_y \in \mathfrak{P}$ such that $y \in Q_y$ and $Q_y \perp M_f$, i.e. $Q_y \cap M_f = \{y'\}$ for some $y' \in \mathbb{R}^2$. We obviously have $y' = f(x')$ for a unique $x' \in M$. Next, there is a unique $P_{x'} \in \mathfrak{P}$ such that $x' \in P_{x'}$ and $P_{x'} \perp M$, i.e. $P_{x'} \cap M = \{x'\}$. In turn, we have $(P_{x'})_f \cap M_f = f(P_{x'}) \cap f(M) = \{y'\}$. Consequently, we conclude that $Q_y \parallel (P_{x'})_f$ and $y' \in Q_y \cap (P_{x'})_f$, whence $Q_y = (P_{x'})_f = f(P_{x'})$, and so, there is $x \in P_{x'}$ such that $y = f(x)$. Thus, f is surjective because y can be an arbitrary point of $\mathbb{R}^2$. Next, since f is injective, it has to be bijective. ∎

2.19. THEOREM. *Let f be a principal map of $\mathbb{R}^2$. Then the following conditions are equivalent*:

(a) *$f(\mathbb{R}^2)$ is not contained in any principal cross*;

(b) *f strictly preserves orthogonality*;

(c) *$f(\mathbb{R}^2)$ is not contained in any principal line and f preserves parallelism.*

Proof. Observe that implications (a)⇒(b) and (b)⇒(c) follow from Theorem 2.9 and Lemma 2.7, respectively. Therefore it suffices to prove implications (b)⇒(a) and (c)⇒(b).

(b)⇒(a). Suppose to the contrary that $f(\mathbb{R}^2)$ is contained in a principal cross K with origin q, i.e. $K = \mathbb{K}_q$. Let us take an arbitrary point $p = (a, b)$ of $\mathbb{R}^2$ and consider the principal cross $\mathbb{K}_p = V_a \cup H_b$. From Lemma 2.3 and since f strictly preserves orthogonality, it follows that f is totally P-noncontracting and the principal lines $(V_a)_f$ and $(H_b)_f$ are orthogonal. Thus we get a principal cross $K' = (V_a)_f \cup (H_b)_f$ with origin $f(p)$, i.e. $K' = \mathbb{K}_{f(p)}$, such that $f(\mathbb{K}_p) \subseteq K'$. Moreover, note that K' is a unique principal cross in $\mathbb{R}^2$ for which $f(\mathbb{K}_p) \subseteq K'$. On the other hand, since $f(\mathbb{R}^2) \subseteq K$ by our hypothesis, it follows that $K' = K$, so $f(p)$=q. Next, since p can be an arbitrary point of $\mathbb{R}^2$, we conclude that f is constant. But this contradicts the fact that f strictly preserves orthogonality.

(c)⇒(b). Observe first that f is totally P-noncontracting. Indeed, for otherwise there is a principal line P in $\mathbb{R}^2$ such that $f(P) = \{f(p)\}$ where p is a fixed point of P. Next, for an arbitrary point $x \in \mathbb{R}^2$ there is a principal line Q_x in $\mathbb{R}^2$ such that $x \in Q_x$ and $Q_x \perp P$, so $Q_x \cap P = \{x'\}$. Clearly, $\mathfrak{A} = \{Q_x : x \in \mathbb{R}^2\}$ is a maximal principal class of parallelism. Thus since f is nonconstant, it follows from Proposition 2.5 that there is $Q \in \mathfrak{A}$ such that f is noncontracting for Q. This means that there are $a, a' \in Q$ such that $f(a) \neq f(a')$ and $Q \cap P = \{a'\}$. Let $\overline{Q}$ be a unique principal line in $\mathbb{R}^2$

such that $f(Q) \subseteq \overline{Q}$. Since $Q_x \| Q$ for $x \in \mathbb{R}^2$ and f preserves parallelism, we infer that for each $x \in \mathbb{R}^2$ there is a principal line $\overline{Q}_x$ in $\mathbb{R}^2$ such that $f(Q_x) \subseteq \overline{Q}_x$ and $\overline{Q}_x \| \overline{Q}$. In turn, we have $f(x') = f(p) = f(a')$, which implies that $f(p) \in \overline{Q}_x \cap \overline{Q}$. Consequently, we conclude that $\overline{Q}_x = \overline{Q}$ for each $x \in \mathbb{R}^2$. Therefore we have

$$f(\mathbb{R}^2) = f\Big(\bigcup\{Q_x : x \in \mathbb{R}^2\}\Big) \subseteq \bigcup\{\overline{Q}_x : x \in \mathbb{R}^2\} = \overline{Q},$$

a contradiction. This proves that f is totally P-noncontracting, and so, the principal assignment $P \mapsto P_f$ is defined.

To prove that f strictly preserves orthogonality, suppose to the contrary that there are principal lines P and Q such that $P \perp Q$ but P_f and Q_f are not orthogonal. This implies that $P_f \cap Q_f \neq \emptyset$ and $P_f \| Q_f$, so $P_f = Q_f$. Clearly, for an arbitrary point $x \in \mathbb{R}^2$ there is a principal line Q_x in $\mathbb{R}^2$ such that $x \in Q_x$, $Q_x \| Q$ and $Q_x \cap P = \{x'\}$. Hence we conclude that $f(x') \in (Q_x)_f \cap P_f = (Q_x)_f \cap Q_f$ and $(Q_x)_f \| Q_f$ because f preserves parallelism. Consequently, we infer that $(Q_x)_f = Q_f$ for each $x \in \mathbb{R}^2$. Therefore we have

$$f(\mathbb{R}^2) = f\Big(\bigcup\{Q_x : x \in \mathbb{R}^2\}\Big) \subseteq \bigcup\{(Q_x)_f : x \in \mathbb{R}^2\} = Q_f,$$

a contradiction. This completes the proof of (c)$\Rightarrow$(b). ∎

Let f be a principal map of $\mathbb{R}^2$. We say that f is *upper principal* if it satisfied at least one of the equivalent conditions of Theorem 2.19. Otherwise, f is called *lower principal*. It is seen that if f and g are principal maps of $\mathbb{R}^2$ such that at least one of them is lower principal, then so is the composition $f \circ g$. In particular, the family of all lower principal map of $\mathbb{R}^2$ is a semigroup under multiplication given by the composition of maps. The following example shows that the composition of upper principal maps of $\mathbb{R}^2$ can be lower principal, even a constant map.

2.20. EXAMPLE. Let $a, b, c, d \in \mathbb{R}$ where $a \neq b$ and $c \neq d$. Define the function $\phi : \mathbb{R} \to \mathbb{R}$ ($\psi : \mathbb{R} \to \mathbb{R}$) by $\phi(x) = a$ ($\psi(x) = c$) if $x \leq 0$ and $\phi(x) = b$ ($\psi(x) = d$) if $x > 0$. Next, we set $f(x,y) = (\phi(x), \psi(y))$ for $(x,y) \in \mathbb{R}^2$. It is easily seen that f is an upper principal map of $\mathbb{R}^2$ satisfying the following conditions:

$$f(V_x) = \{(a,c),(a,d)\}, \quad f(H_y) = \{(a,c),(b,c)\} \quad \text{for } x, y \leq 0;$$
$$f(V_x) = \{(b,c),(b,d)\}, \quad f(H_y) = \{(a,d),(b,d)\} \quad \text{for } x, y > 0.$$

In particular, we have $f(x,y) = (a,c)$ for $x, y \leq 0$. This implies that if $a, b, c, d \leq 0$, then $(f \circ f)(x,y) = (\phi(\phi(x)), \psi(\psi(y))) = (a,c)$ for each $(x,y) \in \mathbb{R}^2$, which means that the principal map $f \circ f$ is constant, and so, lower principal. ∎

3. Descriptions and further properties

Let $X \in \{V, H, P\}$. Obviously, there are principal maps of $\mathbb{R}^2$ which are not V-principal and H-principal. The simplest example of such a map is given by the symmetry map $\mathfrak{s}$ defined by $\mathfrak{s}(x, y) = (y, x)$. Note that the assignment $f \mapsto \mathfrak{s} \circ f \circ \mathfrak{s}$ defines a one-to-one correspondence between V-principal (H-principal) and H-principal (V-principal) maps of $\mathbb{R}^2$. Clearly, a V-principal map of $\mathbb{R}^2$ need not be H-principal, and conversely. For instance, as we observed, the smooth quasi principal map f of $\mathbb{R}^2$ defined in Example 1.1 is V-principal but not H-principal.

Let f be a map of $\mathbb{R}^2$. We say that f is *0-principal* if it is V-principal and H-principal simultaneously. In turn, we say that f is *1-principal* in case the map $f \circ \mathfrak{s}$, or equivalently, the map $\mathfrak{s} \circ f$ is 0-principal, that is, if f is (V, H)-principal and (H, V)-principal. Clearly, the assignment $f \mapsto f \circ \mathfrak{s}$ defines a one-to-one correspondence between 0-principal (1-principal) and 1-principal (0-principal) maps of $\mathbb{R}^2$. Note that if f is constant, then it is 0-principal and 1-principal. For arbitrary maps ϕ and ψ of $\mathbb{R}$, denote by $\phi \times \psi$ the map of $\mathbb{R}^2$ defined by $(\phi \times \psi)(x, y) = (\phi(x), \psi(y))$ and observe that it is 0-principal.

3.1. THEOREM. *Let f be a 0-principal (1-principal) map of $\mathbb{R}^2$. Then f is of the form $\phi \times \psi$ ($\mathfrak{s} \circ (\phi \times \psi) = (\psi \times \phi) \circ \mathfrak{s}$) where ϕ and ψ are maps of $\mathbb{R}$. Furthermore, if f is 0-principal and 1-principal, then it is constant.*

Proof. Suppose first that f is a 0-principal map of $\mathbb{R}^2$, i.e. V-principal and H-principal. In this case, for any $x, y \in \mathbb{R}$ there are unique $x', y' \in \mathbb{R}$ such that $f(V_x) \subseteq V_{x'}$ and $f(H_y) \subseteq H_{y'}$. We have thus defined the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = x'$ and $\psi(y) = y'$. For any $p = (a, b) \in \mathbb{R}^2$ we have $\mathbb{K}_p = V_a \cup H_b$, so $f(\mathbb{K}_p) = f(V_a) \cup f(H_b) \subseteq V_{a'} \cup H_{b'} = \mathbb{K}_{p'}$ where $p' = (a', b') = (\phi \times \psi)(a, b)$. On the other hand, since $p \in V_a \cap H_b$, we conclude that $f(p) \in V_{a'} \cap H_{b'} = \{p'\}$, and so, $f(p) = p' = (\phi \times \psi)(p)$. Consequently, since p can be an arbitrary point of $\mathbb{R}^2$, it follows that $f = \phi \times \psi$.

Suppose now that f is a 1-principal map of $\mathbb{R}^2$. Then $f \circ \mathfrak{s}$ is 0-principal, so by the first part of our proof we infer that $f \circ \mathfrak{s} = \phi \times \psi$ where ϕ and ψ are the corresponding maps of $\mathbb{R}$. Obviously, this implies $f = (\phi \times \psi) \circ \mathfrak{s} = \mathfrak{s} \circ (\psi \times \phi)$.

Finally, suppose that f is 0-principal and 1-principal map of $\mathbb{R}^2$. It follows that if V is a vertical line in $\mathbb{R}^2$, there is a vertical (horizontal) one $V'(H')$ such that $f(V) \subseteq V'$ ($f(V) \subseteq H'$). Hence $f(V) \subseteq V' \cap H'$, so $\operatorname{card} f(V) = \operatorname{card}(V' \cap H') = 1$ because $V' \perp H'$. Consequently, f is contracting for each vertical line in $\mathbb{R}^2$, and similarly, it is such for each horizontal one. Thus by Proposition 2.5, f is constant. ∎

A map f of $\mathbb{R}^2$ is said to be *regularly principal* if it is 0-principal or 1-principal. One can observe that the composition of regularly principal maps is regularly principal. More precisely, if f and g are such maps, then the composition $g \circ f$ is 0-principal provided that both f and g are 0-principal or 1-principal, and it is 1-principal otherwise. From Theorem 3.1 we obviously get

3.2. COROLLARY. *If f is a regularly lower principal map of $\mathbb{R}^2$, then $f(\mathbb{R}^2)$ is contained in a principal line. More exactly, f is of the form $\phi \times \psi$ or $(\phi \times \psi) \circ \mathfrak{s} = \mathfrak{s} \circ (\psi \times \phi)$ where ϕ and ψ are maps of $\mathbb{R}$ such that at least one of them is constant.* ∎

A map f of $\mathbb{R}^2$ is said to be *strictly 0-principal* (*1-principal*) if it is 0-principal (1-principal) but not 1-principal (0-principal). We call f *strictly principal* if it is strictly 0-principal or strictly 1-principal. From Theorem 3.1 it follows that every strictly principal map of $\mathbb{R}^2$ is nonconstant. Moreover, one can see that the composition of strictly principal maps need not be strictly principal. For instance, if p_1 and p_2 are the projective maps of $\mathbb{R}^2$ defined by

$$p_1 : (x, y) \mapsto (x, 0) \quad \text{and} \quad p_2 : (x, y) \mapsto (0, y),$$

then they are strictly 0-principal but the composition $p_2 \circ p_1 = p_1 \circ p_2$ is not strictly principal as a constant map. However, observe that the composition of strictly principal maps is always regularly principal. Furthermore, note that every regularly principal map of $\mathbb{R}^2$ is either strictly principal or constant. By an easy verification we get

3.3. PROPOSITION. *Any upper principal map of $\mathbb{R}^2$ is strictly principal, and so, regularly principal.* ∎

Remark that by this proposition Theorem 3.1 gives a full description of any upper principal map of $\mathbb{R}^2$.

Observe that a strictly principal map of $\mathbb{R}^2$ need not be upper principal. For example, such a map can be given by the above-mentioned p_1 or p_2. By this proposition, Corollary 2.15 and since every bijective principal map of $\mathbb{R}^2$ is upper principal, we obviously get

3.4. COROLLARY. *Every bijective principal map of $\mathbb{R}^2$ is strictly principal. Moreover, so is the inverse map of it.* ∎

If f is a bijective strictly 0-principal (1-principal) map of $\mathbb{R}^2$, then from Theorem 3.1 it follows that $f = \phi \times \psi$ ($f = \mathfrak{s} \circ (\phi \times \psi) = (\psi \times \phi) \circ \mathfrak{s}$) where ϕ and ψ are the corresponding bijective maps of $\mathbb{R}$. In this case we have

$$f^{-1} = \phi^{-1} \times \psi^{-1} \quad (f^{-1} = \mathfrak{s} \circ (\psi^{-1} \times \phi^{-1}) = (\phi^{-1} \times \psi^{-1}) \circ \mathfrak{s})),$$

which means that f^{-1} is also strictly 0-principal (1-principal). Clearly, the composition of bijective principal maps of $\mathbb{R}^2$ is again a bijective principal one. Moreover, if f is such a map, then so is the inverse f^{-1} by Corollary 2.15. For this reason the set $\mathcal{P}_b$ of all bijective principal maps of $\mathbb{R}^2$ is a group under multiplication given by the composition of maps. In turn, one can see that every continuous bijective principal map of $\mathbb{R}^2$ is a homeomorphism. Thus the set of all such maps determines a subgroup of $\mathcal{P}_b$, denoted by $\mathcal{P}_h$. On the other hand, there are smooth bijective principal maps of $\mathbb{R}^2$ which are not diffeomorphisms. For example, such a map can be given by the assignment $(x,y) \mapsto (x^3, y)$. Denote by $\mathcal{P}_d$ the subgroup of $\mathcal{P}_b$ consisting of all principal diffeomorphisms of $\mathbb{R}^2$. Clearly, $\mathcal{P}_h\,(\mathcal{P}_d)$ is a subgroup of the group of all homeomorphisms (diffeomorphisms) of $\mathbb{R}^2$. For any $f \in \mathcal{P}_b$ we write $\tau(f) = 0$ ($\tau(f) = 1$) in case f is strictly 0-principal (1-principal). It can easily be seen that the assignment $f \mapsto \tau(f)$ defines a homomorphism $\tau : \mathcal{P}_b \to \mathbb{Z}_2$ of groups. This means that for any $f, g \in \mathcal{P}_b$ we get

$$\tau(f \circ g) \equiv \tau(f) + \tau(g) \pmod 2.$$

Let $x \in \{b, h, d\}$ and $\iota \in \{0, 1\}$. We set $\tau_x = \tau \,|\, \mathcal{P}_x$ and $\mathcal{P}_{x\iota} = \{f \in \mathcal{P}_x : \tau(f) = \iota\}$. Clearly, $\mathcal{P}_{x\iota}$ consists of all strictly ι-principal maps from $\mathcal{P}_x$. It is seen that the symmetry map $\mathfrak{s}$ belongs to $\mathcal{P}_{x1}$. Denote by $\mathfrak{a}_{x,\mathfrak{s}}$ the inner automorphism of $\mathcal{P}_x$ determined by $\mathfrak{s}$, i.e. $\mathfrak{a}_{x,\mathfrak{s}}(f) = \mathfrak{s} \circ f \circ \mathfrak{s}$. By an easy verification we get (compare [1], Proposition 4.9)

3.5. Proposition. *The map $\tau_x : \mathcal{P}_x \to \mathbb{Z}_2$ is an epimorphism of groups such that $\mathcal{P}_{x\iota} = \tau_x^{-1}(\iota)$. In particular, $\mathcal{P}_{x0}$ is a normal divisor of $\mathcal{P}_x$ and the factor group $\mathcal{P}_x/\mathcal{P}_{x0}$ is isomorphic to $\mathbb{Z}_2$. Moreover, $\mathfrak{s} \circ \mathcal{P}_{x0} = \mathcal{P}_{x0} \circ \mathfrak{s} = \mathcal{P}_{x1}$ and the cosets $\mathcal{P}_{x0}$ and $\mathcal{P}_{x1}$ are invariant under $\mathfrak{a}_{x,\mathfrak{s}}$, i.e. $\mathfrak{s} \circ \mathcal{P}_{x\iota} \circ \mathfrak{s} = \mathcal{P}_{x\iota}$.* ∎

Let $\mathfrak{M}$ be an arbitrary family of maps of $\mathbb{R}^2$. Let $\mathcal{K}$ be a class of subsets of $\mathbb{R}^2$ (curves in $\mathbb{R}^2$). We say that $\mathcal{K}$ is *invariant under* $\mathfrak{M}$ in case the condition $A \in \mathcal{K}$ ($c \in \mathcal{K}$) involves that $f(A) \in \mathcal{K}$ ($f_\#(c) \in \mathcal{K}$) for each $f \in \mathfrak{M}$. Denote by $\mathcal{P}_c\,(\mathcal{P}_s)$ the family of all continuous (smooth) principal maps of $\mathbb{R}^2$. Clearly, $\mathcal{P}_c$ and $\mathcal{P}_s$ are semigroups under multiplication defined by the composition of maps. Let $\operatorname{lso}(K)$ ($\operatorname{gr}(K)$) stand for the class of all locally K-subordinate subsets of $\mathbb{R}^2$ (principal K-graphs in $\mathbb{R}^2$). Moreover, let $\operatorname{cur}(K)$ ($\operatorname{cur}^\infty(K)$) denote the class of all continuous (smooth) locally K-subordinate curves in $\mathbb{R}^2$ (see [1], the corresponding definitions). It is easy to verify

3.6. Proposition. *The following statements hold:*

(1) *The class* $\operatorname{lso}(K)$ ($\operatorname{gr}(K)$) *is invariant under the group $\mathcal{P}_h$ (semigroup $\mathcal{P}_c$);*

(2) *The class* $\operatorname{cur}(K)$ $(\operatorname{cur}^{\infty}(K))$ *is invariant under the semigroup* $\mathcal{P}_c$ $(\mathcal{P}_s)$. ∎

Remark that the class $\operatorname{lso}(K)$ is not invariant under the semigroups $\mathcal{P}_c$ and $\mathcal{P}_s$. For example, the map f of $\mathbb{R}^2$ given by the assignment $(x,y) \mapsto (\sin x, y)$ is smooth and principal. On the other hand, the set $\{\sin n : n \in \mathbb{N}\}$ is a dense subset of $[-1;1]$, which means that $\mathbb{N} \times \mathbb{R} \in \operatorname{lso}(K)$ but $f(\mathbb{N} \times \mathbb{R}) \notin \operatorname{lso}(K)$.

3.7. LEMMA. *Let f be a principal map of $\mathbb{R}^2$ which is not* 0*-principal. Then f is subject to exactly one of the following statements*:

(1) *$f(\mathbb{R}^2)$ is contained in a principal line*;

(2) *f is neither V-principal nor H-principal.*

Proof. Suppose that $f(\mathbb{R}^2)$ is not contained in any principal line. Next, suppose to the contrary that f is V-principal. It follows that there are vertical lines $V^1, V^2, \overline{V}^1$ and $\overline{V}^2$ for which $f(V^1) \subseteq \overline{V}^1$ and $f(V^2) \subseteq \overline{V}^2$ where $V^1 \neq V^2$ and $\overline{V}^1 \neq \overline{V}^2$. In turn, any horizontal line H in $\mathbb{R}^2$ satisfies $H \cap V^1 \neq \emptyset$ and $H \cap V^2 \neq \emptyset$, and so, $f(H) \cap \overline{V}^1 \neq \emptyset$ and $f(H) \cap \overline{V}^2 \neq \emptyset$, which implies that $f(H)$ is contained in a horizontal line. Consequently, f would be 0-principal, a contradiction. This means that f is not V-principal. Similarly, one can show that f is not H-principal. Finally, note that statements (1) and (2) cannot be satisfied simultaneously. ∎

This lemma immediately implies

3.8. LEMMA. *Let f be a principal map of $\mathbb{R}^2$ which is neither* 0*-principal nor* 1*-principal. Then f is subject to exactly one of the following statements*:

(1) *$f(\mathbb{R}^2)$ is contained in a principal line*;

(2) *There are distinct $V^1, V^2 \in \mathfrak{P}_V$ such that f is noncontracting for both V^1 and V^2. Moreover, if $f(V^1) \subseteq P$ and $f(V^2) \subseteq Q$ for some $P, Q \in \mathfrak{P}$, then $P \perp Q$.* ∎

We call subsets A and B of $\mathbb{R}^2$ *principal-disjoint* if there is no principal line L such that $L \cap A \neq \emptyset$ and $L \cap B \neq \emptyset$.

3.9. THEOREM. *Suppose that Γ and Δ are nonempty principal-disjoint subsets of $\mathbb{R}^2$. If $\mathbf{K}_p = V_a \cup H_b$ is a fixed principal cross with origin $p = (a,b)$, then there exists a principal map $f : \mathbb{R}^2 \to \mathbf{K}_p$ such that*

$$\Gamma = f^{-1}(H_b \setminus \{p\}) \quad \textit{and} \quad \Delta = f^{-1}(V_a \setminus \{p\}),$$

which implies that f is neither 0*-principal nor* 1*-principal and $f(\mathbb{R}^2)$ is not contained in any principal line. Furthermore, $f = (\phi, \psi)$ where ϕ and ψ are* *real functions on* $\mathbb{R}^2$ *such that*

$$\Gamma = \{x \in \mathbb{R}^2 : \phi(x) \neq a\} \quad \textit{and} \quad \Delta = \{x \in \mathbb{R}^2 : \psi(x) \neq b\}.$$

Moreover, f *can be continuous (smooth) if* Γ *and* Δ *are open.*

Conversely, if f *is a principal map of* $\mathbb{R}^2$ *satisfying the following statements:*

(1) f *is neither* 0*-principal nor* 1*-principal;*
(2) $f(\mathbb{R}^2)$ *is not contained in any principal line,*

then f *is of the above-described form.*

Proof. We can take real functions ϕ and ψ on $\mathbb{R}^2$ such that

$$\Gamma = \{x \in \mathbb{R}^2 : \phi(x) \neq a\} \quad \text{and} \quad \Delta = \{x \in \mathbb{R}^2 : \psi(x) \neq b\}.$$

Clearly, the map $f = (\phi, \psi)$ is principal, transforms $\mathbb{R}^2$ into $\mathbb{K}_p$ and satisfies $\Gamma = f^{-1}(H_b \setminus \{p\})$ and $\Delta = f^{-1}(V_a \setminus \{p\})$. In particular, $f(\mathbb{R}^2)$ is not contained in any principal line.

To prove that f is neither 0-principal nor 1-principal, suppose to the contrary that f is 0-principal. Let us take $p \in \Gamma$ and $q \in \Delta$ and note that $f(p) \neq f(q)$. Let $V\,(H)$ be the vertical (horizontal) line passing through $p\,(q)$. Since f is 0-principal, we conclude that $f(V) \subseteq V'$ and $f(H) \subseteq H'$ where V' and H' are corresponding vertical and horizontal lines in $\mathbb{R}^2$, respectively. On the other hand, note that if P is a principal line in $\mathbb{R}^2$ such that $P \cap \Gamma \neq \emptyset$ $(P \cap \Delta \neq \emptyset)$, then we have $f(P) \subseteq H_b$ $(f(P) \subseteq V_a)$. Consequently, since $p \in V$ and $q \in H$, it follows that $f(V) \subseteq V' \cap H_b$ and $f(H) \subseteq H' \cap V_b$, so $f(V) = \{f(p)\}$ and $f(H) = \{f(q)\}$, which implies that $f(V) \cap f(H) = \emptyset$ because $f(p) \neq f(q)$. But this contradicts the fact that $\emptyset \neq f(V \cap H) \subseteq f(V) \cap f(H)$. Summarizing, f is not 0-principal. Moreover, by applying this result to the map $\mathfrak{s} \circ f = (\psi, \phi)$, we also conclude that f is not 1-principal.

If in addition Γ and Δ are nonempty open subsets of $\mathbb{R}^2$, one can construct continuous (smooth) real functions ϕ and ψ on $\mathbb{R}^2$ such that

$$\Gamma = \{x \in \mathbb{R}^2 : \phi(x) \neq a\} \quad \text{and} \quad \Delta = \{x \in \mathbb{R}^2 : \psi(x) \neq b\}.$$

In this case the map $f = (\phi, \psi)$ is clearly continuous (smooth).

Let now f be an arbitrary principal map of $\mathbb{R}^2$ satisfying statements (1) and (2). First, we prove that there is a principal cross $K = V_a \cup H_b$ such that $f(\mathbb{R}^2) \subseteq K$ and the sets

$$\Gamma = f^{-1}(V_a \setminus \{p\}) \quad \text{and} \quad \Delta = f^{-1}(H_b \setminus \{p\})$$

are nonempty subsets of $\mathbb{R}^2$ where $p = (a, b)$. Indeed, by statement (2) of Lemma 3.8, there are $a, b \in \mathbb{R}$ and distinct $V^1, V^2 \in \mathfrak{P}_V$ such that f is noncontracting for both V^1 and V^2, moreover, $f(V^1) \subseteq V_a$ and $f(V^2) \subseteq H_b$, whence $\Gamma \neq \emptyset$ and $\Delta \neq \emptyset$. To prove that $f(\mathbb{R}^2) \subseteq K$, let us take an arbitrary horizontal line H in $\mathbb{R}^2$. Obviously, we have $H \cap V^1 \neq \emptyset$ and $H \cap V^2 \neq \emptyset$, which implies that $f(H) \cap V_a \neq \emptyset$ and $f(H) \cap H_b \neq \emptyset$. Therefore if $f(H)$ is contained in a vertical (horizontal) line in $\mathbb{R}^2$, then $f(H) \subseteq V_a (f(H) \subseteq H_b)$. Consequently, in each case $f(H) \subseteq V_a \cup H_b = K$, so $f(\mathbb{R}^2) \subseteq \bigcup\{f(H) : H \in \mathfrak{P}_H\} \subseteq K$. Finally, note that Γ and Δ are principal-disjoint subsets of $\mathbb{R}^2$. Moreover, the map f has to be of the above-described form (ϕ, ψ). ∎

From Theorems 3.1 and 3.9 we obviously get

3.10. COROLLARY. *Let f be a principal map of $\mathbb{R}^2$ such that the image $f(\mathbb{R}^2)$ is not contained in any principal line. Then f is either regularly principal or lower principal.* ∎

Remark that Theorem 3.9 gives a full description of a lower principal map f of $\mathbb{R}^2$ in the case when the image $f(\mathbb{R}^2)$ is not contained in any principal line. Otherwise, it can easily be seen that $f(\mathbb{R}^2)$ is contained in a principal line if and only if f is of the form (ϕ, ψ) where ϕ and ψ are real functions on $\mathbb{R}^2$ such that at least one of them is constant.

By a *principal retract* of $\mathbb{R}^2$ we shall mean subset R of $\mathbb{R}^2$ for which there is a principal map $f : \mathbb{R}^2 \to R$ such that f restricted to R is the identity map of R. In this case f is called a *principal retraction* from $\mathbb{R}^2$ to R. If f is continuous, we call R a *topological principal retract* of $\mathbb{R}^2$. It is seen that if ϕ and ψ are retractions from $\mathbb{R}$ to its subsets A and B, respectively, then $\phi \times \psi$ is a principal retraction from $\mathbb{R}^2$ to $A \times B$. If in addition ϕ and ψ are continuous, then A and B are connected closed subsets of $\mathbb{R}$ and $A \times B$ is a topological retract of $\mathbb{R}^2$. Clearly, any point of $\mathbb{R}^2$ and any principal line in $\mathbb{R}^2$ are topological principal retracts of $\mathbb{R}^2$. One can ask whether a principal cross in $\mathbb{R}^2$ is such a retract. Note first that if f is a principal retraction from $\mathbb{R}^2$ to a principal cross $K = V_a \cup H_b$, then f satisfies statements (1) and (2) of Theorem 3.9, and so, by this theorem we infer that $f = (\phi, \psi)$, $H_b \setminus \{(a, b)\} \subseteq \Gamma = \{p \in \mathbb{R}^2 : \phi(p) \neq a\}$ and $V_a \setminus \{(a, b)\} \subseteq \Delta = \{p \in \mathbb{R}^2 : \psi(p) \neq b\}$ where $\phi(x, b) = x$ and $\psi(a, y) = y$ for any $x, y \in \mathbb{R}$. In fact, we have $H_b \setminus \{(a, b)\} = \Gamma$ and $V_a \setminus \{(a, b)\} = \Delta$, for otherwise Γ and Δ would not be principal-disjoint. It follows that $f(p) = (a, b)$ for each $p \in \mathbb{R}^2 \setminus K$, so f is unique. Moreover, note that it is discontinuous. We have thus proved

3.11. COROLLARY. *For any principal cross K in $\mathbb{R}^2$ there is a unique discontinuous principal retraction from $\mathbb{R}^2$ to K. In particular, K is a principal retract of $\mathbb{R}^2$ which, however, cannot be a topological principal one.* ∎

Remark that in the purely topological sense every principal cross in $\mathbb{R}^2$ is obviously a topological retract of $\mathbb{R}^2$.

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