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FIXED POINT THEOREMS ON PRODUCT OF COMPACT METRIC SPACES

Results generalizing and unifying fixed point theorems of Edelstein, Fisher, Jungck, Matkowski, Rhoades, Seghal and others are obtained for four systems of maps on a finite product of compact metric spaces.

1. Coordinatewise weakly / asymptotically commuting maps

Throughout this paper we shall follow the following notations (cf. [1]-[2], [9], [14]-[15]).

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for } i \neq k, \\ 1 - a_{ik} & \text{for } i = k, \end{cases} \qquad i,k = 1,\dots,n, \tag{1.1}$$

and $c_{ik}^{(t)}$ are defined recursively by

$$c_{ik}^{t+1} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & \text{for } i \neq k \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & \text{for } i = k, \end{cases} \tag{1.2}$$
$$i,k = 1,\dots,n-t-1, \quad t = 0,\dots,n-2;$$

$$c_{ii}^{(t)} > 0, \quad i = 1,\dots,n-t, \quad t = 0,\dots,n-1; \quad n \geq 2. \tag{1.3}$$

If $n = 1$, we define $c_{11}^{(0)} = a_{11}$.

Let (X_i, d_i), $i = 1,\dots,n$, be metric spaces,

$$X := X_1 \times X_2 \times \dots \times X_n,$$
$$x(1,n) := (x_1, x_2, \dots, x_n), \quad x_i \in X_i, \quad i = 1,\dots,n,$$
$$s_n^m := x^m(1,n) = (x_1^m, x_2^m, \dots, x_n^m), \quad x_i^m \in X_i, \quad i = 1,\dots,n,$$
$$m = 1,2,3,\dots;$$

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and

$$P_i, S_i : X \to X_i, \quad i = 1, \ldots, n.$$

Note that here $\{s_n^m\}_{m=1}^{\infty}$ or simply $\{s_n^m\}$ denotes a sequence from the product space X of n metric spaces $X_i, i = 1, \ldots, n$.

DEFINITION 1. Two systems of maps $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise commuting at a point $x(1,n) \in X$ if and only if

$$P_i(S_1(x(1,n)), \ldots, S_n(x(1,n))) \\ = S_i(P_1(x(1,n)), \ldots, P_n(x(1,n))), \quad i = 1, \ldots, n.$$

Two systems of maps $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise commuting on X (cf. [15]) if and only if they are coordinatewise commutating at every point of X.

DEFINITION 2. Two systems of maps $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise weakly commuting at a point $x(1,n) \in X$ if and only if

$$d_i(P_i(S_i(x(1,n)), \ldots, S_n(x(1,n))), S_i(P_1(x(1,n), \ldots, P_n(x(1,n)))) \\ \leq d_i(P_i(x(1,n)), S_i(x(1,n))), \quad i = 1, \ldots, n.$$

Two systems of maps $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise weakly commuting on X (cf. [15]), if and only if they are coordinatewise weak commutating at every point of X.

DEFINITION 3. Two systems of maps $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise $u(1,n)$-asymptotically commuting (or simply coordinatewise asymptotically commuting or, following the terminology of Jungck [6], coordinatewise commuting-compatible) if and only if

$$\lim_m d_i(P_i(S_1(s_n^m), \ldots, S_n(s_n^m)), S_i(P_1(s_n^m), \ldots, P_n(s_n^m))) = 0$$

as soon as $\lim_m P_i(s_n^m) = \lim_m S_i(s_n^m) = u_i$ for some $u_i \in X_i$, $i = 1, \ldots, n$.

Notice that Definitions 1–3 with $n = 1$ are the standard ones of commuting, weakly commuting (see [13]) and asymptotically commuting [17] (also called compatible [6]) maps.

We remark that the following examples show that (i) weakly commuting systems of maps need not be commuting, and (ii) asymptotically commuting systems of maps need not be weakly commuting.

EXAMPLE 1. Let $X_1 = [1, \infty)$ and $X_2 = [0,1]$ be metric spaces with absolute value metrics and

$$P_1, S_1 : X_1 \times X_2 \to X_1; \; P_2, S_2 : X_1 \times X_2 \to X_2 \text{ such that}$$
$$P_1(x_1, x_2) = 1 + x_1, \quad P_2(x_1, x_2) = x_2/2,$$
$$S_1(x_1, x_2) = 1 + 2x_1, \quad S_2(x_1, x_2) = x_2/4.$$

The following two inequalities show that $\{P_1, P_2\}$ and $\{S_1, S_2\}$ are coordinatewise weakly commuting but not commuting systems of maps

$$d_1(P_1(S_1(x_1,x_2),S_2(x_1,x_2)),S_1(P_1(x_1,x_2),P_2(x_1,x_2))) \\ = 1 \leq x_1 = d_1(P_1((x_1,x_2),S_1(x_1,x_2));$$

and

$$d_2(P_2(S_1(x_1,x_2),S_2(x_1,x_2)),S_2(P_1(x_1,x_2),P_2(x_1,x_2))) \\ = 0 \leq d_2(P_2(x_1,x_2),S_2(x_1,x_2)).$$

The first inequality shows that there does not exist a sequence $\{(s_2^m)\}$ in $X_1 \times X_2$ for which $\lim_m d_1(P_1(S_1(s_2^m),S_2(s_2^m),S_1(P_1(s_2^m),P_2(s_2^m))) = 0$ is not satisfied. Therefore the systems of maps $\{P_1, P_2\}$ and $\{S_1, S_2\}$ are (vacuously) coordinatewise asymptotically commuting (or compatible).

EXAMPLE 2. Let $X_1 = [0,\infty)$ and $X_2 = (-\infty,\infty)$ be metric spaces with absolute value metrics on them. Consider the following coordinatewise non-commutative systems of maps $\{P_1, P_2\}$ and $\{S_1, S_2\}$ such that

$$P_1, S_1 : X_1 \times X_2 \to X_1, \quad P_1(x_1,x_2) = 4x_1^3, \quad S_1(x_1,x_2) = 2x_1^3;$$
$$P_2, S_2 : X_1 \times X_2 \to X_2, \quad P_2(x_1,x_2) = 4x_2^2, \quad S_2(x_1,x_2) = 2x_2^2.$$

Then

$$d_1(P_1(x_1,x_2), \quad S_1(x_1,x_2)) = 2x_1^3 \to 0 \quad \text{iff } x_1 \to 0$$

and

$$d_1(P_1(S_1(x_1,x_2),S_2(x_1,x_2)),S_1(P_1(x_1,x_2),P_2(x_1,x_2))) \\ = 96x_1^9 \to 0 \quad \text{iff } x_1 \to 0.$$

Similarly

$$d_2(P_2(x_1,x_2),S_2(x_1,x_2)) \to 0 \quad \text{iff } x_2 \to 0$$

and

$$d_2(P_2(S_1(x_1,x_2),S_2(x_1,x_2)),S_2(P_1(x_1,x_2),P_2(x_1,x_2))) \to 0 \quad \text{iff } x_2 \to 0.$$

Consequently the systems $\{P_1, P_2\}$ and $\{S_1, S_2\}$ are coordinatewise asymptotically commuting but not coordinatewise weakly commuting.

2. Fixed point theorems

We need the following generalization of Fisher [4], Kubiak [8], Matkowski [9], Sessa-Rhoades-Khan [13], and Singh-Singh [16] to prove our main result.

THEOREM 1 [15]. *Let (X_i, d_i) be complete metric spaces and $P_i, Q_i, S_i, T_i : X \to X_i$, $i = 1,\ldots,n$ such that*

(2.1) $\quad P_i(X) \subseteq T_i(X), \quad Q_i(X) \subseteq S_i(X), \quad i = 1,\ldots,n;$

(2.2) *systems of maps* $\{P_1,\dots,P_n\}$, $\{S_1,\dots,S_n\}$ *and* $\{Q_1,\dots,Q_n\}$, $\{T_1,\dots,T_n\}$ *are coordinatewise weakly commuting pairs*;

(2.3) S_i *or* T_i *is continuous*, $i = 1,\dots,n$.

If there exist nonnegative numbers b *and* a_{ik}, $i,k = 1,\dots,n$ *defined in* (1.1) *such that* (1.2), (1.3) *and the following hold*:

$$0 \le b < 1 - h \quad \text{and} \quad h = \max\Big(r_i^{-1}\sum_{k=1}^{n} a_{ik} r_k\Big), \tag{2.4}$$

and

$$\begin{aligned} &d_i(P_i(x(1,n)), Q_i(y(1,n))) \\ &\le \max\Big\{\sum_{k=1}^{n} a_{ik} d_k(S_k(x(1,n)), T_k(y(1,n))), b\max\{d_i(S_i(x(1,n)), \\ &\quad P_i(x(1,n))), d_i(T_i(y(1,n)), Q_i(y(1,n))), \tfrac{1}{2}[d_i(S_i(x(1,n)), Q_i(y(1,n))) \\ &\quad + d_i(T_i(y(1,n)), P_i(x(1,n)))]\}\Big\} \end{aligned} \tag{2.5}$$

for all $x(1,n), y(1,n)$ *in* X, $i = 1,\dots,n$, *then the system of equations*

$$P_i(x(1,n)) = S_i(x(1,n)) = x_i = Q_i(x(1,n)) = T_i(x(1,n))$$

has a unique common solution $x_1,\dots,x_n$ *such that*

$$x_i \in X_i, \quad i = 1,\dots,n.$$

THEOREM 1 bis. *Theorem* 1 *wherein the condition* (2.2) *is replaced by* (2.2a):

(2.2a) *Systems* $\{P_1,\dots,P_n\}$, $\{S_1,\dots,S_n\}$ *and* $\{Q_1,\dots,Q_n\}$, $\{T_1,\dots,T_n\}$ *are coordinatewise asymptotically commuting pairs.*

Proof. The proof of Theorem 1 works.

Remark 1. An examination of the proof of Theorem 1 (see[15]) suggests that the condition (2.4) of the above theorems may be replaced by (2.4a):

$$0 \le b < 1. \tag{2.4a}$$

Matkowski [9] has shown that the system of inequalities $\sum_{k=1}^{n} a_{ik} r_k < r_i$, $i = 1,\dots,n$, has a solution $r_i > 0$, $i = 1,\dots,n$, iff (1.3) holds. This clearly explains the definition of h in (2.4). For a detailed analysis on this aspect, refer to [9] (see also [2], [14] and [15, p. 795]).

THEOREM 2. *Let* (X_i, d_i) *be compact metric spaces, and let* P_i, Q_i, S_i, T_i *be continuous maps from* $X \to X_i$, $i = 1,\dots,n$. *If there exist nonnegative numbers* a_{ik}, $i,k = 1,\dots,n$, *defined in* (1.1), *such that* (1.2), (1.3), (2.1), (2.2a) *and the following hold*:

(2.6) $\quad d_i(P_i(x(1,n)),\ Q_i(y(1,n))) < M_i(x(1,n),\ y(1,n)),\quad i=1,\dots,n$

for such $x(1,n)$, $y(1,n)\in X$ *that the right hand side of the inequality is positive*, *where*

$$M_i(x(1,n),y(1,n)) = \max\Big\{\sum_{k=1}^{n} a_{ik}d_k(S_k(x(1,n)),T_k(y(1,n))),$$
$$d_i(S_i(x(1,n)),P_i(x(1,n))),d_i(T_i(y(1,n)),Q_i(y(1,n))),$$
$$\tfrac{1}{2}[d_i(S_i(x(1,n)),Q_i(y(1,n)))+d_i(T_i(y(1,n)),P_i(x(1,n)))]\Big\},$$
$$i=1,\dots,n.$$

Then the system of equations

$$P_i(x(1,n)) = S_i(x(1,n)) = x_i = Q_i(x(1,n)) = T_i(x(1,n))$$

has a unique common solution $x_1,\dots,x_n$ *such that* $x_i\in X_i$, $i=1,\dots,n$.

Proof. We assert that $M_i(x(1,n),y(1,n))=0$ for some $x(1,n),y(1,n)$, otherwise functions $f_i(x(1,n),y(1,n)) = \frac{d_i(P_i(x(1,n)),Q_i(y(1,n)))}{M_i(x(1,n),y(1,n))}$, $i=1,\dots,n$, are continuous and satisfy $f_i(x(1,n),y(1,n))<1$ on $X\times X$, $i=1,\dots,n$. Since $X\times X$ is compact, there exist $u(1,n),v(1,n)\in X$ such that $f_i(x(1,n), y(1,n))\le\lambda_i = f_i(u(1,n),v(1,n))<1$ for $x(1,n),y(1,n)\in X$. Consequently, $d_i(P_i(x(1,n),y(1,n)))\le\lambda_i M_i(x(1,n),y(1,n))$, $i=1,\dots,n$, on X with $\lambda_i<1$. That is

$$d_i(P_i(x(1,n),Q_i(y(1,n)))$$
$$\le\max\Big\{\sum_{k=1}^{n}(\lambda_i a_{ik})d_k(S_k(x(1,n),T_k(y(1,n))),$$
$$b\max\{d_i(S_i(x(1,n)),\ P_i(x(1,n))),\ d_i(T_i(y(1,n)),\ Q_i(y(1,n))),$$
$$\tfrac{1}{2}[d_i(S_i(x(1,n)),\ Q_i(y(1,n)))+d_i(T_i(y(1,n)),\ P_i(x(1,n)))]\}\Big\},$$
$$i=1,\dots,n,\ \text{where } b=\max\{\lambda_1,\dots,\lambda_n\}.$$

So, by Theorem 1 bis and Remark 1, there exists a $z(1,n)\in X$ such that

$$P_i(z(1,n)) = S_i(z(1,n)) = z_i = T_i(z(1,n))$$
$$= Q_i(z(1,n)),\quad i=1,\dots,n.$$

Consequently we have the contradiction $M_i(z(1,n),z(1,n))>0$, and $M_i(z(1,n),z(1,n))=0$. Since $M_i(x(1,n),y(1,n))=0$ for some $x(1,n)$,

$y(1,n) \in X$, then (2.6) implies

$$S_i(x(1,n)) = T_i(y(1,n)) = P_i(x(1,n)) = Q_i(y(1,n)) = w_i, \tag{2.7}$$

say $i = 1, \ldots, n$.

Since the systems $\{P_1, \ldots, P_n\}$ and $\{S_1, \ldots, S_n\}$ are coordinatewise asymptotically commuting, then the equality $S_i(x(1,n)) = P_i(x(1,n))$ (cf. (2.7)) implies, by considering the sequence $\{s_n^m\}$ where $s_n^m = x(1,n)$ for $m \in N$, that

$$P_i(S_1(x(1,n)), \ldots, S_n(x(1,n))) = S_i(P_i(x(1,n)), \ldots, P_n(x(1,n)))$$

i.e. $P_i(w(1,n)) = S_i(w(1,n))$. Similarly

$$Q_i(w(1,n)) = T_i(w(1,n)).$$

Now let

$$d_i(P_i(w(1,n)), w_i) = d_i(P_i(w(1,n)), Q_i(y(1,n))) \neq 0.$$

Then

$$\begin{aligned} d_i(P_i(w(1,n)), w_i) &= d_i(P_i(w(1,n)), Q_i(y(1,n))) \\ < \max\Big\{ &\sum_{k=1}^{n} a_{ik} d_k(S_k(w(1,n)),\ T_k(y(1,n))),\ d_i(S_i(w(1,n)),\ P_i(w(1,n))), \\ &d_i(T_i(y(1,n)),\ Q_i(y(1,n)),\ \tfrac{1}{2}[d_i(S_i(w(1,n)),\ Q_i(y(1,n))) \\ &\quad + d_i(T_i(y(1,n)),\ P_i(w(1,n)))]\Big\}, \quad i = 1, \ldots, n. \end{aligned}$$

Since without any loss of generality we can assume $d_i(P_i(w(1,n)), w_i) \leq r_i$, the above inequalities yield

$$d_i(P_i(w(1,n)), w_i) < \max\{hr_i, d_i(P_i(w(1,n)), w_i)\}$$

that is $d_i(P_i(w(1,n)), w_i) < hr_i$, $i = 1, \ldots, n$, wherein

$$h = \max_i \Big\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \Big\}.$$

Inductively

$$d_i(P_i(w(1,n)), w_i) < h^m r_i.$$

Since $h \in (0,1)$ (see, [2,p.137], [9, p.11-12] and [14, Lemma 1]), making $m \to \infty$ we have

$$P_i(w(1,n)) = w_i, \quad i = 1, \ldots, n.$$

Similarly $Q_i(w(1,n)) = w_i$, $i = 1, \ldots, n$.

The uniqueness of $w_i, i = 1, \ldots, n$, follows easily.

R e m a r k 2. In view of Theorem 1(xxviii) and the proof of Theorem 1(xxvi) of Rhoades [11], it easily follows that (2.5) implies (2.6) but not conversely.

R e m a r k 3. Theorem 3 of Rhoades [11] is proved under his contractive condition (22), and generalizes the results of Edelstein [3] and Sehgal [12]. The condition (22) is our condition (2.6) with $n = 1$, $a_{11} = 1$, $P_1 = Q_1$ and $S_1 x = T_1 x = x$ for every x in X_1.

R e m a r k 4. Recently Jungck [6] discussing the relative roles of commutativity and compatibility (or asymptotic commutativity) has obtained a generalization of Fisher's theorem [5] under very tight hypotheses. Theorem 2 extends and unifies the main results of Fisher [5] and Jungck [6].

R e m a r k 5. Contractive conditions studied by Czerwik [1], Kasahara-Rhoades [7], Naimpally-Singh-Whitfield [10, cf. Cor.3] and several others on compact setting may be obtained as special cases of (2.6).

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