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ALMOST r-CONTACT SEMI-SYMMETRIC METRIC FINSLER CONNECTION ON VECTOR BUNDLE

1. Introduction

The purpose of the present paper is to define almost r-contact semi-symmetric metric Finsler connection on the total space V of the vector bundle $V(M) = (V, \Pi, M)$ and to study its various properties. In particular, they are studied for almost r-Sasakian semi-symmetric metric and r-Sasakian semi-symmetric metric Finsler connection on the total space of the vector bundle.

Let $V(M) = \{V, \Pi, M\}$ be a vector bundle whose total space V is an $(n+m)$-dimensional C^∞-manifold and the base space M is an n-dimensional C^∞-manifold. The projection map $\Pi : V \to M$, that is $\Pi(u) = X \in M$, for $u \in V$, where $u = (X, Y)$ and $Y \in R^m = \Pi^{-1}(X)$, is the fibre of $V(M)$ over X.

A non-linear connection N on the total space V of $V(M)$ is a differentiable distribution $u \mapsto N_u \in T_v(V)$ for $u \in V$ such that

$$T_u(V) = N_u \otimes V_u^v, \tag{1.1}$$

where

$$V_u^v = \{X \in T_u(V) : \Pi_*(X) = 0\}.$$

Now N_u is called the horizontal and V^v the vertical distribution. Thus for each $X \in T_u(V)$ we can write

$$X = X^H + X^V, \quad X^H \in N_u \quad \text{and} \quad X^V \in V_u^v. \tag{1.2}$$

Let X^i, $i = 1, 2, \ldots, n$, and y^a, $a = 1, 2, \ldots, m$, be the coordinates of $u \in V$. The local base of N_u is $(\delta/\delta x^i) = (\partial/\partial x^i - N_i^a(x,y)\partial/\partial y^a)$ and that of V_u^v is $(\partial/\partial y^a)$, where N_i^a are the coefficients of N. Their dual basis is $(dx^1, \delta y^a)$, where $\delta y^a = dy^a + N_i^a(X,Y)\, dX^i$.

Let $X = X^i\delta/\delta x^i + X^a\partial/\partial y^a$, $X \in T_u(V)$, then

(1.3) $$X^H = X^i\delta/\delta x^i, \quad V^V = X^a\partial/\partial y^a, \quad \widetilde{X}^a = X^a + N_i^a X^i.$$

Let ω be a 1-form, $\omega = \omega_i\, dx^i + \omega_a \delta y^a$. Then

(1.4) $$\omega^H = \overline{\omega}_i\, dx^i, \quad \overline{\omega}_i = \omega_i - N_i^a\omega_a, \quad \omega^v = \omega_a\delta y^a$$

which gives $\omega^H(X^V) = 0$, $\omega^V(X^H) = 0$, where $\omega = \omega^H + \omega^V$.

DEFINITION 1.1. A Finsler connection $F\Gamma$ on V is a linear connection ∇ on V which is preserved by the parallelism of N_u the horizontal and V^v the vertical distribution.

A Finsler connection ∇ on V is characterised by the conditions $(\nabla_X Y^H)^V = 0$, $(\nabla_X Y^V)^H = 0$ for $X, Y \in Y_u(V)$.

A linear connection ∇ on V is a Finsler connection on V if

$$\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V$$
$$\nabla_x\omega = (\nabla_X\omega^H)^H + (\nabla_X\omega^V)^V \quad \text{for } X, Y \in T_u(V),$$

and $\omega \in T_u^*(V)$.

For any Finsler connection ∇ on V we define

$$\nabla_X^H Y = \nabla_X Y^H, \quad \nabla_X^V Y = \nabla_X Y^V, \quad \nabla_X^H f' = \nabla_{Xf'}^H,$$
$$\nabla_X^V f' = \nabla_{Xf'}^V, \quad \text{for } X, Y \in T_u(V), \quad f' \in \Im(V),$$

where ∇^H is called the h-covariant and ∇^V the V-covariant derivative.

The torsion tensor field T of a Finsler connection on V is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad \text{for } X, Y \in T_u(V),$$

and is characterized by five Finsler connection ∇ on V is torsion free, then we have [4]

$$T(X^H, Y^H) = 0, \quad T(X^H, Y^V) = 0, \quad T(X^V, Y^V) = 0$$
$$\text{for each } X, Y \in T_u(V).$$

2. Almost r-Sasakian structure on vector bundle

Let f be an almost r-contact Finsler structure on V given by the Finsler tensor field of type $\binom{1\ 1}{1\ 1}$ with the property

(2.1) $$\begin{cases} f^2 = -I_n + \sum_{p=1}^r \eta_p^H \otimes (\xi^p)^H + \sum_{p=1}^r \eta_p^V(\xi^p)^V, \\ f(\xi^p)^H = 0, \quad f(\xi^p)^V = 0, \quad \eta_p^H(\xi^q)^H + \eta_p^V(\xi^q)^V = \delta_q^p, \\ \eta_p^H(fX^H) = 0, \quad \eta_p^H(fX^V) = 0, \quad \eta_p^V(fX^H) = 0, \\ \qquad p, q = 1, 2, \ldots, r, \end{cases}$$

where η_p is r, 1-form, ξ^p is r-vector fields and δ_q^p is the Kronecker symbol.

Let G be the Finsler metric structure on V which is symmetric, positive definite and non-degenerate on V. The metric structure G on V can be decomposed as

$$G = G^H + G^V, \tag{2.2}$$

where G^H is of type $\binom{0\ 0}{2\ 0}$, symmetric positive definite and non-degenerate on N_u and G^V is of type $\binom{0\ 0}{0\ 2}$, symmetric, positive definite and non-degenerate on N^V, that is, for $X, Y \in T_u(V)$,

$$G(X,Y) = G^H(X,Y) + G^V(X,Y), \tag{2.3}$$

where

$$G^H(X,Y) = G(X^H, Y^H), \quad G^V(X,Y) = G(X^V, Y^V).$$

Let Finsler metric structure G on V satisfy

$$\begin{cases} G(fX, fY) = G(X,Y) - \sum_{p=1}^r \eta_p(X)\eta_p(Y), \\ G^H(fX, fY) = G^H(X,Y) - \sum_{p=1}^r \eta_p^H(X)\eta_p^H(Y), \\ G^V(fX, fY) = G^V(X,Y) - \sum_{p=1}^r \eta_p^V(X)\eta_p^V(Y) \end{cases} \tag{2.4}$$

which is equivalent to

$$\begin{cases} G^H(X, \xi^p) = \eta_p^H(X), \quad G^V(X, \xi^p) = \eta_p^V(X), \\ G^H(fX, fY) = -G^H(f^2X, Y), \quad G^V(fX, fY) = -G^V(f^2X, Y). \end{cases} \tag{2.5}$$

Then, (f, η_0, ξ^p, G), $p = 1, \ldots, r$, is called almost r-contact metrical Finsler structure on V.

Now, let us define

$$F(X,Y) \stackrel{\text{def}}{=} G(fX, Y), \tag{2.6}$$

that is

$$F(X^H, Y^H) = G^H(fX, Y) \quad F(X^V, Y^V) = G^V(fX, Y).$$

Then, from (2.6) we obtain

$$\begin{cases} F(fX^H, fY^H) = F(X^H, Y^H), \quad F(fX^V, fY^V) = F(X^V, Y^V), \\ F(X^H, Y^H) = -F(Y^H, X^H), \quad F(X^V, Y^V) = -F(Y^V, X^V), \end{cases} \tag{2.7}$$

for each $X, Y \in T_u(V)$.

DEFINITION 2.1. Let ∇ be a Finsler connection on V and F be the fundamental 2-form which satisfies

$$F(X,Y) = d\eta_p(X,Y), \quad p = 1, 2, \ldots, r, \text{ that is}$$

$$\begin{cases} F(X^H, Y^H) = (\nabla_X^H \eta_p)(Y^H) - (\nabla_Y^H \eta_p)(X^H) + \eta_p(T(X^H, Y^H)) \\ F(X^V, Y^V) = (\nabla_X^H \eta_p)(Y^V) - (\nabla_Y^V \eta_p)(X^V) + \eta_p(T(X^V, Y^V)) \\ F(X^H, Y^H) = (\nabla_X^H \eta_p)(Y^H) - (\nabla_Y^H \eta_p)(X^V) + \eta_p(T(X^V, Y^H)). \end{cases} \tag{2.8}$$

Then the almost r-contact metrical Finsler structure is called almost r-Sasakian Finsler connection on V.

THEOREM 2.1. *Let F be the fundamental 2-form and almost r-Sasakian Finsler connection ∇ on V be torsion free. Then we have for each $X, Y \in T_u(V)$.*

$$
(2.9)\qquad \begin{cases} F(X^H, Y^H) = (\nabla^H_X \eta_p)(Y^H) - (\nabla^H_Y \eta_p)(X^H) \\ F(X^V, Y^V) = (\nabla^V_X \eta_p)(Y^V) - (\nabla^V_Y \eta_p)(X^V) \\ F(X^V, Y^H) = (\nabla^V_X \eta_p)(Y^H) - (\nabla^H_Y \eta_p)(X^V). \end{cases}
$$

Proof follows from (1.1), (2.9).

DEFINITION 2.2. An almost r-Sasakian Finsler structure on V is said to be r-Sasakian Finsler structure if the r, 1-form η_p is a Killing vector field, that is

$$
(2.10)\qquad \begin{cases} (\nabla^H_X \eta_p)(Y^H) + (\nabla^H_Y \eta_p)(X^H) = 0 \\ (\nabla^V_X \eta_p)(Y^V) + (\nabla^V_Y \eta_p)(X^V) = 0 \\ (\nabla^H_X \eta_p)(Y^V) + (\nabla^V_Y \eta_p)(X^H) = 0 \end{cases}
$$

for each $X, Y \in T_u(V)$ and the Finsler connection ∇ on V is torsion free, which is called r-Sasakian Finsler connection.

THEOREM 2.2. *Let ∇ be the torsion free Finsler connection together with an r-Sasakian Finsler structure on V and F be the fundamental 2-form. Then for each $X, Y \in T_u(V)$ we have*

$$
(2.11)\qquad \begin{cases} F(X^H, Y^H) = 2(\nabla^H_X \eta_p)(Y^H) = -2(\nabla^H_Y \eta_p)(X^H) \\ F(X^V, Y^V) = 2(\nabla^V_X \eta_p)(Y^V) = -2(\nabla^V_Y \eta_p)(X^V) \\ F(X^H, Y^V) = 2(\nabla^H_X \eta_p)(Y^V) = -2(\nabla^V_Y \eta_p)(X^H). \end{cases}
$$

Proof follows easily from (2.9), (2.10).

3. Almost r-contact semi-symmetric Finsler connection

DEFINITION 3.1. A Finsler connection ∇ on V is said to be semisymmetric, if the torsion tensor T satisfies

$$
(3.1)\qquad \begin{cases} [T(X^H, Y^H)]^H = X^H \omega^H(Y^H) - Y^H \omega^H(X^H) \\ [T(X^V, Y^V)]^V = X^V \omega^V(Y^V) - Y^V \omega^V(X^V) \end{cases}
$$

for each $X, Y \in T_u(V)$ and $\omega \in T^*_u(V)$.

DEFINITION 3.2. An almost r-contact Finsler connection ∇ on V is semi-symmetric, if the torsion tensor T of ∇ satisfies

$$
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = X\eta_p(r) - Y\eta_p(X)
$$

which gives

$$(3.2)\qquad \begin{cases} [T(X^H,Y^H)]^H = X^H\eta_p^H(Y^H) - Y^H\eta_p^H(X^H) \\ [T(X^V,Y^V)]^V = X^V\eta_p^V(Y^V) - Y^V\eta_p^V(X^V) \end{cases}$$

for each $X, Y \in T_u(V)$.

THEOREM 3.1. *For an almost r-contact semi-symmetric Finsler connection ∇ on V, we have*

$$(3.3)\qquad \begin{cases} [T(X^H,(\xi^p)^H)]^H = X^H - \eta_p^H(X^H)(\xi^p)^H \\ [T(X^V,(\xi^p)^V)]^V = X^V - \eta_p^V(X^V)(\xi^p)^V, \end{cases}$$

$$(3.3')\qquad \begin{cases} f^2[T(X^H,Y^H)]^H + [T(X^H,Y^H)]^H = 0 \\ f^2[T(X^V,Y^V)]^H + [T(X^V,Y^V)]^V = 0. \end{cases}$$

Proof. Follows easily from definitions 3.1, 3.2 and from (3.1).

THEOREM 3.2. *Let ∇ be an almost r-contact semi-symmetric Finsler connection on V. Then for every $X, Y \in T_u(V)$, there exists a Finsler tensor field A such that*

$$f^2[A(X^H,Y^H)]^H = [T(X^H,Y^H)]^H,$$
$$f^2[A(X^V,Y^V)]^V = [T(X^V,Y^V)]^V,$$
$$f^2[A(X^H,Y^H)]^H + [A(X^H,Y^H)]^H = 0,$$
$$f^2[A(X^V,Y^V)]^V + [A(X^V,Y^V)]^V = 0.$$

Proof. Putting $[A(X^H,Y^H)]^H = f[T(Y^H,X^H)]^H$. In view of (3,3′), we have the required result.

4. Almost r-contact semi-symmetric metric Finsler connection

DEFINITION 4.1. An almost r-contact semi-symmetric Finsler connection ∇ on V is said to be an almost r-contact semi-symmetric Finsler connection on V if and only if $\nabla_X^{G^H} = 0$, $\nabla_X^{G^V} = 0$.

If $\overset{\circ}{\nabla}$ is the almost r-contact Finsler metric connection on V which is torsion free, then any almost r-contact semi-symmetric metric Finsler connection $\overline{\nabla}$ is given by

$$(4.1)\qquad \overline{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + H(X,Y) \quad \text{for } X, Y \in T_u(V),$$

where

$$(4.2)\qquad \begin{cases} H(X,Y) = \frac{1}{2}[T(X,Y) + P(X,Y) + P(Y,X)] \\ P(X,Y) = \eta_p(X)Y - G(X,Y)\xi^p. \end{cases}$$

PROPOSITION 4.1. *The Finsler tensor field $H(X,Y)$ satisfies for each $X,Y \in T_u(V)$ the following relations*

$$(4.3)\qquad \begin{cases} H(X^H,Y^H)]^H = \eta_p^H(Y^H)(\xi_P)^H - G(X^H,Y^H)(\xi^p)^H \\ [H(X^H,Y^H)]^V = 0, \quad H(X^H,Y^V) = 0 \\ [H(X^V,Y^V)]^V = \eta_p^V(Y^V)(\xi^p)^V - G(X^V,Y^V)(\xi^p)^V. \end{cases}$$

Proof. In view of $T(X,Y) = \eta_p(X)Y - \eta_p(Y)X$, we have

$$H(X,Y) = \frac{1}{2}[T(X,Y) + P(X,Y) + P(Y,X)] = -G(X,Y)\xi + \eta_p(Y)X$$

which gives (4.3).

THEOREM 4.2. *Let $\overline{\nabla}$ be an almost r-contact semi-symmetric metric Finsler connection and $\overset{\circ}{\nabla}$ be any fixed torsion free Finsler connection. Then, for each $X,Y \in T_u(V)$, we have*

$$(4.4)\qquad \begin{cases} \overline{\nabla}_X^H fY^H = \overset{\circ}{\nabla}{}_X^H fY^H + F(X^H,Y^H)(\xi^p)^H \\ \overline{\nabla}_X^V fY^V = \overset{\circ}{\nabla}{}_X^V fY^V + F(X^V,Y^V)(\xi^p)^V, \quad p = 1,2,\dots r. \end{cases}$$

Proof. In view of (4.1), (4.2), we have

$$(4.5)\qquad \overline{\nabla}_X^Y = \overset{\circ}{\nabla}{}_X^Y + \eta_p(Y) - G(X,Y)\xi^p.$$

Replacing Y by fY, we get

$$\overline{\nabla}_X fY = \overset{\circ}{\nabla}_X fY - G(X,fY)\xi^p = \overset{\circ}{\nabla}_X fY + F(X,Y)\xi^p,$$

since $\overline{\nabla}$ and $\overset{\circ}{\nabla}$ are Finsler connections. Therefore,

$$\overline{\nabla}_X^H fY^H + \overline{\nabla}_X^V fY^V =$$
$$= \overset{\circ}{\nabla}{}_X^H fY^H + \overset{\circ}{\nabla}{}_X^V fY^V + F(X^H,Y^H)(\xi^p) + F(X^V,Y^V)(\xi^p)^V$$

which gives (4.4).

COROLLARY 4.1. *We have*

$$(4.6)\qquad \begin{cases} (\overline{\nabla}_X^H \eta_p)(fY^H) = (\overset{\circ}{\nabla}{}_X^H \eta_p)(fY^H) - F(X^H,Y^H) \\ (\overline{\nabla}_X^V \eta_p)(fY^V) = (\overset{\circ}{\nabla}{}_X^V \eta_p)(fY^V) - F(X^V,Y^V), \end{cases}$$

for each $X,Y \in T_u(V)$ and $p = 1,2,\dots,r$.

Proof. From (4.5) we obtain

$$(\overline{\nabla}_X \eta_p)(Y) = (\overset{\circ}{\nabla}_X \eta_p)(Y) + \eta_p(X)\eta_p(Y) - G(X,Y).$$

Replacing Y by fY, we get,

$$(\overline{\nabla}_X \eta_p)(fY) = (\overset{\circ}{\nabla}_X \eta_p)(fY) - G(X,fY) = (\overset{\circ}{\nabla}_X \eta_p)(fY) - F(X,Y).$$

Since $\overline{\nabla}$ and $\mathring{\nabla}$ are Finsler connections, we can write

$$(\overline{\nabla}_X\eta_p)(fY) = (\overline{\nabla}^H_X\eta_p)(fY^H) + (\overline{\nabla}^V_X\eta_p)(fY^V)$$
$$= (\mathring{\nabla}^H_X\eta_p)(fY^H) + (\mathring{\nabla}^V_X\eta_p)(fY^V) - F(X^H, Y^H) - F(X^V, Y^V)$$

which gives the desired result (4.6).

THEOREM 4.3. *Let $\overline{\nabla}$ be an almost r-contact semi-symmetric metric Finsler connection and $\mathring{\nabla}$ be r-Sasakian torsion free Finsler connection. Then, for each $X, Y \in T_u(V)$ we have*

$$(\overline{\nabla}^H_{fX}\eta_p)(fY^H) + (\overline{\nabla}^H_{fY}\eta_p)(fX^H) = 2G(fX^H, fY^H),$$
$$(\overline{\nabla}^V_{fX}\eta_p)(fY^V) + (\overline{\nabla}^V_{fY}\eta_p)(fX^V) = 2G(fX^V, fY^V), \quad p = 1, 2, \ldots, r$$

Proof. Using Corollary 4.1, we obtain

$$(\overline{\nabla}_{fX}\eta_p)(fY) + (\overline{\nabla}_{fY}\eta_p)(fX) =$$
$$= (\mathring{\nabla}_{fX}\eta_p)(fY) + (\mathring{\nabla}_{fY}\eta_p)(fX) - 2G(fX, fY).$$

Since ∇ is r-Sasakian Finsler connection, we have

$$(\mathring{\nabla}_{fX}\eta_p)(fY) + (\mathring{\nabla}_{fY}\eta_p)(fX) = 0$$

which gives the required result.

THEOREM 4.4. *Let $\overline{\nabla}$ be an almost r-contact semi-symmetric metric Finsler connection. Then, for each $X, Y \in T_u(V)$, we have*

$$(\overline{\nabla}^H_X\eta_p)(Y^H) - (\overline{\nabla}^H_Y\eta_p)(X^H) = F(X^H, Y^H),$$
$$(\overline{\nabla}^V_X\eta_p)(Y^V) - (\overline{\nabla}^V_Y\eta_p)(X^V) = F(X^V, Y^V).$$

Proof. From (4.5), we have

$$\begin{cases} (\overline{\nabla}_X\eta_p)(Y) = (\mathring{\nabla}_X\eta_p)(Y) + \eta_p(X)\eta_p(Y) - G(X, Y) \\ (\overline{\nabla}_Y\eta_p)(X) = (\mathring{\nabla}_Y\eta_p)(X) + \eta_p(X)\eta_p(X) - G(X, Y). \end{cases} \tag{4.7}$$

From (4.7), (2.9), we obtain

$$(\overline{\nabla}_X\eta_p)(Y) - (\overline{\nabla}_Y\eta_p)(X) = F(X, Y).$$

Since $\overline{\nabla}$ is a Finsler connection, we get

$$(\overline{\nabla}^H_X\eta_p)(Y^H) + (\overline{\nabla}^V_X\eta_p)(Y^V) - (\overline{\nabla}^H_Y\eta_p)(X^H) - (\overline{\nabla}^V_Y\eta_p)(X^V) =$$
$$= F(X^H, Y^H) + F(X^V, Y^V),$$

which proves our theorem.

THEOREM 4.5. *Let $\overline{\nabla}$ be an almost r-contact semi-symmetric metric Finsler connection an $\overset{\circ}{\nabla}$ be r-Sasakian torsion free Finsler connection. Then for each $X, Y \in T_u(V)$, we have*

$$(\overline{\nabla}^H_X \eta_p)(Y^H) + (\overline{\nabla}^H_Y \eta_p)(X^H) = -2G(fX^H, fY^H),$$
$$(\overline{\nabla}^V_X \eta_p)(Y^V) + (\overline{\nabla}^H_Y \eta_p)(Y^V) = -2G(fX^V, fY^V).$$

Proof. Using

$$(\overline{\nabla}_X \eta_p)(Y) + (\overline{\nabla}_Y \eta_p)(X) = (\overset{\circ}{\nabla}_X \eta_p)(Y) + (\overset{\circ}{\nabla}_Y \eta_p)(X) - 2G(fX, fY)$$

and (2.10), we get

$$(\overline{\nabla}_X \eta_p)(Y) + (\overline{\nabla}_Y \eta_p)(X) = -2G(fX, fY)$$

and, since $\overline{\nabla}$ is a Finsler connection, we obtain the desired result.

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