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ON SOME CLASSES OF SYSTEMS OF INTEGRAL INEQUALITIES

1. Introduction and main results

In this paper we establish some generalizations of systems of integral inequalities due to Greene [2] and Pachpatte [4]. These inequalities can be used for solution's estimates of system integral, differential and integrodifferential equations.

In this section we present our results on systems of integral inequalities. Their proofs will be given in section 2. Examples are considered in section 3. At last, in section 4 the further generalizations of our results are discussed.

At first we formulate the following result for linear systems.

THEOREM 1.1. *Let* $f(t), g(t), C_1(t), C_9(t), \sigma(t), h_i(t)$ $(i = 1,2,7,8)$ $(t \geq 0)$; $h_j(t,\tau)$ $(j = 3,4,9,10)$ $(t \geq \tau \geq 0)$; $h_k(t,\tau)$ $(k = 5,6,11,12)$ $(t \geq s \geq \tau \geq 0)$ *be real-valued nonnegative continuous functions and* $c_i (i = 2,\dots,16)$, $\mu_1, \mu_2 (\mu_1 \geq \mu_2 \geq 0)$ *be nonnegative constants such that*

$$(1.1)\quad \begin{aligned} f(t) \leq C_1(t) + \sigma(t)\Big\{ & C_2 + C_3 \int_0^t h_1(s)f(s)\,ds \\ & + C_4 \int_0^t h_2(s)e^{\mu_2 s}g(s)ds + C_5 \int_0^t \int_0^s h_3(s,\tau)f(\tau)d\tau\,ds \\ & + C_6 \int_0^t \int_0^s h_4(s,\tau)e^{\mu_2 \tau}g(\tau)d\tau\,ds \\ & + C_7 \int_0^t \int_0^s \int_0^\tau h_5(s,\tau,x)f(x)dx\,d\tau\,ds \\ & + C_8 \int_0^t \int_0^s \int_0^\tau h_6(s,\tau,x)e^{\mu_2 x}g(x)dx\,d\tau\,ds \Big\}, \end{aligned}$$

$$(1.2)\quad g(t) \le C_9(t) + \sigma(t)\{C_{10} + C_{11} \int_0^t h_7(s) e^{-\mu_1 s} f(s)\, ds$$
$$+ C_{12} \int_0^t h_8(s) g(s) ds + C_{13} \int_0^t \int_0^s h_9(s,\tau) g(\tau)\, d\tau\, ds$$
$$+ C_{14} \int_0^t \int_0^s h_{10}(s,\tau) e^{-\mu_1 \tau} f(\tau)\, d\tau\, ds$$
$$+ C_{15} \int_0^t \int_0^s \int_0^\tau h_{11}(s,\tau,x) g(x) dx\, d\tau\, ds$$
$$+ C_{16} \int_0^t \int_0^s \int_0^\tau h_{12}(s,\tau,x) e^{-\mu_1 x} f(x) dx\, d\tau\, ds\}$$

for all $t \ge 0$.

Then for all $t \ge 0$

$$(1.3)\qquad f(t) \le e^{\mu_1 t} Q(t), \qquad g(t) \le Q(t),$$

where

$$(1.4)\quad Q(t) = C_{17}(t) + \sigma(t)\Big\{C_{18} \exp \int_0^t \Big[h_{13}(s)\sigma(s)$$
$$+ \int_0^s h_{14}(s,\tau)\sigma(\tau) d\tau + \int_0^s \int_0^\tau h_{15}(s,\tau,x)\sigma(x) dx\, d\tau\Big]\, ds$$
$$+ \int_0^t \Big\{\Big[h_{13}(s) C_{17}(s) + \int_0^s h_{14}(s,\tau) C_{17}(\tau) d\tau$$
$$+ \int_0^s \int_0^\tau h_{15}(s,\tau,x) C_{17}(x) dx\, d\tau\Big] \exp \int_s^t \Big[h_{13}(\tau)\sigma(\tau)$$
$$+ \int_0^\tau h_{14}(\tau,x)\sigma(x) dx + \int_0^\tau \int_0^y h_{15}(\tau,y,x)\sigma(x) dx\, dy\Big]\, d\tau\Big\}\, ds\Big\}$$

and

$$(1.5)\qquad \begin{aligned} C_{17}(t) &= C_1(t) + C_9(t);\\ C_{18} &= C_2 + C_{10};\\ h_{13}(t) &= \max\{C_{11} h_7(t) + C_3 h_1(t); C_{12} h_8(t) + C_4 h_2(t)\};\\ h_{14}(t,s) &= \max\{C_{13} h_9(t,s)\\ &\qquad + C_6 h_4(t,s); C_{14} h_{10}(t,s) + C_5 h_3(t,s)\}; \end{aligned}$$

$$h_{15}(t,s,\tau) = \max\{C_{15}h_{11}(t,s,\tau) + C_8h_6(t,s,\tau); C_{16}h_{12}(t,s,\tau) + C_7h_5(t,s,\tau)\}$$

for all $t \geq s \geq \tau \geq 0$.

Next, we formulate the following nonlinear generalizations of Pachpatte's inequalities.

THEOREM 1.2. *Let the conditions of Theorem 1.1 be true. Let $G(t)$ be a continuous strictly increasing, convex and submultiplicative function for $r \geq 0$, $G(0) = 0$; $\lim_{r\to\infty} G(r) = \infty$; $\alpha_1(t), \alpha_2(t)$ be continuous functions for all $t \geq 0$; $\alpha_1(t)$ and $\alpha_2(t) > 0$, $\alpha_1(t) + \alpha_2(t) = 1$.*

Let, next, instead of (1.1) and (1.2) the following inequalities be true

$$\begin{aligned}(1.6)\quad f(t) \leq C_1(t) + \sigma(t)G^{-1}\Big\{&C_2 + C_3\int_0^t h_1(s)G(f(s))\,ds \\ &+ C_4\int_0^t h_2(s)e^{\mu_2 s}G(g(s))\,ds + C_5\int_0^t\int_0^s h_3(s,\tau)G(f(\tau))\,d\tau\,ds \\ &+ C_6\int_0^t\int_0^s h_4(s,\tau)e^{\mu_2\tau}G(g(\tau))d\tau\,ds \\ &+ C_7\int_0^t\int_0^s\int_0^\tau h_5(s,\tau,x)G(f(x))dx\,d\tau\,ds \\ &+ C_8\int_0^t\int_0^s\int_0^\tau h_6(s,\tau,x)e^{\mu_2 x}G(g(x))dx\,d\tau\,ds\Big\},\end{aligned}$$

$$\begin{aligned}(1.7)\quad g(t) \leq C_9(t) + \sigma(t)G^{-1}\Big\{&C_{10} + C_{11}\int_0^t h_7(s)e^{-\mu_1 s}G(f(s))\,ds \\ &+ C_{12}\int_0^t h_8(s)G(g(s))ds + C_{13}\int_0^t\int_0^s h_9(s,\tau)G(g(\tau))d\tau\,ds \\ &+ C_{14}\int_0^t\int_0^s h_{10}(s,\tau)e^{-\mu_1\tau}G(f(\tau))d\tau\,ds \\ &+ C_{15}\int_0^t\int_0^s\int_0^\tau h_{11}(s,\tau,x)G(g(x))dx\,d\tau\,ds \\ &+ C_{16}\int_0^t\int_0^s\int_0^\tau h_{12}(s,\tau,x)e^{-\mu_1 x}G(f(x))dx\,d\tau\,ds\Big\}\end{aligned}$$

for all $t \geq 0$, where G^{-1} is the inverse function of G.

Then

$$f(t) \leq G^{-1}(e^{\mu_1 t} Q_0(t)); \quad g(t) \leq G^{-1}(Q_0(t)) \tag{1.8}$$

for all $t \geq 0$, in which

$$\begin{aligned} Q_0(t) = \widetilde{C}_{17}(t) + \widetilde{\sigma}(t)\Big\{ & C_{18} \exp \int_0^t \Big[h_{13}(s)\widetilde{\sigma}(s) \\ & + \int_0^s h_{14}(s,\tau)\widetilde{\sigma}(\tau)d\tau + \int_0^s \int_0^\tau h_{15}(s,\tau,x)\widetilde{\sigma}(x)dx\,d\tau\Big]\,ds \\ & + \int_0^t \Big\{\Big[h_{13}\widetilde{C}_{17}(s) + \int_0^s h_{14}(s,\tau)\widetilde{C}_{17}(\tau)\,d\tau \\ & + \int_0^s \int_0^\tau h_{15}(s,\tau,x)\widetilde{C}_{17}(x)\,dx\,d\tau\Big] \exp \int_s^t \Big[h_{13}(\tau)\widetilde{\sigma}(\tau) \\ & + \int_0^\tau h_{14}(\tau,x)\widetilde{\sigma}(x)dx \\ & + \int_0^\tau \int_0^y h_{15}(\tau,y,x)\widetilde{\sigma}(x)dx\,dy\Big]\,d\tau\Big\}\,ds\Big\}, \end{aligned} \tag{1.9}$$

where

$$\widetilde{C}_{17}(t) = \alpha_1(t)G(C_1(t)\alpha_1^{-1}(t)) + \alpha_1(t)G(C_9(t)\alpha_1^{-1}(t)),$$
$$\widetilde{\sigma}(t) = \alpha_2(t)G(\sigma(t)\alpha_2^{-1}(t)).$$

THEOREM 1.3. *Let the conditions of Theorem* 1.1 *be true. Let $F(\tau)$ be a continuous strictly increasing, submultiplicative and subadditive for $r > 0$; $F(0) = 0$. Let instead of* (1.1) *and* (1.2) *the following inequalities be true*

$$\begin{aligned} f(t) \leq C_1(t) + \sigma(t)F^{-1}\Big\{ & C_2 + C_3 \int_0^t h_1(s)F(f(s))\,ds \\ & + C_4 \int_0^t h_2(s)e^{\mu_2 s}F(g(s))ds + C_5 \int_0^t \int_0^s h_3(s,\tau)F(f(\tau))d\tau\,ds \\ & + C_6 \int_0^t \int_0^s h_4(s,\tau)e^{\mu_2 \tau}F(g(\tau))d\tau\,ds \\ & + C_7 \int_0^t \int_0^s \int_0^\tau h_5(s,\tau,x)F(f(x))dx\,d\tau\,ds \end{aligned} \tag{1.10}$$

$$+ C_8 \int_0^t \int_0^s \int_0^\tau h_6(s,\tau,x)e^{\mu_2 x}F(g(x))dx\, d\tau\, ds\Big\},$$

$$g(t) \le C_9(t) + \sigma(t)F^{-1}\Big\{C_{10} + C_{11}\int_0^t h_7(s)e^{-\mu_1 s}F(f(s))\,ds \tag{1.11}$$

$$+ C_{12}\int_0^t h_8(s)F(g(s))ds + C_{13}\int_0^t\int_0^s h_9(s,\tau)F(g(\tau))\,d\tau\, ds$$

$$+ C_{14}\int_0^t\int_0^s h_{10}(s,\tau)e^{-\mu_1\tau}F(f(\tau))\,d\tau\, ds$$

$$+ C_{15}\int_0^t\int_0^s\int_0^\tau h_{11}(s,\tau,x)F(g(x))dx\, d\tau\, ds$$

$$+ C_{16}\int_0^t\int_0^s\int_0^\tau h_{12}(s,\tau,x)e^{-\mu_1 x}F(f(s))dx\, d\tau\, ds\Big\},$$

for all $t \ge 0$, where F^{-1} is the inverse function of F.

Then

$$f(t) \le F^{-1}(e^{\mu_1 t}Q_1(t)); g(t) \le F^{-1}(Q_1(t)), \tag{1.12}$$

where $Q_1(t)$ is defined in (1.9)

$$(\widetilde{C}_{17}(t) = F(C_1(t)) + F(C_9(t)); \widetilde{\sigma}(t) = F(\sigma(t))).$$

2. Proofs of main results

At first the following assertion is to be proved.

LEMMA 2.1. *Let $C_1(t), v(t), W(t), \sigma(t)$ $(t \ge 0)$, $w(t,s)$ $(t \ge s \ge 0)$, $H(t,s,\tau)$ $(t \ge s \ge \tau)$ be real-valued, nonnegative, continuous functions and $C_i (i = 2,3,4)$ be nonnegative constants such that*

$$W(t) \le C_1(t) \tag{2.1}$$

$$+ \sigma(t)\Big\{C_2 + C_3\int_0^t \Big[v(s)W(s) + \int_0^s w(s,\tau)W(\tau)d\tau\Big]\,ds$$

$$+ C_4\int_0^t\int_0^s\int_0^\tau H(s,\tau,x)W(x)dx\, d\tau\, ds\Big\}.$$

Then

$$W(t) \leq C_1(t) + \sigma(t)\Big[C_2 \exp \int_0^t v(s)\,ds + \int_0^t \Big[v_1(s) \exp \int_s^t v_2(\tau)d\tau\Big]\,ds\Big], \tag{2.2}$$

where

$$\begin{aligned} v_1(t) &= C_1(t)C_3v(t) + C_3 \int_0^t w(t,\tau)C_1(\tau)d\tau \\ &\quad + C_4 \int_0^t \int_0^s H(t,s,\tau)C_1(\tau)d\tau\,ds; \\ v_2(t) &= C_3v(t)\sigma(t) + C_3 \int_0^t w(t,s)\sigma(s)ds \\ &\quad + C_4 \int_0^t \int_0^s H(t,s,\tau)\sigma(\tau)d\tau\,ds \end{aligned} \tag{2.3}$$

for all $t \geq 0$.

Proof. Let's denote

$$\begin{aligned} U(t) &= C_2 + C_3 \int_0^t \Big[v(s)W(s) + \int_0^s w(s,t)W(\tau)\,d\tau\Big]\,ds \\ &\quad + C_4 \int_0^t \int_0^s \int_0^\tau H(s,\tau,x)W(x)\,dx\,d\tau\,ds. \end{aligned} \tag{2.4}$$

Now we rewrite (2.1) as

$$W(t) \leq C_1(t) + \sigma(t)U(t). \tag{2.5}$$

Under the assumption of this lemma $U(t)$ is a nonnegative, monotone nondecreasing, continuously differentiable function for all $t \geq 0$. Using (2.4) and (2.5) we have

$$\begin{aligned} U'(t) &\leq C_3v(t)[C_1(t) + \sigma(t)U(t)] + C_3 \int_0^t w(t,\tau)\Big[C_1(t) + \sigma(\tau)U(\tau)\Big]d\tau \\ &\quad + C_4 \int_0^t \int_0^s H(t,s,\tau)[C_1(\tau) + \sigma(\tau)U(\tau)]d\tau\,ds \\ &\leq v_1(t) + v_2(t)U(t), \end{aligned} \tag{2.6}$$

where $v_1(t)$ and $v_2(t)$ are as defined in (2.3).

From the differential inequality (2.5) we obtain the desired result. The lemma is proved.

Remark 1. If $C_1(t)$ and $\sigma(t)$ are nonnegative constants, we come to the result of Theorem 2.1 [1]. If $C_1(t)$ is a nonnegative constant and $H(t,s,\tau) \equiv 0$, we obtain the result of Theorem 2.2. [1].

Proof of Theorem 1.1. Multiplying both parts of (1.1) by $e^{-\mu_1 t}$ we have

$$(2.7)\qquad \begin{aligned} e^{-\mu_1 t} f(t) \le C_1(t) + \sigma(t)\Big\{ & C_2 + C_3 \int_0^t h_1(s) e^{-\mu_1 s} f(s)\, ds \\ & + C_4 \int_0^t h_2(s) g(s) ds + C_5 \int_0^t \int_0^s h_3(s,\tau) e^{-\mu_1 \tau} f(\tau)\, d\tau\, ds \\ & + C_6 \int_0^t \int_0^s h_4(s,\tau) g(\tau)\, d\tau\, ds \\ & + C_7 \int_0^t \int_0^s \int_0^\tau h_5(s,\tau,x) e^{-\mu_1 x} f(x) dx\, d\tau\, ds \\ & + C_8 \int_0^t \int_0^s \int_0^\tau h_6(s,\tau,x) g(x) dx\, d\tau\, ds \Big\}. \end{aligned}$$

Adding (2.7) and (1.2) and setting $W(t) = e^{-\mu_1 t} f(t) + g(t)$ we obtain the inequality

$$(2.8)\qquad \begin{aligned} W(t) \le C_1(t) + C_9(t) + \sigma(t)\Big\{ & C_2 + C_{10} \\ & + \int_0^t [C_{11} h_7(s) + C_3 h_1(s)] e^{-\mu_1 s} f(s) ds \\ & + \int_0^t [C_4 h_2(s) + C_{12} h_8(s)] g(s) ds \\ & + \int_0^t \int_0^s [C_{13} h_9(s,\tau) + C_6 h_4(s,\tau)] g(\tau) d\tau\, ds \\ & + \int_0^t \int_0^s \int_0^\tau [C_{16} h_{12}(s,\tau,x) + C_7 h_5(s,\tau,x)] e^{-\mu_1 x} f(x) dx \Big\} \end{aligned}$$

or

$$(2.9)\quad W(t)\le C_{17}(t)+\sigma(t)\Big\{C_{18}+\int_0^t h_{13}(s)W(s)ds$$
$$+\int_0^t\int_0^s h_{14}(s,\tau)W(\tau)d\tau\,ds+\int_0^t\int_0^s\int_0^\tau h_{15}(s,\tau,x)W(x)dx\,d\tau\,ds\Big\},$$

where $C_{17}(t), C_{18}, h_{13}(t), h_{14}(t,s), h_{15}(t,s,\tau)$ are as defined in (1.5). From (2.9) by virtue of Lemma we have

$$(2.10)\qquad W(t)\le Q(t),$$

where $Q(t)$ is as defined in (1.4). The inequality (2.10) implies that the desired bounds in (1.3) are obtained.

Proof of Theorem 1.2. Rewrite (1.6) and (1.7) as below

$$(2.11)\quad f(t)\le \alpha_1(t)C_1(t)\alpha_1^{-1}(t)$$
$$+\alpha_2(t)\sigma(t)\alpha_2^{-1}G^{-1}\Big\{C_2+C_3\int_0^t h_1(s)G(f(s))ds$$
$$+C_4\int_0^t e^{\mu_2 s}G(g(s))ds+C_5\int_0^t\int_0^s h_3(s,\tau)G(f(\tau))d\tau\,ds$$
$$+C_6\int_0^t\int_0^s h_4(s,\tau)e^{\mu_2 t}G(g(\tau))d\tau\,ds$$
$$+C_7\int_0^t\int_0^s\int_0^\tau h_5(s,\tau,x)G(f(x))dx\,d\tau\,ds$$
$$+C_8\int_0^t\int_0^s\int_0^\tau h_6(s,\tau,x)e^{\mu_2 x}G(g(x))dx\,d\tau\,ds\Big\},$$

$$(2.12)\quad g(t)\le \alpha_1(t)C_9(t)\alpha_1^{-1}(t)+\alpha_2(t)\sigma(t)\alpha_2^{-1}G^{-1}\Big\{C_{10}$$
$$+C_{11}\int_0^t e^{-\mu_1 s}h_7(s)G(f(s))\,ds+C_{12}\int_0^t h_8(s)G(g(s))\,ds$$
$$+C_{13}\int_0^t\int_0^s h_9(s,\tau)G(g(\tau))\,d\tau\,ds$$
$$+C_{14}\int_0^t\int_0^s h_{10}(s,\tau)e^{-\mu_1\tau}G(f(\tau))\,d\tau\,ds$$

$$+ C_{15} \int_0^t \int_0^s \int_0^\tau h_{11}(s,\tau,x) G(g(x))\, dx\, d\tau\, ds$$

$$+ C_{16} \int_0^t \int_0^s \int_0^\tau h_{12}(s,\tau,x) e^{-\mu_1 x} G(f(x)) dx\, d\tau\, ds \Big\}.$$

Since $G(r)$ is submultiplicative, converse [5] and strictly increasing, we have from (2.11) and (2.12)

$$\begin{aligned}(2.13)\quad G(f(t)) \le{}& \alpha_1(t) G(C_1(t)\alpha_1^{-1}(t)) + \alpha_2(t) G(\sigma(t)\alpha_2^{-1}(t)) \Big\{ C_2 \\ &+ C_3 \int_0^t h_1(s) G(f(s)) ds + C_4 \int_0^t h_2(s) e^{\mu_2 s} G(g(s))\, ds \\ &+ C_5 \int_0^t \int_0^s h_3(s,\tau) G(f(\tau))\, d\tau\, ds \\ &+ C_6 \int_0^t \int_0^s h_4(s,\tau) e^{\mu_2 \tau} G(g(\tau))\, d\tau\, ds \\ &+ C_7 \int_0^t \int_0^s \int_0^\tau h_5(s,\tau,x) G(f(x))\, dx\, d\tau\, ds \\ &+ C_8 \int_0^t \int_0^s \int_0^\tau h_6(s,\tau,x) e^{\mu_2 x} G(g(x))\, dx\, d\tau\, ds \Big\},\end{aligned}$$

$$\begin{aligned}(2.14)\quad G(g(t)) \le{}& \alpha_1(t) G(C_9(t)\alpha_1^{-1}(t)) \\ &+ \alpha_2(t) G(\sigma(t)\alpha_2^{-1}(t)) \Big\{ C_{10} + C_{11} \int_0^t h_7(s) e^{-\mu_1 s} G(f(s))\, ds \\ &+ C_{12} \int_0^t h_8(s) G(g(s))\, ds + C_{13} \int_0^t \int_0^s h_9(s,\tau) G(g(\tau))\, d\tau\, ds \\ &+ C_{14} \int_0^t \int_0^s h_{10}(s,\tau) e^{-\mu_1 \tau} G(f(\tau))\, d\tau\, ds \\ &+ C_{15} \int_0^t \int_0^s \int_0^\tau h_{11}(s,\tau,x) G(g(x))\, dx\, d\tau\, ds \\ &+ C_{16} \int_0^t \int_0^s \int_0^\tau h_{12}(s,\tau,x) e^{-\mu_1 x} G(f(x)) dx\, d\tau\, ds \Big\}.\end{aligned}$$

Let's denote

$$\widetilde{f}(t) = G(f(t)), \widetilde{g}(t) = G(g(t)),$$

$$\widetilde{C}_1(t) = \alpha_1(t)G(C_1(t)a_1^{-1}(t)), \widetilde{C}_9(t) = \alpha_1(t)G(C_9(t)\alpha_1^{-1}(t)), \tag{2.15}$$

$$\widetilde{\sigma}(t) = \alpha_2(t)G(\sigma(t)\alpha_2^{-1}(t)).$$

It follows from (2.13) and (2.14) that for the functions $\widetilde{f}(t)$ and $\widetilde{g}(t)$ the inequalities (1.1)-(1.2) are true (there are $\widetilde{C}_1(t), \widetilde{C}_9(t)$ and $\widetilde{\sigma}(t)$ instead of $C_1(t), C_9(t)$ and $\sigma(t)$ correspondingly).

Then by Theorem 1.1 we conclude that

$$\widetilde{f}(t) \leq e^{\mu_1 t}Q_0(t), \widetilde{g}(t) \leq Q_0(t),$$

where the function $Q_0(t)$ is as defined in (1.9). Q.E.D.

Proof of Theorem 1.3 is analogous to that of Theorem 1.2. The bounds in (1.12) follow by first using the fact that F is subadditive [5], submultiplicative and strictly monotonous and then applying Theorem 1.1 with $\widetilde{f}(t) = F(f(t)), \widetilde{g}(t) = F(g(t)), \widetilde{\sigma}(t) = F(\sigma(t)), \widetilde{C}_1(t)) = F(C_1(t)), \widetilde{C}_9(t) = F(C_9(t))$.

3. Examples

It's known that the equations of motion of viscoelastic medium can be written as [3]

$$p\frac{\partial^2 u}{\partial t^2} = pf_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i = 1, 2, \ldots.). \tag{3.1}$$

Here $u = (u_i)$ is a vector of some transfer along the direction of the i^{th} coordinate axis (there is the fixed orthogonal coordinate system x_1, x_2, x_3); σ_{ij} are the stress tensor's components; F_i is a projection of the outer mass-strength; p is a medium density.

There is the next relation [3] between the stress tensor and the strain tensor

$$\begin{aligned} \sigma_{ij} = & \int_0^1 \Gamma^{(1)}_{iji_1j_1}(t, \tau_1)\varepsilon_{i_1j_1}(\tau_1)d\tau \\ & + \int_0^t \int_0^t \Gamma^{(2)}_{iji_1j_1}t, (\tau_1, \tau_2)\varepsilon_{i_1j_1}(\tau_1)\varepsilon_{i_2j_2}(\tau_2)d\tau_1\, d\tau_2 + \ldots, \end{aligned} \tag{3.2}$$

where ε_{ij} are the strain tensor's components; $\Gamma^{(l)}_{iji_1j_1}$ are integral kernels.

Substituting the value of (3.2) in (3.1) and using the fact that

$$2\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

we come to the nonlinear integrodifferential equations describing motion of visco-elastic medium. These equations contain some multiple integrals which are much similar to ones of the right-hand sides in (1.6)–(1.7) or (1.10)–(1.11) (if $G(r)$ or $F(r)$ are chosen appropriately).

Let, for example, $G(r) = r^\rho (\rho \geq 1)$. r^ρ is a convex function [5] and it's clear that this function satisfies all the hypothesises of Theorem 1.2., i.e., from the inequalities

$$\begin{aligned} f(t) \leq C_1(t) + \sigma(t)\Big\{ & C_2 + C_3 \int_0^t h_1(s) f^\rho(s)\, ds \\ & + C_4 \int_0^t h_2(s) e^{\mu_2 s} g^\rho(s)\, ds \\ & + C_5 \int_0^t \int_0^s h_3(s,\tau) f^\rho(\tau) d\tau\, ds + C_6 \int_0^t \int_0^s h_4(s,\tau) e^{\mu_2 \tau} g^\rho(\tau)\, d\tau\, ds \\ & + C_7 \int_0^t \int_0^s \int_0^\tau h_5(s,\tau,x) f^\rho(x) dx\, d\tau\, ds \\ & + C_8 \int_0^t \int_0^s \int_0^\tau h_6(s,\tau,x) e^{\mu_2 x} g^\rho(x) dx\, d\tau\, ds \Big\}^{1/\rho}, \end{aligned}$$

$$\begin{aligned} g(t) \leq C_9(t) + \sigma(t)\Big\{ & C_{10} + C_{11} \int_0^t h_7(s) e^{-\mu_1 s} f^\rho(s)\, ds \\ & + C_{12} \int_0^t h_8(s) g^\rho(s)\, ds + C_{13} \int_0^t \int_0^s h_9(s,\tau) g^\rho(\tau) d\tau\, ds \\ & + C_{14} \int_0^t \int_0^s h_{10}(s,\tau) e^{-\mu_1 x} f^\rho(\tau)\, d\tau\, ds \\ & + C_{15} \int_0^t \int_0^s \int_0^\tau h_{11}(s,\tau,x) g^\rho(x)\, dx\, d\tau\, ds \\ & + C_{16} \int_0^t \int_0^s \int_0^\tau h_{12}(s,\tau,x) e^{-\mu_1 x} f^\rho(x) dx\, d\tau\, ds \Big\}^{1/\rho}, \end{aligned}$$

we conclude that

$$f(t) \leq (e^{\mu_1 t} Q_0(t))^{\frac{1}{\rho}}, g(t) \leq (Q_0(t))^{\frac{1}{\rho}},$$

where $Q_0(t)$ is as defined in (1.9)

$$(\widetilde{C}_{17}(t) = \alpha_1(t)(C_1(t)\alpha_1^{-1}(t))^{\rho} + \alpha_1(t)(C_9(t)\alpha_1^{-1}(t))^{\rho},$$
$$\widetilde{\sigma}(t) = \alpha_2(t)(\sigma(t)\alpha_2^{-1}(t))^{\rho}).$$

4. Concluding remarks

Remark 2. Let

$$h_i(i = 3,4,5,6,9,10,11,12) \equiv 0, \quad C_2 = C_{10} = 0, \quad \mu_1 = \mu_2 = \mu$$

in Theorems 1.1–1.3. Then we come to the inequalities of Pachpatte [4]. If in addition $\sigma(t) = 1, C_1(t) = K_1, C_9(t) = K_2$, where K_1, K_2 are nonnegative constants and the functions $h_i(i = 1,2,7,8)$ are bounded in Theorem 1.1, we obtain the inequality of Greene [2].

Remark 3. The lemma proved in the paper can be generalized for the inequality

$$\begin{aligned}
(4.1)\quad W(t) \leq C_1(t) + \sigma(T)\Big\{C_2 + C_3 \int_0^t \Big[v(s)W(s) \\
+ \int_0^s w(s,\tau)W(\tau)d\tau\Big]ds + C_4 \int_0^t \int_0^s \int_0^\tau H_1(s,\tau,x)W(x)dx\,d\tau\,ds \\
+ C_5 \int_0^t \int_0^s \int_0^\tau \int_0^{x_1} H_2(s,t,x,x_1)W(x_1)dx_1\,dx\,d\tau\,ds + \ldots + \\
+ C_{n+3} \int_0^t \int_0^s \int_0^\tau \int_0^{x_1} \cdots \\
\cdots \int_0^{x_{n-1}} H_n(s,t,x,x_1\ldots x_{n-1})W(x_{n-1})dx_{n-1}\ldots dx_1\,dx\,d\tau\,ds\Big\},
\end{aligned}$$

where $W(t), v(t), C_1(t), \sigma(t)(t \geq 0), w(t,s)(t \geq s \geq 0)$, $H_j(j = \overline{1n})$ are nonnegative and continuous functions defined for $t \geq s \geq \tau \geq x \geq x_1 \geq \ldots \geq x_{n-1} \geq 0$ and C_j $(j = \overline{2n+3})$ are nonnegative constants.

In fact an estimate analogous to (2.2) is also valid here. The proof of this assertion follows as above (see the proof of the lemma) and is therefore omitted. So, our results can be established for systems of integral inequalities given in the form of (4.1) as well.

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