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UMBILICAL POINTS ON ANALYTIC SURFACES DIFFEOMORPHIC WITH THE 2-DIMENSIONAL SPHERE IN EUCLIDEAN 3-DIMENSIONAL SPACE

1. Introduction

This paper is an information about some results on sets of umbilical points on analytic surfaces diffeomorphic with the 2-dimensional sphere S^2 in the 3-dimensional Euclidean space E^3. Let

$$x : S^2 \to E^3, x(S^2) \neq S^2(r), r > 0, \tag{1.1}$$

denotes an analytic diffeomorphism, where $S^2(r) \subset E^3$ denotes the metric sphere of radius r, and $S^2(1) = S^2$. In the following we identify $p \in S^2$ with $x(p) \in x(S^2)$, and if $x(\phi(t_1, t_2))$, $t_2 = \text{const}$ respectively $t_1 = \text{const}$, are lines of curvature on $x(S^2)$, then the underlying curves $\phi(t_1, t_2)$, $t_2 = \text{const}$ respectively $t_1 = \text{const}$, are called lines of curvature on S^2. Since (1.1) is an imbedding and we do not consider properties of lines of curvature in E^3, such an identification is possible.

We distinguish on S^2 regular and chief umbilical points. Geometrically, regular and chief umbilical points can be described as follows. A regular umbilical point has the property that there exists a neighbourhood $Y \subset S^2$ of this point such that the lines of curvature in Y behaves as in a non umbilical point. A chief umbilical point is such at which the parametrization of a neighbourhood $Y \subset S^2$ of such a point by means of lines of curvature admits a singularity at this point. The set of chief umbilical points of (1.1) is denoted by $\Xi_c \subset S^2$.

We assume that (1.1) besides $x(S^2) \neq S^2(r)$ satisfies a further condition which roughly speaking asserts that Ξ_c does not contain arcs. Such imbeddings (1.1) are called punctured. For punctured imbeddings (1.1) the set $\Xi_c \subset S^2$ is compact and totally disconnected, and therefore Ξ_c form a "puncture" on the orthogonal net of lines of curvature on S^2. Punctured imbeddings (1.1) are the only one considered in this paper.

In the following by $\overline{A}$ we denote the closure an by $\mathrm{Int}(A)$ the interior of a set A.

Let $\phi : P \to S^2$ denotes an analytic diffeomorphism of an open set $P \subset E^2$, where E^2 denotes the Euclidean plane, such that there exists a prolongation $\overline{\phi} : \overline{P} \to S^2$ of ϕ to a continuous mapping $\overline{\phi}$ defined on the closure $\overline{P}$ of P. The mapping $\overline{\phi}$ is called analytic at a point $(t_{10}, t_{20}) \in \in \overline{P} \setminus P$, if there exists a neighbourhood $A \subset E^2$ of (t_{10}, t_{20}) and an analytic diffeomorphism $\phi_1 : A \to S^2$ such that $\overline{\phi}(t_1, t_2) = \phi_1(t_1, t_2)$ for every $(t_1, t_2) \in \overline{P} \cap A$.

Unit vectors of E^2 added to E^2 are called points "at infinity" of E^2, and E^2 completed by these vectors is homeomorphic with the unit closed disk $\overline{D}^2 \subset E^2$. A point "at infinity" of $P \subset E^2$ is such a unit vector of E^2 that there exists an unbounded sequence $(t_1^n, t_2^n) \in P$, $1 \le n < \infty$, which defines an oriented asymptotic straightline with the direction defined by this vector.

The aim of this paper is the following

PARAMETRIZATION THEOREM. *For every punctured imbedding* (1.1) *there exists an open, connected and simply connected set* $P \subset E^2$ *and a continuous mapping*

$$\overline{\phi} : \overline{P} \to S^2, \overline{\phi}(\overline{P}) = S^2, \tag{1.2}$$

with the following properties; a) the restriction $\overline{\phi}/_P = \phi$ *is an analytic diffeomorphism, and* $\overline{\phi}$ *is analytic for every* $(t_1, t_2) \in \overline{P} \setminus P$ *such that* $\overline{\phi}(t_1, t_2) \in S^2 \setminus \Xi_c$; *if* $\overline{\phi}(t_1, t_2) \in \Xi_c$, *then* $\overline{\phi}$ *is continuous but not analytic at* $(t_1, t_2) \in \overline{P} \setminus P$, *b)* $\Xi_c \subset \overline{\phi}(\overline{P}) \setminus \phi(P)$ *and* Ξ_c *is compact and totally disconnected and every point "at infinity" of* P *belongs to* $\overline{\phi}^{-1}(\Xi_c) \subset \overline{P} \setminus P$, *c)* $P = \mathrm{Int}(\overline{P})$, *d)* $\phi(t_1, t_2)$, $t_2 = \mathrm{const}$ *respectively* $t_1 = \mathrm{const}$, $(t_1, t_2) \in P$, *are lines of curvature of* (1.1) *on* S^2.

By specializing punctured imbeddings (1.1) to periodic imbeddings (1.1) we get a further result which asserts that the set of chief umbilical points of a periodic imbedding (1.1) contains at least 2 and at most 7 elements.

2. Preliminaries

By $\{(a, U), (b, V)\}$ we denote an analytic atlas on S^2 with chartes (a, U), (b, V) such that $U \cup V = S^2$ and $a : U \to D_u, b :\to D_v$ are homeomorphisms of $U, V \subset S^2$ on open sets $D_u, D_v \subset E^2$. We assume that D_u and D_v are analytic diffeomorphic with the unit open disk $D^2 \subset E^2$ and $D_u, D_v \subset D^2$. The transition functions

$$(u_1, u_2) = ab^{-1}(v_1, v_2), (v_1, v_2) = ba^{-1}, \tag{2.1}$$

$(u_1, u_2) \in D_u$, $(v_1, v_2) \in D_v$, are analytic. We suppose that (a, U), (b, V) are orthogonal with respect to the Riemannian metric induced on S^2 by (1.1), i. e. that

$$(2.2)\quad ds^2 = g_{11}du_1^2 + g_{22}du_2^2 \quad \text{in } (a, U), ds^2 = h_{11}dv_1^2 + h_{22}dv_2^2 \quad \text{in } (b, V)$$

and the orthogonality condition of (2.1) has the form

$$(2.3)\qquad h_{12} = g_{11}\frac{\partial u_1}{\partial v_1}\frac{\partial u_1}{\partial v_2} + g_{22}\frac{\partial u_2}{\partial v_1}\frac{\partial u_2}{\partial v_2} = 0, g_{12} = 0.$$

In this case that (2.3) are satisfied (2.1) are called orthogonal transformations. The second quadratic form of (1.1) in (a, U) has the form

$$(2.4)\qquad n.d^2x = \sum_{i,j=1}^{2} L_{ij}du_idu_j,$$

where n denotes the unit normal vector to $x(S^2)$ at $x(p)$, $p = a^{-1}(u_1, u_2)$. The equation of lines of curvature in orthogonal coordinates in (a, U) has the form

$$(2.5)\qquad -\alpha_u du_1 du_2 + \frac{\beta_u}{2\sqrt{g_{11}g_{22}}}(g_{22}du_2^2 - g_{11}du_1^2) = 0,$$

where

$$(2.6)\qquad \alpha_u(u_1, u_2) = -\left(\frac{L_{11}}{g_{11}} - \frac{L_{22}}{g_{22}}\right), \quad \beta_u(u_1, u_2) = \frac{2L_{12}}{\sqrt{g_{11}g_{22}}}.$$

We define

$$(2.7)\quad H_u^+ = \{(u_1, U_2) \in D_u \mid \alpha_u > 0\}, \quad H_u^- = \{(u_1, u_2) \in D_u \mid \alpha_u < 0\},$$

$$(2.8)\qquad H_u = H_u^+ \cup H_u^-, H_u^0 = \{(u_1, u_2) \in D_u \mid \alpha_u(u_1, u_2) = 0\},$$

$$(2.9)\qquad G_u = \{(u_1, u_2) \in D_u \mid \beta_u(u_1, u_2) \neq 0\}.$$

Analogously, denoting by M_{ij}, $i, j = 1, 2$, the components of the second quadratic form of (1.1) in (b, V) we define α_v, β_v, H_v^+, H_v^-, H_v^0, G_v and (2.5) in (b, V).

The set of umbilical points in (a, U) respectively (b, V) is defined by $H_u^0 \setminus G_u \subset D_u$ respectively $H_v^0 \setminus G_v \subset D_v$. The points $a^{-1}(u_{10}, u_{20}) = p_0$, $b^{-1}(v_{10}, v_{20}) = q_0$ where $(u_{10}, u_{20}) \in H_u^0 \setminus G_u$, $(v_{10}, v_{20}) \in H_v^0 \setminus G_v$, $p_0, q_0 \in S^2$, are called umbilical points of (1.1). If $a^{-1}(u_{10}, u_{20}) \in V$, then (v_{10}, v_{20}) defined by the second formula of (2.1) is an umbilical point in (b, V), and we identify $a(p_0) = (u_{10}, u_{20})$ and $b(p_0) = (v_{10}, v_{20})$ with $p_0 \in S^2$. The set of umbilical points of (1.1) is denoted by $\Xi \subset S^2$.

3. The system of partial differential equations of lines of curvature

PROPOSITION 3.1. *The set $\Xi \subset S^2$ is closed and nowhere dense.*

Proof. The proof we get by the contrary argument using analycity of (1.1). From Proposition 3.1 it follows that $S^2 \setminus \Xi \neq \emptyset$.

The equation (2.5) can be written in general orthogonal coordinates with the following property.

LEMMA 3.1. *There exists an orthogonal transformation $\mu : D_w \to D_u$ of $D_w \subset D^2$ such that $G_w \subset D_w$ is open and dense in D_w, where D_w is referred to coordinates (w_1, w_2).*

Proof. If there exists an open set $A \subset G_u$, then since β_u defined in (2.6) is analytic, it follows that $G_u \subset D_u$ is open and dense in D_u. In this case μ is the identity transformation. Let us suppose

$$\beta_u(u_1, u_2) = 0 \quad \text{for every } (u_1, u_2) \in D_u. \tag{3.1}$$

There exists an analytic function $u_2(w_1, w_2)$ $(w_1, w_2) \in D^2$, such that

$$\frac{\partial u_1}{\partial w_1}\frac{\partial u_2}{\partial w_2} \neq 0 \quad \text{for every } (w_1, w_2) \in D^2. \tag{3.2}$$

The orthogonality condition (2.3), where instead of $(v_1, v_2) \in D_v$ we set $(w_1, w_2) \in D^2$, is a partial differential equation of the form

$$\frac{\partial u_1}{\partial w_1}\frac{\partial u_2}{\partial w_2} = f(w_1, w_2), f(w_1, w_2) \neq 0 \quad \text{for every } (w_1, w_2) \in D^2. \tag{3.3}$$

A solution of (3.3) can be find as follows. We set

$$\frac{\partial u_1}{\partial w_1} = \frac{f}{g}, \quad \frac{\partial u_1}{\partial w_2} = g, \quad g(w_1, w_2) \neq 0, \quad (w_1, w_2) \in D^2. \tag{3.4}$$

The linear system (3.4) is equivalent with (3.3) for every choice of $g \neq 0$. The system (3.4) is completely integrable, if

$$g^2\frac{\partial g}{\partial w_1} + f\frac{\partial g}{\partial w_2} = \frac{\partial f}{\partial w_2}g. \tag{3.5}$$

Since f is a known function, it follows that (3.5) is a quasilinear, partial differential equation with the seeked function g. Hence, (3.5) can be solved by means of the method of characteristics. By such a choice of g the system (3.4) is completely integrable, and therefore by the theorem of Fröbenius there exists a solution $u_1(w_1, w_2)$, $(w_1, w_2) \in D^2$, of (3.4). This proves that $\mu_1 : D^2 \to E^2$ defined by $\mu_1(w_1, w_2) = (u_1(w_1, w_2), u_2(w_1, w_2))$ satisfies the orthogonality condition (2.3), where instead of $(v_1, v_2) \in D_v$ we set $(w_1, w_2) \in D^2$. We prove that $\mu_1 : D^2 \to \mu_1(D^2)$ is a diffeomorphism. From

(2.3), where $(v_1, v_2) \in D_v$ is replaced by $(w_1, w_2) \in D^2$, it follows that the Jacobi determinant Δ of μ_1 multiplied by $\dfrac{\partial u_2}{\partial w_2}$ has the form

$$\Delta \frac{\partial u_2}{\partial v_2} = \frac{1}{g_{22}} \left(g_{11} \left(\frac{\partial u_1}{\partial w_2} \right)^2 + g_{22} \left(\frac{\partial u_2}{\partial w_2} \right)^2 \right). \tag{3.6}$$

From (2.3), where $(v_1, v_2) \in D^2$ is replaced by $(w_1, w_2) \in D^2$, and (3.2) it follows

$$\Delta(w_1, w_2) \neq 0 \quad \text{for every } (w_1, w_2) \in D^2. \tag{3.7}$$

Since D^2 is connected and simply connected, it follows from (3.7) that μ_1 is a diffeomorphism. Hence, it follows that μ_1 is an orthogonal transformation. By (2.3) , where $(v_1, v_2) \in D_v$ is replaced by $(w_1, w_2) \in D^2$, and (3.3) it follows easely that together with μ_1 orthogonal is every transformation $\mu_2 = \alpha\mu_1 + (\gamma_1, \gamma_2)$, where α is a positive constant and (γ_1, γ_2) is a constant vector. By assumption we have $D_u \subset D^2$. Therefore there exist α and (γ_1, γ_2) such that $D_u \subset \mu_2(D^2)$. We define $D_w = \mu_2^{-1}(d_u)$ and $\mu = \mu_2/D_w$. By k_{ii}, N_{ij}, $i, j = 1, 2$, we denote the components of the first and second quadratic forms of (1.1) in the coordinates $(w_1, w_2) \in D_w$. From the transformation rule of the components of the second quadratic form, (2.4), the second formula of (2.6) and (3.1) we get

$$N_{12} = -\frac{\partial u_2}{\partial w_1} \frac{\partial u_2}{\partial w_2} g_{22} \left(\frac{L_{11}}{g_{11}} - \frac{L_{22}}{g_{22}} \right), \qquad (w_1, w_2) \in D_w. \tag{3.8}$$

If $\alpha_u(u_1, u_2) = 0$ for every $(u_1, u_2) \in D_u$, then from (3.1) it follows that every point $a^{-1}(u_1, u_2) \in S^2$, $(u_1, u_2) \in D_u$, is umbilical, and therefore $x(S^2) = S^2(r)$, $r > 0$, contrary to the condition in (1.1). Thus, there exists an open set $A \subset H_u$. From the first formula (2.6), (2.7), (3.1) and (3.8) it follows

$$N_{12}(w_1, w_2) \neq 0 \quad \text{for every } (w_1, w_2) \in \mu^{-1}(A), \tag{3.9}$$

where $\mu^{-1}(A)$ is an open set diffeomorphic with A. We define

$$G_w = \{(w_1, w_2) \in D_w \mid \beta_w \neq 0\}, \quad \beta_w(w_1, w_2) = \frac{2N_{12}}{\sqrt{k_{11}k_{22}}}. \tag{3.10}$$

Then from (3.9) and (3.10) it follows that $G_w \subset D_w$ is open and dense in D_w. This ends the proof.

As an immediate consequence of Lemma 3.1 we get

PROPOSITION 3.2. *For every analytic imbedding* (1.1) *there exists an atlas* $\{(a, U), (b, V)\}$ *of orthogonal charts on* S^2 *such that* $G_u \subset D_u$ *and* $G_v \subset D_v$ *are open and dense sets in* D_u *and* D_v *respectively.*

In the following we suppose that the atlas $\{(a, U), (b, V)\}$ has the property of Proposition 3.2. Hence, we set $\overline{G_u} = D_u$, $\overline{G_v} = D_v$, where $\overline{G_u}$, $\overline{G_v}$ denote the closures of G_u, G_v relative to D_u, D_v respectively.

For every point of $H_u \cup G_u \subset D_u$ as a consequence of (2.5) we get the following system of partial differential equations. If $(u_1, u_2) \in G_u$, then it follows from (2.5)

$$\frac{\partial u_1}{\partial t_1} = f_1, \quad \frac{\partial u_2}{\partial t_1} = f_1 \sqrt{\frac{g_{11}}{g_{22}}} \frac{\alpha_u \pm \sqrt{\alpha_u^2 + \beta_u^2}}{\beta_u}, \tag{3.11}$$

$$\frac{\partial u_1}{\partial t_2} = f_2 \frac{g_{22}}{g_{11}} \frac{\beta_u}{\alpha_u \mp \sqrt{\alpha_u^2 + \beta_u^2}}, \quad \frac{\partial u_2}{\partial t_2} = f_2, \tag{3.12}$$

where f_1, f_2 are analytic functions defined on $H_u \cup G_u \subset D_u$, different from zero and such that the system (3.11), (3.12) is completely integrable.

If $(u_{10}, u_{20}) \in H_u \setminus G_u$, then we set

$$\frac{\partial u_1}{\partial t_1} = a_1, \quad \frac{\partial u_2}{\partial t_1} = 0, \quad \frac{\partial u_1}{\partial t_2} = 0, \quad \frac{\partial u_2}{\partial t_2} = a_2, \tag{3.13}$$

where $a_i = f_i(u_{10}, u_{20})$, $i = 1, 2$. This definition is correct in the following sense. If $(u_1, u_2) \in G_u$ tends to (u_{10}, u_{20}), then directly from (2.3) we get (3.13). The same result we get from (3.11), (3.12) provided the signs before the square roots in these equations are chosen in such a manner that the right sides in (3.11), (3.12) define analytic, principal vector fields in $H_u \cup G_u$. This is the case, if the following

Signs convention. For $(u_1, u_2) \in H_u^- \cap G_u$ we choose the upper signs before the square roots in (3.11), (3.12) and the lower signs, if $(u_1, u_2) \in H_u^+ \cap G_u$ holds. This signs convention leads to the following

PROPOSITION 3.3. *Let $G \subset H_u \cup G_u$ denotes an open and connected set and*

$$(H_u^- \setminus G_u) \cap G \neq \emptyset \quad \textit{respectively} \quad (H_u^+ \setminus G_u) \cap G \neq \emptyset. \tag{3.14}$$

If we choose the upper signs respectively the lower signs before the square roots in (3.11), (3.12), *then the right side in* (3.11), (3.12) *are analytic for every* $(u_1, u_2) \in G$ *and*

$$(H_u^+ \setminus G_u) \cap G = \emptyset \quad \textit{respectively} \quad (H_u^- \setminus G_u) \cap G = \emptyset, \tag{3.15}$$

If besides (3.15) *we have* $(H_u^- \setminus G_u) \cap G = \emptyset$ *respectively* $(H_u^+ \setminus G_u) \cap G = \emptyset$, *then the right sides of* (3.11), (3.12) *in the chart* (a, U), *i.e. it can be choosed the upper or the lower signs before the square roots in* (3.11), (3.12) *or they are determined by the signs before the square roots in the corresponding to* (3.11), (3,12) *system written in* (b, V) *and the transformation of coordinates* (2.1).

Proof. In the first case of (3.14), if $(u_1, u_2) \in H_u^- \cap G_u \cap G$ tends to $(u_{10}, u_{20}) \in (H_u^- \setminus G_u) \cap G$, then choosing the upper signs in (3.11), (3.12) before the square roots we get (3.13). The proof of (3.15) is indirect. For $(u_1, u_2) \in H_u^- \cap G_u \cap G$ respectively $(u_1, u_2) \in H_u^+ \cap G_u \cap G$ tending to $(u_{11}, u_{21}) \in H_u^0 \cap G_u \cap G$ it follows from (3.11), (3.12)

$$\frac{\partial u_2}{\partial t_1} = f_1 \sqrt{\frac{g_{11}}{g_{22}}} \frac{|\beta_u|}{\beta_u}, \quad \frac{\partial u_1}{\partial t_2} = -f_2 \sqrt{\frac{g_{22}}{g_{11}}} \frac{\beta_u}{|\beta_u|} \tag{3.16}$$

respectively

$$\frac{\partial u_2}{\partial t_1} = -f_1 \sqrt{\frac{g_{11}}{g_{22}}} \frac{|\beta_u|}{\beta_u}, \quad \frac{\partial u_1}{\partial t_2} = f_2 \sqrt{\frac{g_{22}}{g_{11}}} \frac{\beta_u}{|\beta_u|}. \tag{3.17}$$

From (3.16), (3.17) it follows that the right sides in (3.11), (3.12) are discontinuous at (u_{11}, u_{21}). This proves the first formula in (3.15), the proof of the second is analogous.

Remark 3.1. Although the system (3.11), (3.12) completed by (3.13) is defined for every $(u_1, u_2) \in H_u \cup G_u$, from Proposition 3.3 we get that the signs before the square roots in (3.11), (3.12) can be uniquely foxed according to the signs convention only on any open, connected set $G \subset \subset H_u \cup G_u$. Therefore the integrating factors f_1, f_2 in fact are defined only on G. However, in the following we prove (for punctured imbeddings (1.1)) that $H_u \cup G_u$ itself is connected, and therefore the properties of f_1, f_2 in (3.11), (3.12) anticipate that what follows.

As a further consequence of Proposition 3.3 we get that, if $(u_{10}, u_{20}) \in \overline{G}$, where $\overline{G}$ denotes the closure of (u_{10}, u_{20}) the signs before the square roots in (3.11), (3.12) in general cannot be chosen in such a way that the right sides of (3.11), (3.12) remain bounded, if $(u_1, u_2) \in G$ tends to $(u_{10}, u_{20}$; they can get unbounded.

Therefore in the following besides the system (3.11), (3.12) completed by (3.13) we consider the equivalent on $G_u \subset D_u$ system

$$\frac{\partial u_1}{\partial \tau_1} = e_1 \beta_u, \frac{\partial u_2}{\tau_1} = e_1 \sqrt{\frac{g_{11}}{g_{22}}} (\alpha_u \pm \sqrt{\alpha_u^2 + \beta_u^2}), \tag{3.18}$$

$$\frac{\partial u_1}{\partial \tau_2} = e_2 \beta_u, \quad \frac{\partial u_2}{\partial \tau_2} = e_2 \sqrt{\frac{g_{22}}{g_{11}}} (\alpha_u \mp \sqrt{\alpha_u^2 + \beta_u^2}), \tag{3.19}$$

where e_1, e_2 are analytic functions in $H_u \cup G_u$, different from zero in $H_u \cup G_u$ and such that the system (3.18), (3.19) is completely integrable. The signs before the square roots in (3.18), (3.19) are chosen according to the signs convention (see also Proposition 3.3 and Remark 3.1). The systems (3.11), (3.12) completed by (3,13) and (3.18), (3.19) define on G_u the same lines of curvature, and since $\overline{G} = D_u$, it follows $H_u \subset \overline{G_u}$. Therefore, although the systems are not equivalent on $H_u \cup G_u$, their solutions coincide on $H_u \cup G_u$ in the following sense. There exists a scale transformation

$$t_1 = \chi_1(\tau_1), t_2 = \chi_2(\tau_2), \tag{3.20}$$

defined by

$$f_1 dt_1 = e_1 \beta_u d\tau_1, \quad f_2 dt_2 = e_2 \sqrt{\frac{g_{22}}{g_{11}}}(\alpha_u \mp \sqrt{\alpha_u^2 + \beta_u^2}) d\tau_2, \tag{3.21}$$

such that, if $(u_1(t_1,t_2), u_2(t_1,t_2)) \in H_u \cup G_u$ is a solution of (3.11), (3.12) completed by (3.13), defined on an open, connected set $P' \subset E^2$, then $(x_1(\tau_1,\tau_2), x_2(\tau_1,T_2)) \in H_u \cup G_u$, $(\tau_1,\tau_2) \in \Pi'$, $\Pi' \subset E^2$, where $x_i(\tau_1,\tau_2)$, $u_i(\chi_1(\tau_1),\chi_2(\tau_2))$, $i = 1,2$, is a solution of (3.18), (3.19), and $\Pi' = \chi^{-1}(P')$, $\chi^{-1} = (\chi_1^{-1}, \chi_2^{-1})$. In general the functions (3.20) are not analytic diffeomorphisms ($\beta_u \neq 0$), but analytic homeomorphisms; there can appear points (τ_1,τ_2) such that $\beta_u = 0$.

The systems of partial differential equations (3.11), (3.12) completed by (3.13) and (3.18), (3.19) are consequences of (2.5), but they are not equivalent with (2.5). This is so because a direction $du_1 : du_2 = 0 : 0$ can be definite as a limit of definite directions, while $(du_1, du_2) = (0,0)$ always defines an indefinite direction.

DEFINITION 3.1. The analytic imbedding (1.1) is called a punctured imbedding, if the equation (2.5) respectively the equation (2.5) written in (b,V) is equivalent with the system (3.18), (3.19) written in (b,V).

Punctured imbeddings (1.1) have the following property: on S^2 there does not exist a differentiable proper arc $L \subset \Xi \subset S^2$. Indeed, let $a(L) = (u_1(\tau_1(t),\tau_2(t)), u_2(\tau_1(t),\tau_2(t)))$, $(\tau_1(t),\tau_2(t)) \in \Pi'$, $0 < t < 1$. By (3.18), (3.19) we have

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial \tau_1}\frac{d\tau_1}{dt} + \frac{\partial u_i}{\partial \tau_2}\frac{d\tau_2}{dt} = 0, \quad i = 1,2.$$

Hence, it follows that the structure of the set of umbilical points on surfaces which contain arcs of umbilical points cannot be investigated by means of the systems (3.11), (3.12) completed by (3.13) or (3.18), (3.19).

4. A property of the set Ξ of umbilical points

As a direct application of the theorem of Fröbenius to the system (3.18), (3.19) we get the following

LEMMA 4.1. *For every point* $(u_{10}, u_{20}) \in H_u \cup G_u$ *and every point* $(\tau_{10}, \tau_{20}) \in E^2$ *there exists an open set* $Q_0 \subset H_u \cup G_u$, *an open rectangle*

$$\Pi_0 = \{(\tau_1,\tau_2) \in E^2 \mid |\tau_1 - \tau_{10}| < \gamma_{10}, |\tau_2 - \tau_{20}| < \gamma_{20}\}, \tag{4.1}$$

where $0 < \gamma_{10}, \gamma_{20} < \infty$, *and a uniquely defined analytic, orthogonal* (*i.e.* (2.3) *holds in the independent variables* $\tau_i, i = 1, 2$) *homeomorphism*

$$\psi_0 : \pi_0 \to Q_0, \psi_0(\Pi_0) = Q_0, \psi_0 = (\psi_{10}, \psi_{20}), \tag{4.2}$$

such that the functions (ψ_{10}, ψ_{20}) *are solutions of the system* (3.18), (3.19) *and* $\psi_0(\tau_{10}, \tau_{20}) = (u_{10}, u_{20})$.

The mapping (4.2) is in general not a diffeomorphism, since at points $(\tau_1, \tau_2) \in \Pi_0$ such that $\psi(\tau_1, \tau_2) \in \Pi_u \setminus G_u$ the determinant of the Jacobi matrix of (4.2) is zero.

As a consequence of Lemma 4.1 we get the following

THEOREM 4.1. *For every point* $(u_{10}, u_{20}) \in H_u \cup G_u$ *and* $(\tau_{10}, \tau_{20}) \in E^2$, *and every open, connected set* $G \subset H_u \cup G_u$ *such that* $(u_{10}, u_{20}) \in G$, *there exists an open, connected set* $\Gamma \subset E^2$ *such that*$(\tau_{10}, \tau_{20}) \in \Gamma$ *and a uniquely defined analytic, orthogonal homeomorphism*

$$\psi : \Gamma \to G, \psi(\Gamma) = G, \psi = (\psi_1, \psi_2), \tag{4.3}$$

such that the functions (ψ_1, ψ_2) *are solutions of the system* (3.18), (3.19) *and* $\psi(\tau_{10}, \tau_{20}) = (u_{10}, u_{20})$.

If $\overline{\psi}$ denotes a prolongation of (4.3) to a continuous mapping defined on the closure $\overline{\Gamma}$ of Γ, then $\overline{\psi}$ is called analytic at $(\tau_{10}, \tau_{20}) \in \overline{\Gamma} \setminus \Gamma$, if there exists a rectangle (4.1) and a homeomorphism (4.2) such that $\overline{\psi}(\tau_{10}, \tau_{20}) = \psi_0(\tau_1, \tau_2)$ for every $(\tau_1, T_2) \in \overline{\Gamma} \cap \Pi_0$.

By the mean value theorem of differential calculus we get

THEOREM 4.2. *Let* $G \subset H_u \cup G_u$ *denotes an open, connected set such that* $\overline{G} \subset D_u$. *There exists a unique prolongation of* (*4.3*) *to a continuous mapping*

$$\overline{\psi} : \overline{\Gamma} \to \overline{G}, \overline{\psi}(\overline{\Gamma}) = \overline{G}. \tag{4.4}$$

For every $(\tau_1, \tau_2) \in \overline{\Gamma} \setminus \Gamma$ *such that* $\overline{\psi}(\tau_1, \tau_2) \in H_u \cup G_u$ *the function* $\overline{\psi}$ *is analytic at* (τ_1, τ_2).

DEFINITION 4.1. A point $(\tau_{10}, \tau_{20}) \in \overline{\Gamma} \setminus \Gamma$ is called a parametric umbilical point, if

$$\lim \frac{\partial \psi}{\partial \tau_1} = 0, \quad \text{where } (\tau_1, \tau_2) \in \Gamma \text{ tends to } (\tau_{10}, \tau_{20}). \tag{4.5}$$

Also $(t_{10}, t_{20}) = \lim(\chi_1(\tau_1), \chi_2(\tau_2))$, where $(\tau_1, \tau_2) \in \Gamma$ tends to $(\tau_{10}, \tau_{20}$, is called in the following a parametric umbilical point.

From (3.18), (3.19) it follows that if $(\tau_{10}, \tau_{20}) \in \overline{\Gamma} \setminus \Gamma$ is a parametric umbilical point, then$\overline{\psi}(\tau_{10}, \tau_{20}) \in H_u^0 \setminus G_u$.

As a consequence of the mean value theorem for one variable we get the following

LEMMA 4.2. *Let* $h(\tau_1,\tau_2),(\tau_1,\tau_2) \in \Gamma$ *where* $\Gamma \subset E^2$ *is an open, connected set, denotes a differentiable function such that*

$$h(\tau_{12},\tau_{22}) - h(\tau_{11},\tau_{21}) = d, d \geq 0, \tag{4.6}$$

where $(\tau_{12},\tau_{22}),(\tau_{11},\tau_{21}) \in \Gamma$ *are different points. Let*

$$\varrho = \mathrm{dist}((\tau_{11},\tau_{21}),(\tau_{12},\tau_{22}), \tag{4.7}$$

where dist denotes the Euclidean distance function. By $(\tau_1(s),\tau_2(s)) \in \Gamma$, $0 \leq s \leq 1$, *we denote a differentiable arc which joins* (τ_{11},τ_{21}) *and* (τ_{12},τ_{22}). *Then there exists such a number* s_0, $0 < s_0 < 1$, *that*

$$\left(\frac{\partial h}{\partial \tau_1}\right)^2 + \left(\frac{\partial h}{\partial \tau_2}\right)^2 \geq \frac{d^2}{\varrho^2}, \tag{4.8}$$

where $\dfrac{\partial h}{\partial \tau_i}$, $i = 1,2$, *in* (4.8) *are evaluated at* $(\tau_1(s_0),\tau_2(s_0))$.

THEOREM 4.3. *For every punctured imbedding* (1.1) *the set* $\Xi \subset S^2$ *of umbilical points is compact and totally disconnected.*

Proof. The proof of Theorem 4.3 is indirect. We suppose that there exists a component $C \subset \Xi$ which contains more than one point and $C \subset U \subset S^2$. We denote $C_u = a(C)$, and we have $C_u \subset H_u^0 \setminus G_u$. There exist different points $(u_{11},u_{21}),(u_{12},u_{22}) \in C_u$. There exists an open, connected set $G \subset H_u \cup G_u$ such that $\overline{G} \cap C_u$ is a connected set and $(u_{11},u_{21}),(u_{12},u_{22}) \in \overline{G} \cap C_u$. From Theorem 4.1 it follows that there exists an open, connected set $\Gamma \subset E^2$ and analytic, orthogonal homeomorphism (4.3) with properties explained in Theorem 4.1. By Theorem 4.2 we can prolong (4.3) to a continuous mapping (4.4). By $C_\tau \subset \overline{\Gamma} \setminus \Gamma$ we denote a component of $\overline{\psi}^{-1}(C_u)$. There exist different points $(\tau_{11},\tau_{21}),(\tau_{12},\tau_{22}) \in C_\tau$ such that $\overline{\psi}(\tau_{11},\tau_{21}) = (u_{11},u_{21})$, $\overline{\psi}(\tau_{12},\tau_{22}) = (u_{12},u_{22})$. We suppose

$$\overline{\psi}_1(\tau_{12},\tau_{22}) - \overline{\psi}_1(\tau_{11},\tau_{21}) = d_1, d_1 > 0, \overline{\psi} = (\overline{\psi}_1,\overline{\psi}_2), \tag{4.9}$$

and we denote $\varrho_1 = \mathrm{dist}((\tau_{11},\tau_{21}),(\tau_{12},\tau_{22}))$. From definition 4.1 and (4.9) it follows that there exists a neighbourhood $N \subset \overline{\Gamma}$ of C_τ and points $(\tau_1',\tau_1'),(\tau_1'',\tau_2'') \in N$ near to $(\tau_{11},\tau_{21}),(\tau_{12},\tau_{22})) \in C_\tau$ respectively such that for every differentiable arc $(\tau_1(s),(\tau_2(s)) \in N$, $0 \leq s \leq 1$, which joins (τ_1',τ_2') and (τ_1'',τ_2'') we have

$$\left|\frac{\partial \psi_1}{\partial \tau_1}\right| \leq \frac{d}{2\varrho}, i = 1,2, d \geq \frac{d_1}{2}, \tag{4.10}$$

for every point $(\tau_1(s),\tau_2(s)) \in N$, $0 \leq s \leq 1$, and such that besides them $\psi_1(\tau_1'',\tau_2'') - \psi_1(\tau_1',\tau_2') = d$ and $\mathrm{dist}((\tau_1',\tau_2'),\tau_1'',\tau_2'')) = \varrho$, where $0 < \varrho \leq \frac{3}{2}\varrho_1$. Comparing this with Lemma 4.2, where we set $h = \psi_1$, we get a contradiction. The rest of Theorem 4.3 follows from Proposition 3.1.

Remark 4.1. From Theorem 4.3 it follows that the sets $H_u \cup G_u \subset D_u$ and $S^2 \setminus \Xi$ are open, connected and dense in D_u and S^2 respectively ([2], Chap.13, Th.4.). Hence, in Proposition 3.3 we can set $G = H_u \cup G_u$ and choose the signs before the square roots in (3.11), (3.12) and (3.18), (3.19) according to the signs convention on the whole of $H_u \cup G_u$. Hence, for a punctured imbedding (1.1) also the integrating factors f_1, f_2 in (3.11), (3.12) and e_1, e_2 in (3.18), (3.19) are defined on the whole of $H_u \cup G_u$.

5. Types of umbilical points

In the following we suppose that the signs before the square roots in (3.11), (3.12) and (3.18), (3.19) are chosen according to the signs convention on the whole of $H_u \cup G_u$.

Definition 5.1. Let $(u_{10}, u_{20}) \in H_u^0 \setminus G_u$. If the functions

$$\frac{\alpha_u \pm \sqrt{\alpha_u^2 + \beta_u^2}}{\beta_u} \quad \text{and} \quad \frac{\beta_u}{\alpha_u \mp \sqrt{\alpha_u^2 + \beta_u^2}} \tag{5.1}$$

are analytic at (u_{10}, u_{20}), then $a^{-1}(u_{10}, u_{20}) \in \Xi$ is called a regular umbilical point and (u_{10}, u_{20}) is called a regular umbilical point in (a, U). An umbilical point $a^{-1}(u_{10}, u_{20}) \in S^2$ which is not regular is called a chief umbilical point, and $(u_{10}, u_{20}) \in D_u$ is called a chief umbilical point in (a, U).

The set of chief umbilical points is denoted by $\Xi_c \subset \Xi \subset S^2$.

By means of the theorem of Brouwer ([1], Chap.XVI, Th.3.3) which asserts that every vector field $e(p)$, $p \in S^2$, of unit tangent vectors to S^2 admits a singularity, we get the following

Proposition 5.1. *For every punctured imbedding* (1.1) *there exists a chief umbilical point* $s_0 \in \Xi_c$.

In the following we suppose that the chart (a, U) has the property

$$U = S^2 \setminus \{s_0)\}. \tag{5.2}$$

From Definition 5.1 it follows that $\Xi \setminus \Xi_c \subset S^2$ is open in Ξ, and therefore we have the following

Proposition 5.2. *The set* $\Xi_c \subset \Xi \subset S^2$ *of chief umbilical points is compact and totally disconnected.*

The set of chief umbilical points in (a, U) we denote by $\Xi_u \subset H_u^0 \setminus G_u$; hence $\Xi_u = a(\Xi_c)$ and $a^{-1}(\Xi_u) \cup \{s_0\} = \Xi_c$. Now the right sides of (3.11), (3.12) and (3,18), (3.19) can be prolonged from $H_u \cup G_u \subset D_u \setminus \Xi_u$ to $D_u \setminus \Xi_u$, and in Theorem 4.1 we suppose $G \subset D_u \setminus \Xi_u$. Then also the functions (3.20) are prolonged to such points $(\tau_1, \tau_2) \in \Gamma$ that $\psi(\tau_1, \tau_2) \in D_u \setminus \Xi_u$, where ψ denotes the analytic, orthogonal homeomorphism (4.3).

Analogous to Lemma 4.1 we get as a direct consequence of the theorem of Fröbenius the following

LEMMA 5.1. *For every point* $(u_{10}, u_{20}) \in D_u \setminus \Xi_u$ *and every point* $(t_{10}, t_{20}) \in E^2$, *there exists an open set* $Q_0 \subset D_u \setminus \Xi_u$, $(u_{10}, u_{20}) \in Q_0$, *an open rectangle*

$$P_0 = \{(t_1, t_2) \in E^2 \mid |t_1 - t_{10}| < \varepsilon_{10}, |t_2 - t_{20}| < \varepsilon_{20}\}, \tag{5.3}$$

where $0 < \varepsilon_{10}, \varepsilon_{20} < \infty$, *and a uniquely defined analytic, orthogonal diffeomorphism*

$$\varphi_0 : P_0 \to Q_0, \varphi_0(P_0) = Q_0, \varphi_0 = (\varphi_{10}, \varphi_{20}), \tag{5.4}$$

such that the functions $(\varphi_{10}, \varphi_{20})$ *are solutions of the system* (3.11), (3.12) *completed by* (3.13) *and* $\varphi_0(t_{10}, t_{20}) = (u_{10}, u_{20})$.

In analogy to the Theorems 4.1 and 4.2 we get the following theorems

THEOREM 5.1. *For every* $(u_{10}, u_{20}) \in D_u \setminus \Xi_u$, $(t_{10}, t_{20}) \in E^2$ *and every open, connected set* $G \subset D_u \setminus \Xi_u$ *such that* $(u_{10}, u_{20}) \in G$, *there exists an open, connected set* $M \subset E^2$ *such that* $(t_{10}, t_{20}) \in M$ *and a uniquely defined analytic, orthogonal diffeomorphism*

$$h : M \to G, \varphi(M) = G, \varphi = (\varphi_1, \varphi_2), \tag{5.5}$$

such that the functions (φ_1, φ_2) *are solutions of the system* (3.11), (3.12) *completed by* (3.13) *and* $\varphi(t_{10}, t_{20}) = (u_{10}, u_{20})$.

Let $\overline{\varphi}$ denotes a prolongation of φ defined by (5.5) to a continuous mapping defined on the closure $\overline{M}$ of M¿ The mapping $\overline{\varphi}$ is called analytic at $(t_{10}, t_{20}) \in \overline{M} \setminus M$, if there exists a rectangle (5.3) and a diffeomorphism (5.4) such that $\overline{\varphi}(t_1, t_2) = \varphi_0(t_1, t_2)$ for every $(t_1, t_2) \in \overline{M} \cap P_0$.

THEOREM 5.2. *Let* $G \subset D_u \setminus \Xi_u$ *denotes an open, connected set such that* $\overline{G} \subset D_u$. *There exists a unique prolongation of* (5.5) *to a continuous mapping*

$$\overline{\varphi} : \overline{M} \to \overline{G}, \overline{\varphi}(\overline{M}) = \overline{G}. \tag{5.6}$$

For every $(t_1, t_2) \in \overline{M} \setminus M$ *such that* $\overline{\varphi}(t_1, t_2) \in D_u \setminus \Xi_u$ *the function* $\overline{\varphi}$ *is analytic at* (t_1, t_2). *If* $\overline{\varphi}(t_1, t_2) \in \Xi_u$, *then* $\overline{\varphi}$ *is continuous but not analytic at* $(t_1, t_2) \in \overline{M} \setminus M$.

Proof. The proof of Theorem 5.1 is that of Theorem 4.1, where instead of Lemma 4.1 is used Lemma 5.1. The homeomorphism (4.3) prolonged from $G \cap (H_u \cup G_u)$ to $G \cap (D_u \setminus \Xi_u)$ and the diffeomorphism (5.5) are related by the scale transformation (3.20) by $\varphi(t_1, t_2) = \psi(\chi_1^{-1}(t_1), \chi_2^{-1}(t_2))$, $(t_1, t_2) \in M$. Then from Theorem 4.2 it follows that a unique prolongation (5.6) exists.

If the closure of G is not contained in D_u, i.e. $\overline{G} \subset \overline{D_u}$, $\overline{G} \not\subset D_u$, then a prolongation (5.6) has no meaning at points of $\overline{G} \cap (\overline{D_u} \setminus D_u)$. In this case for every sequence $(u_1^n, u_2^n) \in G$, $1 \leq n < \infty$, which converges to a point $(u_{10}, u_{20}) \in \overline{G} \cap (\overline{D_u} \setminus D_u)$ we set

$$\lim_{k\to\infty} \phi(t_1^k, t_2^k) = \overline{\phi}(t_{10}, t_{20}) = s_0, \tag{5.7}$$

where

$$\phi(t_1, t_2) = a^{-1}\varphi(t_1, t_2), \quad (t_1, t_2) \in M, \tag{5.8}$$

and $(t_1^k, t_2^k) \in M$, $1 \leq k < \infty$, denotes a convergent subsequence of $\varphi^{-1}(u_1^n, u_2^n) \in M$ to a point $(t_{10}, t_{20} \in \overline{M} \setminus M$, s_0 denotes the distinguished umbilical point from Proposition 5.1. Thus, in any case there exists a prolongation to a continuous mapping

$$\overline{\phi} : \overline{M} \to \overline{W}, \overline{W} \subset S^2, \tag{5.9}$$

of the diffeomorphism (5.8), where $W = a^{-1}(G)$, $W \subset S^2$. The mapping (5.9) is analytic at every point $(t_1, t_2) \in \overline{M} \setminus M$ such that $\overline{\phi}(t_1, t_2) \in S^2 \setminus \Xi_c$, i.e. a $\overline{\phi}(t_1, t_2) = \overline{\varphi}(t_1, t_2) \in D_u \setminus \Xi_u$ is analytic at $(t_1, t_2) \in \overline{M} \setminus M$. If $\overline{\phi}(t_1, t_2) \in \Xi_c$, then $\overline{\phi}$ is continuous but not analytic at $(t_1, t_2) \in \overline{M} \setminus M$.

6. The parametrization theorems

We have the following

THEOREM 6.1. *For every punctured imbedding* (1.1) *there exists an open, connected and simply connected set $P \subset E^2$ and a continuous mapping*

$$\overline{\phi} : \overline{P} \to S^2, \quad \overline{\phi}(\overline{P}) = S^2, \tag{6.1}$$

with the following properties: a) the restriction $\overline{\phi}/P = \phi$ is an analytic diffeomorphism, and $\overline{\phi}$ is analytic for every $(t_1, t_2) \in \overline{P} \setminus P$ such that $\overline{\phi}(t_1, t_2) \in S^2 \setminus \Xi_c$. If $\overline{\phi}(t_1, t_2) \in \Xi_c$, then $\overline{\phi}$ is continuous but not analytic at $(t_1, t_2) \in \overline{P} \setminus P$, b) $\Xi_c \subset \overline{\phi}(\overline{P}) \setminus \phi(P)$, and every point "at infinity" of P belongs to $\overline{\phi}^{-1}(\Xi_c) \subset \overline{P} \setminus P$, c) the set Int$(\overline{P})$ *is simply connected, d) the functions $a\phi = (\varphi_1, \varphi_2)$ are solutions of the system* (3.11), (3.12) *completed by* (3.13).

Proof (scheme). There exists a prolongation (5.9) of $\phi_0 = a^{-1}\varphi_0$, where φ_0 is defined by (5.4), to a continuous mapping $\overline{\phi_0}$. If $\overline{\phi_0}(\overline{P_0}) = S^2$, then we set $P = P_0$, $\phi = \phi_0$, and the proof is finished. Let us suppose $S^2 \setminus \overline{W_0} \neq \emptyset$, where $W_0 = \phi_0(P_0)$. By O we denote the set of triples (P_1, ϕ_1, W_1) with the following properties. Let $W_1 \subset S^2 \setminus \Xi_c$ denotes an open, connected and simply connected set such that $W_0 \subset W_1$ and that there exists an open, connected and simply connected set $P_1 \subset E^2$ such that Int(P_1) is simply connected and $P_0 \subset P_1$, and a uniquely defined analytic, orthogonal diffeomorphism

$$(6.2) \quad \varphi_1 : P_1 \to G_1, \quad \varphi_1(P_1) = G_1, \quad G_1 \subset D_u \setminus \Xi_u, \quad \varphi_1 = (\varphi_{11}, \varphi_{21}),$$

such that the functions $\varphi_1 = (\varphi_{11}, \varphi_{21})$ are solutions of the system (3.11), (3.12) completed by (3.13) and $\varphi_1(t_{10}, t_{20}) = (u_{10}, u_{20})$. There exists a prolongation (5.9) of the analytic diffeomorphism

$$(6.3) \qquad \phi_1 : P_1 \to W_1, \quad \phi_1(P_1) = W_1, \quad \phi_1 = a^{-1}\varphi_1,$$

to a continuous mapping

$$(6.4) \qquad \overline{\phi}_1 : \overline{P}_1 \to \overline{W}_1, \quad \overline{\phi}_1(\overline{P}_1) = \overline{W}_1, \quad \overline{W}_1 \subset S^2,$$

where $\overline{\phi}_1$ is analytic at every point $(t_1, t_1) \in \overline{P}_1 \setminus P_1$ such that $\overline{\phi}_1(t_1, t_2) \in$ $\in S^2 \setminus \Xi_c$. In the set O we define a partial order as follows

$$(6.5) \qquad (p_1, \phi_1, W_1) \prec (P_2, \phi_2, W_2),$$

if and only if $P_1 \subset P_2$, $W_1 \subset W_2 \subset S^2 \setminus \Xi_C$ and $\phi_2/P_1 = \phi_1$. By the lemma of Zorn there exists in O a maximal element $(P_{\max}, \phi_{\max}, W_{\max})$. We prove that $\overline{W}_{\max} = S^2$.

The proof is indirect. Let us suppose $\overline{W}_{\max} \neq S^2$. Since $W_{\max}$ is connected and simply connected, it follows that the boundary $\overline{W}_{\max} \setminus W_{\max}$ is a connected set which contains more than one point. Since Ξ_c is by Proposition 5.2 compact and totally disconnected, it follows that there exists a point

$$(6.6) \qquad p_1 \in (\overline{W}_{\max} \setminus W_{\max}) \cap (S^2 \setminus \Xi_c)$$

such that p_1 is a limit point of $S^2 \setminus \overline{W}_{\max}$. Therefore for every neighbourhood $A \subset S^2 \setminus \Xi_c$ of p_1 we have

$$(6.7) \qquad W_{\max} \cap A \neq \emptyset \quad \text{and} \quad (S^2 \setminus \overline{W}_{\max}) \cap A \neq \emptyset.$$

We have $a(p_1) = (u_{11}, u_{21}) \in D_u \setminus \Xi_u$, $a(W_{\max}) = G_{\max} \subset D_u \setminus \Xi_u$. By Lemma 5.1 there exists an open set $Q_1 \subset D_u \setminus \Xi_u$, $(u_{11}, u_{21}) \in Q_1$, a rectangle $P_1'' \subset E^2$ with center (t_{11}'', t_{21}'') and a uniquely defined analytic, orthogonal diffeomorphism $\varphi_1'' : P_1'' \to Q_1$, $\varphi_1''(P_1'') = Q_1$, such that $\varphi_1'' =$ $(\varphi_{11}'', \varphi_{21}'')$ are solutions of the system (3.11), (3.12) completed by (3.13) and $\varphi_1(t_{11}'', t_{21}'') = (u_{11}, u_{21})$. Since $(u_{11}, u_{21}) \in \overline{G}_{\max} \setminus G_{\max}$, it follows that there exists a point $(u_1'', u_2'') \in G_{\max} \cap Q_1$. We define

$$(6.8) \qquad \varphi_1''^{-1}(u_1'', u_2'') = (t_1'', t_2''), (t_1'', t_2'') \in P_1'',$$

$$(6.9) \qquad \varphi_{\max}^{-1}(u_1'', u_2'') = (t_1', t_2'), (t_1', t_2') \in P_{\max}.$$

The translation $r : E^2 \to E^2$ defined by the vector $(t_1' - t_1'', t_2' - t_2'')$ transforms the point (6.8) to (6.9). We define $\varphi_1' = \varphi_1'' r^{-1}$, $r(P_1'') = P_1'$, $r(t_{11}'', t_{21}'') =$ $= (t_{11}', t_{21}')$, where (t_{11}', t_{21}') is the center of P_1'. We have $(t_{11}', t_{21}') \in$

$\overline{P}_{\max} \setminus P_{\max}$. There exists s subset $P_2' \subset P_1'$ such that $P_2 = P_{\max} \cup P_2'$ is open, connected and simply connected and $\mathrm{Int}(\overline{P}_2)$ is simply connected and $P_2 \neq P_{\max}$. We define diffeomorphism $\phi_2 : P_2 \to W_2$, $W_2 \subset S^2$, setting $\phi_2(t_1,t_2) = \phi_{\max}(t_1,t_2)$, $(t_1,t_2) \in P_{\max}$, $\phi_2(t_1,t_2) = \phi_1'(t_1,t_2)$, $(t_1,t_2) \in P_2'$, $\phi_1' = a^{-1}\varphi_1'$. Hence it follows

$$(6.10) \qquad \begin{aligned} &(P_{\max}, \phi_{\max}, W_{\max}) \prec (P_2, \phi_2, W_2) \text{ and} \\ &\quad (P_{\max}, \phi_{\max}, W_{\max}) \neq (P_2, \phi_2, W_2). \end{aligned}$$

From (6.10) it follows that the assumption $\overline{W}_{\max} \neq S^2$ leads to a contradiction. This proves $\overline{W}_{\max} = S^2$. There exists a prolongation (5.9) of $\phi_{\max}$ to a continuous mapping $\overline{\phi}_{\max} : \overline{P}_{\max} \to S^2$, $\overline{\phi}_{\max}(\overline{P}_{\max}) = S^2$, and $\overline{\phi}_{\max}$ is analytic at every point $(t_1,t_2) \in \overline{P}_{\max} \setminus P_{\max}$ such that $\overline{\phi}_{\max}(t_1,t_2) \in S^2 \setminus \Xi_c$. Analycity at $(t_{11}',t_{21}') \in \overline{P}_{\max} \setminus P_{\max}$ implies that (t_{11}',t_{21}') cannot be a point "at infinity" of $P_{\max}$, because there exists a translation $r : E^2 \to E^2$ which transforms the center (t_{11}'',t_{21}'') of P_1'' to $(t_{11}',t_{21}') \in \overline{P}_{\max} \setminus P_{\max}$. Hence, both coordinates (t_{11}',t_{21}') are finite. If $\overline{\phi}(t_1,t_2) \setminus \Xi_c$, then $\overline{\phi}$ is continuous but not analytic at $(t_1,t_2) \in \overline{P}_{\max} \setminus P_{\max}$. This ends the proof.

Remark 6.1. This scheme of proof is used in the proof of Theorem 4.1 and implicitly in the proof of Lemma 3.1, where the existence of a global solution $u_1(w_1,w_2)$ of (3.4) is postulated. A (local) transformation of general coordinates $(u_1,u_2) \in D_u$ to general orthogonal (with respect to a Riemannian metric) coordinates $(v_1,v_2) \in E^2$ we get by means of the theorem of Fröbenius as a solution of a system of partial differential equations $\frac{\partial u_i}{\partial v_1} = \alpha_i$, $\frac{\partial u_i}{\partial v_2} = \beta_i$, $i = 1,2$, where (α_1,α_2), (β_1,β_2) are completely integrable, orthogonal vector fields on D_u. By means of the method explained in the scheme of proof this local solution can be extended on the whole of D_u.

In the following we denote $\phi_{\max} = \phi$, $P_{\max} = P$, $W_{\max} = W$, where $\overline{W} = S^2$.

DEFINITION 6.1. A triple (P,ϕ,W) which satisfies Theorem 6.1 is called a parametrization (by means of lines of curvature) of S^2.

DEFINITION 6.2. Let (P,ϕ,W) denotes a parametrization. A point $(t_1,t_2) \in \overline{P} \setminus P$ is called a chief parametric umbilical point, if $\overline{\phi}(t_1,t_2) \in \Xi_c$; $(t_1,t_2) \in \overline{P}$ is called a regular parametric umbilical point, $\overline{\phi}(t_1,t_2) \in \Xi \setminus \Xi_c$ (see Definitions 4.1 and 5.1).

In addition to Theorem 6.1 we have the following

THEOREM 6.2. *For every parametrization (P,ϕ,W) the set P satisfies the equation $P = \mathrm{Int}(\overline{P})$.*

Proof. The proof of Theorem 6.2 is carried out in two steps. At first we prove that if $(t_{10},t_{20}) \in \mathrm{Int}(\overline{P})$ is a parametric umbilical point, then using the fact that P and $\mathrm{Int}(\overline{P})$ are connected and simply connected, it follows that $(t_{10}, t_{20}$ is a regular parametric umbilical point. We prove indirect, passing by (3.20) from the parameters (t_1, t_2) to (τ_1, τ_2), that on the straightlines $t_1 = t_{10}$ and $t_2 = t_{20})$, (t_{10}, t_{20}) is an isolated parametric umbilical point. In the second step we prove that the functions $\overline{\varphi}_1$, $\overline{\varphi}_2$, where $a\overline{\phi} = (\overline{\varphi}_1, \overline{\varphi}_2)$ are analytic at $(t_{10}, t_{20}$ with respect to each of the variables t_1 and t_2 separately. This together with the fact that $\overline{\varphi}_1, \overline{\varphi}_2$ are analytic as functions of 2 variables at points (t_1, t_{20}), (t_{10}, t_2) such that $0 < |t_i - t_{i0}| < r_i$, where $r_i > 0$, $i = 1, 2$, are sufficiently small, implies that $\overline{\varphi}_1, \overline{\varphi}_2$ are analytic at (t_{10}, t_{20}). This ends the proof.

As an immediate consequence of Proposition 5.2, Theorem 6.1 and Theorem 6.2 we get the Parametrization Theorem.

7. The structure of the boundary $\overline{P} \setminus P$

Let (P, ϕ, W) denotes a parametrization. If $\phi(P) = S^2 \setminus \{s_0\}$, where s_0 denotes the distinguished chief umbilical point from Proposition 5.1, then the punctured imbedding (1.1) has a single chief umbilical point s_0. If $\overline{\phi}(\overline{P}) \setminus \phi(P) \neq \{s_0\}$, then $\overline{\phi}(\overline{P}) \setminus \phi(P)$ contains a power continuum of points. Since $\Xi_c \subset S^2$ is by Proposition 5.2 compact and totally disconnected, it follows that the set

$$(\overline{\phi}(\overline{P} \setminus \phi(P)) \cap (S^2 \setminus \Xi_c) \tag{7.1}$$

is dense and open in $\overline{\phi}(\overline{P}) \setminus \phi(P)$ in the induced from S^2 topology of $\overline{\phi}(\overline{P}) \setminus \phi(P)$, and therefore the set (7.1) is non empty.

PROPOSITION 7.1. *For every parametrization (P, ϕ, W), if $\overline{\phi}(\overline{P}) \setminus \phi(P) \neq$ $\neq \{s_0\}$ then the set (7.1) is non empty and for every point $p_1 \in S^2$ of the set (7.1) the counter image $\overline{\phi}^{-1}(p_1)$ contains exactly 2 points of $\overline{P} \setminus P$.*

Proof. We have $a(p_1) = (u_{11}, u_{21}) \in D_u \setminus \Xi_u$. Therefore from Lemma 5.1 it follows that there exists an open set $W_1 \subset S^2 \setminus \Xi_c$ such that $p_1 \in W_1$ and an open rectangle P_1' with center $(t_1', t_2') \in P_1' \cap (\overline{P} \setminus P)$ (see the proof of Theorem 6.1) and a diffeomorphism

$$\phi_1' : P_1' \to W_1, \quad \phi_1'(P_1') = W_1, \quad a\phi_1' = \varphi_1' = (\varphi_{11}', \varphi_{21}'), \tag{7.2}$$

such that the functions $(\varphi_{11}', \varphi_{21}')$ are solutions of the system (3.11), (3.12) completed by (3.13) and $\varphi_1'(t_{11}, t_{21}) = (u_{11}, u_{21})$. We have

$$\overline{\phi}(t_1, t_2) = \overline{\phi}_1'(t_1, t_1) \quad \text{for every } (t_1, t_2) \in P \cap P_1'. \tag{7.3}$$

From the Parametrization Theorem we have

$$P \cap P_1' \neq \emptyset \quad \text{and} \quad (E^2 \setminus \overline{P}) \cap P_1' \neq \emptyset. \tag{7.4}$$

From the definition of a parametrization it follows that for every $(t_1', t_2') \in$ $\in (E^2 \setminus \overline{P}) \cap P_1'$ there exists a unique point $(t_1'', t_2'') \in P$ such that

$$\phi_1'(t_1', t_2') = \phi(t_1'', t_2''). \tag{7.5}$$

There exists a sequence $(t_{1n}^1, t_{2n}^1) \in (E^2 \setminus \overline{P}) \cap P_1'$, $1 \leq n < \infty$, which converges to $(t_{11}', t_{21}') \in \overline{P} \setminus P$. By (7.5) there exists a sequence $(t_{1n}^2, t_{2n}^2) \in P$, $1 \leq n < \infty$, convergent to a point $(t_{11}'', t_{21}'') \in \overline{P} \setminus P$ such that

$$\phi_1'(t_{11}', t_{21}') = \overline{\phi}(t_{11}'', t_{21}'') = \overline{\phi}(t_{11}', t_{21}') = p_1. \tag{7.6}$$

For every n, $1 \leq n < \infty$, $(t_{1n}^2, t_{2n}^2) \notin P_1'$, since otherwise by (7.3) and (7.5) ϕ_1' cannot be a diffeomorphism on P_1'. Hence, $(t_{11}'', t_{21}'') \notin P_1'$ and therefore $(t_{11}', t_{21}') \neq (t_{11}'', t_{21}'')$. This ends the proof.

Remark 7.1. For every $(t_1', t_2') \in P_1'$ we have $(t_1', t_2') \in \overline{P}$ or $(t_1', t_2') \in$ $\in E^2 \setminus \overline{P}$. In the second case there exists such a point $(t_1'', t_2'') \in P$ that (7.5) holds. By fixed t_2' and therefore fixed t_2'' in (7.5) an increasing parameter t_1' on the left side of the equality (7.5) determines an increasing or decreasing parameter t_1'' on the right side of (7.5). Then, since a parametrization is orientation preserving, by fixed t_1', t_1'' an increasing parameter t_2' on the left side of (7.5) determines an increasing or decreasing parameter t_2'' on the right side of (7.5) respectively.

In the following we suppose that the set (7.1) is non empty, i.e. that the set Ξ_c has at least 2 elements.

By $\zeta_1' : E^2 \to E^2$ we denote a translation of E^2 which transforms $(t_{11}', t_{21}') \in \overline{P} \setminus P$ to $(t_{11}'', t_{21}'') \in \overline{P} \setminus P$, where these points are different and satisfy (7.6). We denote $\zeta_1'' = \zeta_1'^{-1}$. By ϱ_1' respectively ϱ_1'' we denote a symmetry of E^2 with respect to (t_{11}', t_{21}') respectively (t_{11}'', t_{21}''). We define $\xi_1' = \varrho_1''\zeta_1'$, $\xi_1'' = \varrho_1'\zeta_1''$. By standard calculations on coordinates we get $\xi_1' = \xi_1'^{-1} = \xi_1'' = \xi_1''^{-1}$, and ξ_1' can be defined as asymmetry of E^2 with respect to the center of the segment with endpoints (t_{11}', t_{21}'), (t_{11}'', t_{21}''). In the following we use this definition of ξ_1'.

By $P_1'' \subset^2$ we denote a rectangle which is obtained from $P_1' \subset E^2$ by means of the translation ζ_1', if in (7.5) an increasing parameter t_1' on the left side of (7.5) determines an increasing parameter t_1'' on the right side of (7.5) and by means of a symmetry ξ_1', if in (7.5) an increasing parameter t_1' on the left side determines a decreasing parameter t_1'' on the right side of (7.5)

LEMMA 7.1. *The sets $(\overline{P} \setminus P) \cap P_1'$ and $(\overline{P} \setminus P) \cap P_1''$ are congruent by means of ζ_1' or ξ_1'.*

Proof. By $Z \subset \overline{P} \setminus P$ we denote the greatest set with the property $\overline{\phi}(Z) \cap \Xi_c = \emptyset$. The set is open in $\overline{P} \setminus P$ in the induced from E^2 topology of $\overline{P} \setminus P$, and it is the counter image of the set (7.1) by means of $\overline{\phi}^{-1}$.

We have $(\overline{P} \setminus P) \cap P_1' \subset Z$, $(\overline{P} \setminus P) \cap P_1'' \subset Z$. In particular we have $(t_{11}', t_{21}'), (t_{11}'', t_{21}'') \in Z$. By $C_1', C_1'' \subset Z$ we denote components of Z such that $(t_{11}', t_{21}') \in C_1'$, $(t_{11}'', t_{21}'') \in C_1''$. There exists an at most countable sequence $(t_{1i}', t_{2i}') \in C_1'$, $1 \leq i < \infty$, of points, rectangles $P_i' \subset E^2$ with centres (t_{1i}', t_{2i}') and diffeomorphisms $\phi_i' : P_i' \to W_i$, $\phi_i'(P_i') = W_i$, $W_i \subset S^2 \setminus \Xi_c$, such that for every i, $1 \leq i < \infty$, the assertions of Lemma 5.1 holds, and such that $\{P_i'\}$, $1 \leq i < \infty$, is a covering of C_1'. by Proposition 7.1 for every i, $1 \leq i < \infty$, there exists a unique point (t_{1i}'', t_{2i}'') different from (t_{1i}', t_{2i}') and such that $\overline{\phi}(t_{1i}', t_{2i}') = \overline{\phi}(t_{1i}'', t_{2i}'')$. In the same way as ζ_1', ξ_1' we define ζ_i', ξ_i', and we set $P_i'' = \zeta_i'(P_i')$ or $P_i'' = \xi_i'(P_i')$ such that for every i, $1 \leq i < \infty$, the sets $(\overline{P} \setminus P) \cap P_i'$ and $(\overline{P} \setminus P) \cap P_i''$ are congruent by means of a translation ζ_i' which sends (t_{1i}', t_{2i}') to (t_{1i}'', t_{2i}'') or a symmetry ξ_i' with respect to the center of the segment with these endpoints.

PROPOSITION 7.2. *The components $C_1', C_1'' \subset Z$ are congruent by means of ζ_1' or ξ_1'.*

Proof. The proof is a verification that $\zeta_i' = \zeta_1'$ or $\xi_i' = \xi_1'$, $2 \leq i < \infty$.

PROPOSITION 7.3. *If $C_1'' = \xi_1'(C_1')$ and $(t_{10}, t_{20}) \in \overline{C}_1' \cap \overline{C}_1''$ is the fixpoint of the symmetry ξ_1', then (t_{10}, t_{20}) is the only common point of $\overline{C}_1' \cap \overline{C}_1''$ and it is a chief parametric umbilical point.*

8. Periodic imbeddings and chief umbilical points

Let $C_i', C_i'' \subset Z$, $1 \leq i < \infty$ denote all components of Z. For every i, $1 \leq i < \infty$, there exists a translation ζ_i' or a symmetry ξ_i' such that $C_i'' = \zeta_i'(C_i')$ or $C_i'' = \xi_i'(C_i')$. We suppose that $P \subset E^2$ is a bounded set and for every integer i, $1 \leq i < \infty$,

$$\zeta_i'(P) \cap P = \zeta_i''(P) \cap P = \emptyset \text{ resp. } \xi_i'(P) \cap P = \xi_i''(P) \cap P = \emptyset. \tag{8.1}$$

Since the area of P is positive, it follows from (8.1) that there exists an integer n such that $1 \leq i \leq n$. In the following we mark the translations with the index i and the symmetries with the index j. Hence, the inequalities $1 \leq i, j \leq n$ means that i and j together take the values from 1 to n. We have

$$\begin{gathered} C_i'' \subset \zeta_i'(\overline{P}) \cap \overline{P}, C_i' \subset \zeta_i''(\overline{P}) \cap \overline{P} \\ \text{resp. } C_j' \cup C_j'' \subset \xi_j'(\overline{P}) \cap \overline{P} = \xi_j''(\overline{P}) \cap \overline{P}. \end{gathered} \tag{8.2}$$

The difference $D = (\overline{P} \setminus P) \setminus Z$ is the set of chief parametric umbilical points. Since the number of components $C_i', C_j', C_i'', C_j'' \subset Z$ of Z is finite, it follows that the number of components of D is also finite. We decompose D in equivalence classes by means of $\overline{\phi}$ setting that 2 points $(t_{11}, t_{21}), (t_{12}, t_{22}) \in$ $\in D$ belong to the same class, if $\overline{\phi}(t_{11}, t_{21}) = \overline{\phi}(t_{12}, t_{22})$. By $D_1 \subset D$ we denote a section in the set of equivalence classes, i.e. D_1 has exactly one common point with every equivalence class and only these points belong to D_1. The set

$$\tilde{P} = P \cup \bigcup_{1 \le I \le n} C_i' \cup \bigcup_{1 \le j \le n} C_j' \cup D_1, \quad P \subset \tilde{P} \subset \overline{P}, \tag{8.3}$$

where $1 \le i \le n$ respectively $1 \le j \le n$ in (8.3) means that i respectively j runs over such integers between 1 and n that ζ_i' respectively ξ_j' are defined, has the property $\overline{\phi}(\tilde{P}) = S^2$.

DEFINITION 8.1. A parametrization (P, ϕ, W) which satisfies (8.1) is called periodic with fundamental domain P, if for every choice of P, i.e. for every choice of a section D_1 there exists a covering of the plane E^2 invariant with respect to every composition γ of mappings $\zeta_i', \zeta_i'', \xi_j', \xi_j'', 1 \le i, j \le n$.

PROPOSITION 8.1. *If (P, ϕ, W) is a periodic parametrization, then D is a finite set of $2n$ elements, where n is the number of components $C_i', C_j' \subset Z$.*

Proof. The proof of Proposition 8.1 is indirect.

In the following a periodic parametrization (P, ϕ, W) with n components $C_i', C_j' \subset Z, 1 \le i, j \le n$, we denote by (P_n, ϕ_n, W_n). In such a case we denote also $D = D_n$, $Z = Z_n$.

There exists an extension $\Psi : E^2 \to S^2$ of $\overline{\phi}_n$ defined by $\Psi(t_1, t_2) = \overline{\phi}_n(\tilde{t}_1, \tilde{t}_2), (\tilde{t}_1, \tilde{t}_2) \in \tilde{P}_n$, $(t_1, t_2) = \gamma(\tilde{t}_1, \tilde{t}_2), (t_1, t_2) \in E^2$. We have $\Psi/P_n = \overline{\phi}_n/P_n$ and $\Psi/\overline{P}_n = \overline{\phi}_n$. By $N_n \subset E^2$ we denote the set equivalent with the points of the set $D_n = (\overline{P}_n \setminus P_n) \setminus Z_n$ of chief umbilical points. By the Parametrization Theorem the mapping Ψ is analytic at every point of $E^2 \setminus N_n$ and in the points of N_n Ψ is continuous but not analytic. The finite set D_n defines an (open) polygon Γ_n and the set N_n defines a covering of E^2 invariant with respect to every composition γ of mappings $\zeta_i', \zeta_i'', \xi_j', \xi_j''$, $1 \le i, j \le n$, whose fundamental domain $\tilde{\Gamma}_n$ is defined by Γ_n. We have $\Gamma_n \subset \tilde{\Gamma}_n \subset \overline{\Gamma}_n$ and $\tilde{\Gamma}_n$ is a polygon with neglected sides of Γ_n which correspond to the components C_i'', C_j'', $1 \le i, j \le n$. We have $\Psi(\tilde{\Gamma}_n) = S^2$. Indeed, for every $(t_1, t_2) \in E^2$ there exists an equivalent unique point $(\tilde{t}_1, \tilde{t}_2) \in \tilde{\Gamma}_n$ and $\Psi(E^2) = S^2$. We denote $\Psi_n = \Psi/\Gamma_n$. The parametrizations (P_n, ϕ_n, W_n) and (Γ_n, Ψ_n, V_n), $V_n = \Psi_n(\Gamma_n)$, $\overline{V}_n = S^2$, have the same set D_n of chief parametric umbilical points and define the same lines of curvature on S^2. Since $N_n \subset E^2$ defines a covering of E^2 with fundamental domain $\tilde{\Gamma}_n$, it follows that the boundary of Γ_n is a simple closed curve (without selfintersections). We now describe more detailed Γ_n. Two points of D_n which belong to $\overline{C}_i'$ or $\overline{C}_i''$ are joined by a segment s_i' or s_i'' respectively. If C_i', C_i'' are such that $\zeta_i'(C_i') = C_i''$, then s_i' and s_i'' are parallel, non colinear and have the same length. If $\xi'_j(C'_j) = C''_j$ and $\overline{C}'_j \cap \overline{C}''_j \neq \emptyset$, then s'_j, s''_j are colinear, have the same length and an endpoint in common (see Proposition 7.3). In this case we denote $s'_j \cup s''_j = \sigma_j$. The polygon bounded by s'_i, s''_i such that $\zeta'_i(s'_i) = s''_i$, by s'_j, s''_j such that $\xi'_j(s'_j) = s''_j$ and $s'_j \cap s''_j = \emptyset$ and by σ_j, if $s'_j \cap s''_j \neq \emptyset$, we denote by $\nabla_n \subset E^2$.

DEFINITION 8.2. A punctured imbedding (1.1) is called periodic, if there exists a periodic parametrization (P_n, ϕ_n, W_n), $n \geq 1$, of (1.1).

This definition is correct in the following sense. If (P', ϕ', W') is an arbitrary parametrization of a punctured imbedding (1.1) with the periodic parametrization (P_n, ϕ_n, W_n), then (P', ϕ', W') is a periodic parametrization (P'_n, ϕ'_n, W'_n) and there exists a translation $r : E^2 \to E^2$ such that $r(\nabla_n) = \nabla'_n$, where ∇'_n is the polygon defined by (P'_n, ϕ'_n, W'_n) in the same way as ∇_n was defined for the parametrization (P_n, ϕ_n, W_n).

Therefore up to a translation $r : E^2 \to E^2$ $\tilde{\Gamma}_n$ can be considered as a period of (1.1).

THEOREM 8.1. *For every periodic imbedding* (1.1) *we have* $1 \leq n \leq 6$ *and the number* κ *of chief umbilical points of* (1.1) *satisfies the inequality* $2 \leq \kappa \leq 7$.

Proof. We estimate the number of sides of $\nabla_n \subset E^2$ as follows. We denote this number by m: we have $m \leq 2n$. Let v_k, $1 \leq k \leq m$, denotes a vertex of ∇_n. There exist 2 sides with this common vertex in ∇_n. By α_k we denote the interior angle of ∇_n at v_k. We have

$$\alpha_i + \alpha_j \leq 2\Pi - \alpha_k, \quad i \neq j \neq k \neq i, \tag{8.4}$$

where α_i and α_j are interior angles of ∇_n by the vertices v_i and v_j such that v_i and v_j are transformed by ζ'_s or ζ''_s or ξ'_s, $1 \leq s \leq n$, to v_k. From (8.4) we get

$$\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{m} \alpha_j \leq 2\Pi m - \sum_{k=1}^{m} \alpha_k. \tag{8.5}$$

We have

$$\sum_{i=1}^{m} \alpha_i = \Pi(m-2).$$

Hence, from (8.5) it follows

$$2\Pi(m-2) \leq 2\Pi m - \Pi(m-2). \tag{8.6}$$

From (8.6) it follows $m \leq 6$. Thus, the maximal number of sides of ∇_n is 6. The vertices of the hexagon $H = \nabla_n$ are chief parametric umbilical points. Besides them the centres of the sides of the hexagon H can be chief parametric umbilical points. Thus, the maximal number of chief parametric umbilical points is 12 for a periodic imbedding (1.1), and we have $1 \leq n \leq 6$ and $H = H_6 = \Gamma_6$. If (H_6, ϕ_6, W_6) denotes a parametrization with this chief parametric umbilical points, then the vertices of H_6 are transformed by $\overline{\phi}_6$ to a single chief umbilical point of (1.1) on S^2. Together with the images of the 6 centres of the sides of H_6 this makes 7 chief umbilical points of (1.1) on S^2. Thus, 7 is the maximal number of chief umbilical points of a periodic imbedding (1.1). On the other hand, since the set (7.1) is non empty, it follows that (1.1) admits at least 2 chief umbilical points. This ends the proof.

Remark 8.1. a) In the proof of Theorem 8.1 we have not assumed that $\overline{C}'_j \cap \overline{C}''_j \neq \emptyset$, if $C''_j = \xi'_j(C'_j)$. Direct verification shows that $\overline{C}'_j \cap \overline{C}''_j = \emptyset$, where $C''_j = \xi'_j(C'_j)$, is not possible. b) In the definition of a periodic parametrization (P_n, ϕ_n, W_n), $n \geq 1$, we have assumed that P_n is a bounded set. In this case the proof that the number of components of $Z_n \subset \overline{P}_n \setminus P_n$ is finite is simple; we avoid the consideration of special cases. This assumption is not necessary, Theorem 8.1 remains valid also, if we allow unbounded fundamental domains; below we describe examples with unbounded fundamental domains.

EXAMPLES. a) If P_1 is the (unbounded) open, connected and simply connected set bounded by 2 parallel straightlines and ζ'_1 transforms the first of the straight lines onto the second, then (P_1, ϕ_1, W_1) is a periodic parametrization with 2 chief parametric umbilical points defined by the points "at infinity" of P_1. Thus, S^2 admits in this case 2 chief umbilical points. b) If P_3 is a bounded triangle and ξ'_1, ξ'_2, ξ'_3 are symmetries with respect to the centres of the sides of the triangle, then (P_3, ϕ_3, W_3) is a periodic parametrization with 6 chief parametric umbilical points, and S^2 admits 4 chief umbilical points which correspond to the centres of the sides of the triangle and to the vertices of the triangle which are transformed by $\overline{\phi}_3$ to a single chief umbilical point of S^2. If the triangle is unbounded, i.e., if 2 sides are parallel rays whose origins are joined by the third side, and ζ'_1 transforms the first ray onto the second and ξ'_1 is a symmetry of E^2 with respect to the center of the bounded side, then (P_2, ϕ_2, W_2) is a periodic parametrization with 4 chief parametric umbilical points, and S^2 admits 3 chief umbilical points which correspond to the vertex "at infinity", to the center of the bounded side and to the remainder 2 vertices of the triangle which are transformed by $\overline{\phi}_2$ to a single chief umbilical point of S^2. c) If P_3 is a parallelogram and ξ'_1, ξ'_2 are symmetries with respect to the centres of 2 parallel sides of the parallelogram and ζ'_1 is a translation of one of the remainder sides onto the other, then (P_3, ϕ_3, W_3) is a periodic parametrization with 6 chief parametric umbilical points, and S^2 admits 4 chief umbilical points; 2 correspond to the centres of 2 parallel sides and the 2 remainder we get from the 4 vertices of the parallelogram. If the parallelogram is a rectangle with sides parallel to the coordinate axes, then the periodic parametrization (P_3, ϕ_3, W_3) can be considered as a parametrization of a non rotational ellipsoide. Another periodic parametrization in the case that P_4 is a parallelogram we get, if we take symmetries $\xi - 1', \xi_2', \xi_3', \xi_4'$ with respect to the centres of the sides of the parallelogram. Then the periodic parametrization (P_4, ϕ_4, W_4) admits 8 chief parametric umbilical points, and S^2 admits 5 chief umbilical points; 4 correspond to the centres of the sides of this parallelogram and the 5-th we get as the image by $\overline{\phi}_4$ of the vertices of the parallelogram. d) id P_4 is a pentagon presented in fig. 1

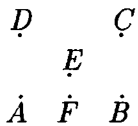

fig. 1

and ζ_1' transforms AD in BC and ξ_1', ξ_2', ξ_3' are symmetries with respect to the centres of the sides AB, CE, ED, where the length of EF is the half length of AD, then (P_4, ϕ_4, W_4) is a periodic parametrization with 8 chief parametric umbilical points, and S^2 admits 5 chief umbilical points; the points A, B and C, E, D define 2 chief umbilical points on S^2 and the centres of the sides AB, CE, ED define the remainder 3 chief umbilical points on S^2. e) In the case of a hexagon considered in the proof of Theorem 8.1 besides 12 chief umbilical points we can have also 10 chief parametric umbilical points such that 2 parallel sides of the hexagon are transformed =by ζ_1' the first on the second. Then S^2 admits 6 chief umbilical points.

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