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ON THE RELATION BETWEEN THE STRATONOVICH AND ITÔ INTEGRALS WITH INTEGRANDS OF DELAYED ARGUMENT

1. Introduction

The relations between different types of stochastic integrals have been examined for years ([1], [5], [7]).

As it is shown in [1] and [7], an additional term occurring when the Itô integral is changed to the Stratonovich integral is the same as the correction term in the corresponding approximation theorem of Wong-Zakai type. Then, the stochastic differential equation is the limit of a sequence of ordinary differential equations perturbed by the coloured noises converging to the white noise.

We consider stochastic delay differential equations with delay on a finite time interval. Our correction term is the same as the term occuring in the approximation theorem of Wong-Zakai type in [8] and [9]. The present paper contains a slight modification of the model in the above papers, where delayed argument is considered on the interval $(-\infty, 0]$ because of some technical reasons.

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2. Definitions and notations

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with $(\mathcal{F}_t)_{t\in[0,T]}$ an increasing family of sub-σ-algebras of the σ-algebra $\mathcal{F}$. Let $I = [-r, 0]$. We define the following Banach spaces $C_- = C(I; \mathbb{R}^d)$, $C_1 = C([-r, T]; \mathbb{R}^d)$ and $C_2^0 = C([-r, T]; \mathbb{R}^m) = \widetilde{\Omega}$ of continuous functions. We assume that all functions of C_2^0 vanish in zero. The above spaces are endowed with the usual norms $\|\ \|_{C_-}$, $\|\ \|_{C_1}$, $\|\ \|_{C_2^0}$ of uniform convergence.

Below we denote by $\mathcal{X}$ one of the above spaces. Let $\mathcal{B}(\mathcal{X})$ denote the topological σ-algebra of the space $\mathcal{X}$.

The smallest Borel algebra that contains $\mathcal{B}_1, \mathcal{B}_2, \ldots$ is denoted by $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$; $\mathcal{B}_{u,v}(X)$ denotes the smallest Borel σ-algebra for which a given stochastic process $X(t)$ is measurable for every $t \in [u,v]$ and $\mathcal{B}_{u,v}(dB)$ denotes the smallest Borel algebra for which $B(s) - B(t)$ is measurable for every (t,s) with $u \le t \le s \le v$.

We introduce the Wiener space $(\mathcal{C}_2^0, \mathcal{B}(\mathcal{C}_2^0), P^W)$, where P^W is the Wiener measure. The coordinate process $B(t,w) = w(t)$, $w \in \mathcal{C}_2^0$, is the m-dimensional Wiener process.

We assume that

(A1) for every $t \in [-r,T]$ the algebra $\mathcal{B}_{-r,t}(X) \cup \mathcal{B}_{-r,t}(dB)$ is independent of $\mathcal{B}_{t,T}(dB)$

to give a meaning to the stochastic integrals below.

For further considerations we need the notion of a segment of a trajectory. Let f be a function of $t \in [-r,T]$. For fixed $t \in [0,T]$ we define a function f_t on $[-r,0]$ by the formula

$$f_t(\theta) = f(t+\theta).$$

Similarly, for the stochastic process $X(t,\omega)$ we define

$$X_t(\theta,\omega) = X(t+\theta,\omega), \qquad \theta \in I.$$

We take a continuous stochastic process $X : [-r,T] \times \widetilde{\Omega} \to \mathbb{R}^d$, that is, $X : \widetilde{\Omega} \to \mathcal{X} = \mathcal{C}_1$, and a fixed initial stochastic process $X_0(\theta,\omega)$.

Let us denote by $\mu = \mu_{f,s,X_s}$ the unique measure satisfying the condition

$$D_x f(s,X_s)(\phi) = \sum_{j=1}^{d} \int_{-r}^{0} \phi_j(v)\, \mu^j(dv), \tag{1}$$

where D_x denotes the Fréchet derivative with respect to x (i.e. the linear operator from $\mathcal{C}_-$ to $L(\mathcal{C}_-, \mathbb{R})$) of the function $f : [0,T] \times \mathcal{C} \to \mathbb{R}^d$, while by $\widetilde{\mu}$ the measure defined by

$$\widetilde{\mu}(A) = \mu(A \cap [-r,0)) \quad \text{for } A \in \mathcal{B}([-r,0]).$$

It is clear that for the Dirac measure δ_0 we have

$$\mu = \widetilde{\mu} + \mu(\{0\})\delta_0. \tag{2}$$

Denote the j-th coordinate of μ taken at $\{0\}$ by

$$(D_j f(s,X_s)) = \mu^j_{f,s,X_s}(\{0\}). \tag{$*$}$$

We shall also use the property of the Dirac measure that for a smooth function $h(\cdot)$ we have $\int_{-r}^{0} h(v)\,\delta_0(dv) = h(0)$.

Remark 1. The measure is understood as in [6], Chapter 6, i.e. it need not be positive.

Now we introduce the operators $b : [0,T] \times \mathcal{C}_- \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathcal{C}_- \to L(\mathbb{R}^m, \mathbb{R}^d)$ (the uniform operator norm in $L(\mathbb{R}^m, \mathbb{R}^d)$ is denoted by $|\ |_L$).

Let us assume that

(A2) b and σ are continuous operators.

We consider the following stochastic integral equation with delayed argument:

$$(3)\quad X^i(t,w) = \\ = X_0^i(w) + \int_0^t b^i(s, X_s(\cdot,w))\,ds + \sum_{p=1}^m \int_0^t \sigma^{ip}(s, X_s(\cdot,w))\,dw^p(s)$$

for $i = 1, \dots, d$. The second integral in (3) is the Itô integral. Apart from (3) we consider the equation

$$(4)\quad X^i(t,w) = X_0^i(w) + \int_0^t b^i(s, X_s(\cdot,w))\,ds + \\ + \sum_{p=1}^m \int_0^t \sigma^{ip}(s, X_s(\cdot,w))\,dw^p(s) + \\ + \frac{1}{2}\sum_{p=1}^m \sum_{j=1}^d \int_0^t (D_j\sigma^{ip}(s, X_s(\cdot,w)))\sigma^{jp}(s, X_s(\cdot,w))\,ds$$

for $i = 1, \dots, d$.

We also assume that

(A3) X_0 is an $\mathcal{F}_0$- adapted process on $[-r, 0]$, $\mathcal{B}_{-r,0}(X_0)$ is independent of $\mathcal{B}_{0,T}(dB)$,

(A4) for every $\varphi, \psi \in \mathcal{C}_-$ we have

$$|b(t,\varphi) - b(t,\psi)|^2 + |\sigma(t,\varphi) - \sigma(t,\psi)|_L^2 \le \\ \le \int_{-r}^0 |\varphi(\theta) - \psi(\theta)|^2 K(d\theta) + L|\varphi(0) - \psi(0)|^2,$$

where K is a certain finite measure on I, and L is a constant,

(A5) for every $\varphi \in \mathcal{C}_-$ we have

$$|b(t,\varphi)|^2 + |\sigma(t,\varphi)|_L^2 \le \int_{-r}^0 (1 + \varphi(\theta)^2) K(d\theta) + L(1 + \varphi(0)^2),$$

(A6) there exists a constant $M > 0$ such that for every $s, t \in I$ and $\varphi \in C_-$ we have

$$|\sigma(s,\varphi) - \sigma(t,\varphi)|_L \leq M|s-t|,$$

(A7) the process X_0 satisfies $E|X_0(\theta)|^4 < \infty$ for every $\theta \in [-r, 0]$, where $|X_0(\theta, w)| = \sum_{i=1}^d |X_0^i(\theta, w)|$.

DEFINITION 1. We say that a d-dimensional continuous stochastic process $X : [-r,T] \times C_2^0 \to \mathbb{R}^d$ is a strong solution to equation (3) for a given process $w(t)$ if conditions (A1), (A2) are satisfied, $P(\int_0^T |b(s,X_s)|\, ds < \infty) = 1$, $P(\int_0^T |\sigma(s,X_s)|_L^2\, ds < \infty) = 1$ and equation (3) is valid with probability 1 for all $t \in [0,T]$.

The uniqueness of strong solutions is understood in the sense of trajectories.

We have the existence and uniqueness of solutions to (3) and (4) under conditions (A2)–(A5) (see [8], [9]).

From [4] it follows that there exists a constant $A > 0$ such that $E|X(t)|^2 + E|X(t)|^4 < A$.

3. Definition of the Stratonovich integral

We have

DEFINITION 2. Given a function $f : [0,T] \times C_- \to \mathbb{R}$, we consider the following limit

$$(5)\quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} [w(t_i^n) - w(t_{i-1}^n)] f(\tfrac{1}{2}(t_i^n + t_{i-1}^n), \tfrac{1}{2}(X_{t_i^n} + X_{t_{i-1}^n})),$$

where $w(t)$ is the one-dimensional Wiener process. The limit is taken in the mean square sense and $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ is a partition of the interval $[a,b]$. We assume that the sequence of partitions is normal, that is, $\max(t_{i-1}^n - t_i^n) \underset{n\to\infty}{\to} 0$. If this limit exists and does not depend on the choice of the partition, it is called the Stratonovich integral and is denoted by $(\mathrm{S})\int_0^T f(t,X_t)\, dw(t)$.

We recall the definition of the Itô integral:

$$(6)\quad (\mathrm{I}) \int_0^T f(t,X_t)\, dw(t) = \lim_{n\to\infty} I_n =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [w(t_i^n) - w(t_{i-1}^n)] f(t_{i-1}^n, X_{t_{i-1}^n}),$$

with the same assumptions as in Definition 2.

4. Relation between the Stratonovich and Itô integrals

We shall prove the following

THEOREM 1. *Let $f : [0,T] \times C_- \to \mathbb{R}$ be continuous in the first variable and differentiable in both variables with bounded derivative $\frac{\partial f}{\partial t}$ and have continuous Fréchet derivative with respect to the second variable. Moreover,*

$$(7) \qquad \int_0^T E|f(s,X_s)|^2\,ds < \infty, \qquad E\int_0^T \left|\frac{d}{ds}f(s,X_s)\right|^2 ds < \infty,$$

where X is the strong solution to stochastic differential equation (3) *and $w(t)$ is the m-dimensional Wiener process. Moreover, we assume that conditions* (A1)–(A7) *are satisfied. Then there exists the Stratonovich integral* (5) *and the following relation is valid:*

$$(8) \qquad (\mathrm{S}) \sum_{p=1}^{m} \int_0^T f(t,X_t)\,dw^p(t) =$$
$$= (\mathrm{I})\sum_{p=1}^{m}\int_0^T f(t,X_t)\,dw^p(t) + \frac{1}{2}\sum_{p=1}^{m}\sum_{j=1}^{d}\int_0^T (D_j f(s,X_s))\sigma^{jp}(s,X_s)\,ds.$$

Remark 2. The condition (7) is satisfied when f is the bounded function together with its first derivatives.

We shall use the following

LEMMA 1. *Let $\sigma : [0,T]\times C_- \to \mathbb{R}$ satisfy assumptions* (A4)–(A6). *Then there exist some positive constants C_1, C_2 such that*

$$E[\sigma(s,X_s) - \sigma(u,X_u)]^2 \le C_1(s-u),$$
$$E[\sigma(s,X_s) - \sigma(u,X_u)]^4 \le C_2(s-u)^2.$$

Proof of Lemma 1. Let $\alpha = 1,2$ and $C_3,\dots,C_7$ be some positive constants. We estimate

$$E[\sigma(s,X_s)-\sigma(u,X_u)]^{2\alpha} = E[\sigma(s,X_s)-\sigma(u,X_s)+\sigma(u,X_s)-\sigma(u,X_u)]^{2\alpha} \le$$
$$\le C_3(\alpha)[E[\sigma(s,X_s)-\sigma(u,X_s)]^{2\alpha} + E[\sigma(u,X_s)-\sigma(u,X_u)]^{2\alpha}] \le$$
$$\le C_4(\alpha)\Big[M(s-u)^{2\alpha} + E\Big[\int_{-r}^{0} |X_s(\theta)-X_u(\theta)|^2 K(d\theta)\Big] + LE|X(s)-X(u)|^2\Big]^{\alpha} \le$$
$$\le C_4(\alpha)\Big[(s-u)^{2\alpha} +$$
$$+E\Big[\int_{-r}^{0} |X(s+\theta)-X(u+\theta)|^2 K(d\theta)\Big]^{\alpha} + LE|X(s)-X(u)|^{2\alpha}\Big] \le$$

$$\leq C_4(\alpha)\Big[(s-u)^{2\alpha}+E\Big[\int\limits_{-r}^{0}\Big|\int\limits_{u+\theta}^{s+\theta} b(v,X_v)dv+\int\limits_{u+\theta}^{s+\theta}\sigma(v,X_v)\,dw(v)\Big|^2 K(d\theta)\Big]^\alpha+$$

$$+LE\Big|\int\limits_{u}^{s}(v,X_v)\,dv+\int\limits_{u}^{s}\sigma(v,X_v)\,dw(v)\Big|^{2\alpha}\Big].$$

So it is sufficient to show that

$$E\Big[\int\limits_{u}^{s} b(v,X_v)\,dv)\Big]^{2\alpha}\leq C_5(\alpha)(s-u)^\alpha$$

and

$$E\Big[\int\limits_{u}^{s}\sigma(v,X_v)\,dw(v))\Big]^{2\alpha}\leq C_6(\alpha)(s-u)^\alpha.$$

First we shall prove the former inequality

$$E\Big[\int\limits_{u}^{s} b(v,X_v)\,dv\Big]^{2\alpha}\leq E\Big[\Big|(s-u)\int\limits_{u}^{s} b(v,X_v)^2\,dv\Big|^\alpha\Big]\leq$$

$$\leq (s-u)^\alpha E\Big|\int\limits_{u}^{s} b(v,X_v)^2\,dv\Big|^\alpha\leq$$

$$\leq (s-u)^\alpha E\Big[\int\limits_{-r}^{0}\int\limits_{u}^{s}(1+X_v(\theta)^2)\,dvK(d\theta)+L\int\limits_{u}^{s}(1+X_v(\theta)^2)\,dv\Big]^\alpha=H.$$

For $\alpha=1$ we have

$$H=\Big\{K([-r,0])(s-u)+\int\limits_{-r}^{0}\int\limits_{u}^{s}E(X(v+\theta)^2)\,dvK(d\theta)+L(s-u)+$$

$$+L\int\limits_{u}^{s}E(X(v)^2)\,dv\Big\}(s-u)\leq C_5(1)(s-u).$$

For $\alpha=2$ we have

$$H=(s-u)^2E\Big[\int\limits_{-r}^{0}\int\limits_{u}^{s}(1+X_v(\theta)^2)\,dvK(d\theta)+L\int\limits_{u}^{s}(1+X_v(\theta)^2)\,dv\Big]^2\leq$$

$$\leq 2(s-u)^2E\Big[\int\limits_{-r}^{0}\int\limits_{u}^{s}(1+X_v(\theta)^2)\,dvK(d\theta)\Big]^2+$$

$$+2(s-u)^2L^2E\Big[\int\limits_{u}^{s}(1+X(v)^2)dv\Big]^2\leq$$

$$\leq 2(s-u)^2 K([-r,0])(s-u)E\Big[\int_{-r}^{0}\int_{u}^{s}(1+X_v(\theta)^2)^2\,dvK(d\theta)\Big]+$$

$$+2(s-u)^2L^2(s-u)E\Big[\int_u^s(1+X(v)^2)^2\,dv\Big]\leq C_5(2)(s-u)^2$$

since $EX(v)^2$ and $EX(v)^4$ are bounded.

Now we shall prove the second inequality mentioned above. For $\alpha = 1$ we have

$$E\Big[\int_u^s \sigma(v,X_v)\,dw(v)\Big]^2 = \int_u^s E[\sigma(v,X_v)^2]\,dv \leq$$

$$\leq E\Big[\int_{-r}^{0}\int_u^s(1+X(v+\theta)^2)\,dvK(d\theta)+L\int_u^s(1+X(v)^2)\,dv\Big]$$

which is already estimated.

For $\alpha = 2$ we have from Itô's formula

$$\Big[\int_u^s \sigma(v,X_v)\,dw(v)\Big]^2 =$$

$$=2\int_u^s\Big[\int_u^v \sigma(\tau,X_\tau)\,dw(\tau)\Big]\sigma(v,X_v)\,dw(v)+\frac{1}{2}\int_u^s\sigma(v,X_v)^2\,dv.$$

So it is sufficient to estimate (using some facts from [4])

$$E\Big[\int_u^s\Big[\int_u^v\sigma(\tau,X_\tau)\,dw(\tau)\Big]\sigma(v,X_v)\,dw(v)\Big]^2 =$$

$$=\int_u^s E\Big[\Big[\int_u^v\sigma(\tau,X_\tau)\,dw(\tau)\Big]\sigma(v,X_v)\Big]^2\,dv =$$

$$=\int_u^s E\Big[\Big[\int_u^v\sigma(\tau,X_\tau)dw(\tau)\Big]^2\sigma(v,X_v)\Big]^2\,dv\leq$$

$$\leq\int_u^s E\Big[\Big[\int_u^v\sigma(\tau,X_\tau)dw(\tau)\Big]^4 E\sigma(v,X_v)^4\Big]^{1/2}\,dv\leq$$

$$\leq C_7\int_u^s\Big[(v-u)\int_u^v E\sigma(\tau,X_\tau)^4\,d\tau E\sigma(v,X_v)^4\Big]^{1/2}\,dv.$$

From the assumptions and boundedness of the moments of X the lemma follows.

Proof of Theorem 1. Consider the difference, for $p = 1, \dots, m$,

$$S_n - I_n = \sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]f(\tfrac{1}{2}(t_i^n + t_{i-1}^n), \tfrac{1}{2}(X_{t_i^n} + X_{t_{i-1}^n}))-$$

$$-\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]f(t_{i-1}^n, X_{t_{i-1}^n}) =$$

$$= \sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)][f(\tfrac{1}{2}(t_i^n + t_{i-1}^n), \tfrac{1}{2}(X_{t_i^n} + X_{t_{i-1}^n})) - f(t_{i-1}^n, X_{t_{i-1}^n})].$$

Hence

$$S_n - I_n = \sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]\sum_{j=1}^{d} D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n}) \cdot \frac{1}{2}(X_{t_i^n}^j - X_{t_{i-1}^n}^j)+$$

$$+\frac{1}{2}\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)](t_i^n - t_{i-1}^n)\frac{\partial}{\partial t} f(t_i^n, X_{t_{i-1}^n})+$$

$$+\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]\int_0^{1/2}[D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n} + \lambda(X_{t_i^n}^j - X_{t_{i-1}^n}^j))-$$

$$-D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})](X_{t_i^n}^j - X_{t_{i-1}^n}^j)\, d\lambda = J_1 + J_2 + J_3.$$

To show that $J_3 \to 0$ with probability one, we shall first notice that for a given trajectory X the function $t \to X_t$ is uniformly continuous and in consequence

$$\sum_{i=1}^{n}\Big|\int_0^{1/2}[D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n} + \lambda(X_{t_i^n}^j - X_{t_{i-1}^n}^j)) - D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})]\, d\lambda =$$

$$= \sum_{i=1}^{n}\Big|\int_{t_{i-1}^n}^{\frac{1}{2}(t_{i-1}^n + t_i^n)}[D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n}^n) - D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})]\, ds\Big| \le$$

$$\le \sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n}|D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n}^n) - D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})|\, ds \to 0,$$

where X_t^n denotes the continuous function $[0,T] \to \mathcal{C}_-$ linear on every segment $[t_{i-1}^n, t_i^n]$ and such that $X_{t_i^n}^n = X_{t_i^n}$.

Therefore, we have

$$|J_3| \le \sup_{i=1,\dots,n}|w^p(t_i^n) - w^p(t_{i-1}^n)|\sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n}|D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})-$$

$$-D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})|\, ds \to 0,$$

which completes the proof of the convergence of J_3.

To estimate J_2 let us notice that by (7) we get

$$E|J_2|^2 \leq \frac{1}{2} \sup_{i=1,\dots,n} (t_i^n - t_{i-1}^n) E\Big[\sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)] \frac{\partial}{\partial t} f(t_{i-1}^n, X_{t_{i-1}^n})\Big]^2 \to 0.$$

To estimate J_1 we first have to notice that for $s \in I$,

$$X^j(t_i^n) - X^j(t_{i-1}^n) = \int_{t_{i-1}^n}^{t_i^n} b^j(u, X_u)\, du + \sum_{p=1}^{m} \int_{t_{i-1}^n}^{t_i^n} \sigma^{jp}(u+s, X_{u+s})\, dw^p(u).$$

Now we have from (1), (2) and (*) for $\mu = \mu_{f, t_{i-1}^n, X_{t_{i-1}^n}}$:

$$J_1 = \sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)] \sum_{j=1}^{d} D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n}) \cdot \frac{1}{2}(X^j_{t_i^n} - X^j_{t_{i-1}^n}) =$$

$$= \frac{1}{2} \sum_{j=1}^{d} \int_{-r}^{0} \sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)][X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]\, \mu(ds) =$$

$$= \frac{1}{2} \sum_{j=1}^{d} \int_{-r}^{0} \sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)][X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]\, \widetilde{\mu}(ds) +$$

$$+ \frac{1}{2} \sum_{j=1}^{d} \sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)][X^j(t_i^n) - X^j(t_{i-1}^n)]((D_{X^j} f(t_{i-1}^n, X_{t_{i-1}^n})) =$$

$$= J_{11} + J_{12}.$$

To prove the convergence of J_{11} to zero we shall first estimate for given $s < 0$ (such that $-s > \max(t_i^n - t_{i-1}^n)$) the L^2-norm of the expression under the integral

$$E\Big[\sum_{i=1}^{n} [w^p(t_i^n) - w^p(t_{i-1}^n)][X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]\Big]^2 =$$

$$= \sum_{i=1}^{n} E[w^p(t_i^n) - w^p(t_{i-1}^n)]^2 [X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]^2 +$$

$$+ \sum_{i<i'} 2E[w^p(t_i^n) - w^p(t_{i-1}^n)][w^p(t_{i'}^n) - w^p(t_{i'-1}^n)] \times$$

$$\times [X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]^2 [X^j(t_{i'}^n + s) - X^j(t_{i'-1}^n + s)]^2 =$$

$$= \sum_{i=1}^{n} E[w^p(t_i^n) - w^p(t_{i-1}^n)]^2 E[X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]^2 =$$

$$= \sum_{i=1}^{n}(t_i^n - t_{i-1}^n)E[X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]^2 \le$$
$$\le T \sup_{i=1,\dots,n} E[X^j(t_i^n + s) - X^j(t_{i-1}^n + s)]^2 \to 0,$$

where we have used the fact that $w^p(t_{i'}^n) - w^p(t_{i'-1}^n)$ is independent of $F_{t_{i-1}^n}$ and from it we conclude that the whole term containing this difference is equal to zero. We also use the continuity of X in L^2. From it follows that the above expression is bounded and in the consequence $J_{11} \to 0$. Now

$$J_{12} = \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)][X^j(t_i^n) - X^j(t_{i-1}^n)]\mu^j(\{0\}) =$$
$$= \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]\int_{t_{i-1}^n}^{t_i^n} b^j(u, X_u)\,du\mu^j(\{0\})+$$
$$+\frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}\sum_{p=1}^{m}[w^p(t_i^n) - w^p(t_{i-1}^n)]\int_{t_{i-1}^n}^{t_i^n}\sigma^{jp}(u, X_u)\,dw^p(u)\mu^j(\{0\}) =$$
$$= K_1 + K_2.$$

We estimate, for a certain constant $M > 0$,

$$\|K_1\|_{L^2} \le \sum_{j=1}^{d}\sum_{i=1}^{n} M(t_i^n - t_{i-1}^n)[E[w^p(t_i^n) - w^p(t_{i-1}^n)]^2\mu^j(\{0\})^2]^{1/2} \le$$
$$\le \sup_i(t_i^n - t_{i-1}^n)\sum_{i=1}^{n}[E[w^p(t_i^n) - w^p(t_{i-1}^n)]^2\mu^j(\{0\})^2]^{1/2}.$$

At last we have

$$K_2 = \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]^2\sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n})\mu^j(\{0\})+$$
$$+\frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}[w^p(t_i^n) - w^p(t_{i-1}^n)]\Big[\int_{t_{i-1}^n}^{t_i^n}(\sigma^{jp}(u, X_u) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))\times$$
$$\times\, dw^p(u)\Big]\mu^j(\{0\}) = K_{21} + K_{22}.$$

We observe that K_{21} gives the correction term. That is,

$$E\Big[K_{21} - \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}\sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n})\mu^j(\{0\})(t_i^n - t_{i-1}^n)\Big]^2 =$$

$$= E\Big[\frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}\sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}})\mu^{j}(\{0\})[[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]^{2} - (t_{i}^{n} - t_{i-1}^{n})]^{2}$$

$$= \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}E[\sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}})\mu^{j}(\{0\})]\times$$

$$\times E[[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]^{2} - (t_{i}^{n} - t_{i-1}^{n})]^{2} \to 0$$

because we have from some properties of the normal distribution,

$$E[[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]^{2} - (t_{i}^{n} - t_{i-1}^{n})]^{2} =$$
$$= E[[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]^{4} + (t_{i}^{n} - t_{i-1}^{n})^{2}] - 2E[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]^{2}(t_{i}^{n} - t_{i-1}^{n}) =$$
$$= 3(t_{i}^{n} - t_{i-1}^{n})^{2} + (t_{i}^{n} - t_{i-1}^{n})^{2} - 2(t_{i}^{n} - t_{i-1}^{n})^{2} = 2(t_{i}^{n} - t_{i-1}^{n})^{2}.$$

Further, we have

$$K_{22} = \frac{1}{2}\sum_{j=1}^{d}\sum_{i=1}^{n}[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]\times$$

$$\times\Big[\int_{t_{i-1}^{n}}^{t_{i}^{n}}(\sigma^{jp}(u, X_{u}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))\,dw^{p}(u)\Big]\mu^{j}(\{0\}).$$

We shall estimate using the Itô formula, and the Itô integral properties

$$[w^{p}(t_{i}^{n}) - w^{p}(t_{i-1}^{n})]\int_{t_{i-1}^{n}}^{t_{i}^{n}}(\sigma^{jp}(u, X_{u}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))\,dw^{p}(u) =$$

$$= \int_{t_{i-1}^{n}}^{t_{i}^{n}} 1\cdot dw^{p}(u)\int_{t_{i-1}^{n}}^{t_{i}^{n}}(\sigma^{jp}(u, X_{u}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))\,dw^{p}(u) =$$

$$= \int_{t_{i-1}^{n}}^{t_{i}^{n}} 1\cdot\Big(\int_{t_{i-1}^{n}}^{u}(\sigma^{jp}(s, X_{s}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))\,dw^{p}(s)\Big)\,dw^{p}(u) +$$

$$+ \int_{t_{i-1}^{n}}^{t_{i}^{n}}(\sigma^{jp}(u, X_{u}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))(w^{p}(u) - w^{p}(t_{i-1}^{n}))\,dw^{p}(u) +$$

$$+ \int_{t_{i-1}^{n}}^{t_{i}^{n}}(\sigma^{jp}(u, X_{u}) - \sigma^{jp}(t_{i-1}^{n}, X_{t_{i-1}^{n}}))\,du$$

and

$$2K_{22} = \sum_{j=1}^{d}\sum_{i=1}^{n} \int_{t_{i-1}^n}^{t_i^n} \Big(\int_{t_{i-1}^n}^{u} (\sigma^{jp}(s, X_s) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))\, dw^p(s)\Big)\, dw^p(u) +$$

$$+\sum_{j=1}^{d}\sum_{i=1}^{n} \int_{t_{i-1}^n}^{t_i^n} (\sigma^{jp}(u, X_u) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))(w^p(u) - w^p(t_{i-1}^n))\, dw^p(u) +$$

$$+\sum_{j=1}^{d}\sum_{i=1}^{n} \int_{t_{i-1}^n}^{t_i^n} (\sigma^{jp}(u, X_u) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))\, du = K_{221} + K_{222} + K_{223}.$$

Let C_8, C_9, C_{10} be some constants. Using Lemma 1 we obtain

$$\|K_{221}\|_{L^2} \le$$

$$\le \sum_{j=1}^{d}\sum_{i=1}^{n} \Big\| \int_{t_{i-1}^n}^{t_i^n} \Big(\int_{t_{i-1}^n}^{u} (\sigma^{jp}(s, X_s) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))\, dw^p(s)\, dw^p(u) \Big\|_{L^2} =$$

$$= \sum_{j=1}^{d}\sum_{i=1}^{n} \sqrt{\int_{t_{i-1}^n}^{t_i^n} E\Big[\int_{t_{i-1}^n}^{u} (\sigma^{jp}(s, X_s) - \sigma^{jp}(t^n, X_{t_{i-1}^n})\, dw^p(s)\Big]^2 du} =$$

$$= \sum_{j=1}^{d}\sum_{i=1}^{n} \sqrt{\int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{u} (E[\sigma^{jp}(s, X_s) - \sigma^{jp}(t^n, X_{t_{i-1}^n})])^2\, ds\, du} \le$$

$$\le \sum_{i=1}^{n} \sqrt{\int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{u} C_1(u - t_{i-1}^n)\, ds\, du} = \sum_{i=1}^{n} \sqrt{C_1} \sqrt{\int_{t_{i-1}^n}^{t_i^n} \frac{1}{3}(u - t_{i-1}^n)^2\, du} =$$

$$= \sum_{i=1}^{n} \sqrt{C_1} \sqrt{\frac{1}{12}(t_i^n - t_{i-1}^n)^3} = C_8 \sum_{i=1}^{n} (t_i^n - t_{i-1}^n)^{\frac{3}{2}} \to 0.$$

We next estimate

$$\|K_{222}\|_{L^2} \le$$

$$\le \sum_{j=1}^{d}\sum_{i=1}^{n} \Big\| \int_{t_{i-1}^n}^{t_i^n} (\sigma^{jp}(u, X_u) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n}))(w^p(u) - w^p(t_{i-1}^n))\, dw^p(u) \Big\|_{L^2} \le$$

$$\le \sum_{j=1}^{d}\sum_{i=1}^{n} \sqrt{\int_{t_{i-1}^n}^{t_i^n} E[\sigma^{jp}(u, X) - \sigma^{jp}(t_{i-1}^n, X_{t_{i-1}^n})]^2 (w^p(u) - w^p(t_{i-1}^n))^2\, du} \le$$

$$\leq \sum_{j=1}^{d}\sum_{i=1}^{n}\sqrt{\int_{t_{i-1}^n}^{t_i^n}\sqrt{E[\sigma^{jp}(u,X_u)-\sigma^{jp}(t_{i-1}^n,X_{t_{i-1}^n})]^4 E[w^p(u)-w^p(t_{i-1}^n)]^4}\,du} \leq$$

$$\leq \sum_{i=1}^{n}\sqrt{\int_{t_{i-1}^n}^{t_i^n} C_2(u-t_{i-1}^n)E[w^p(u)-w^p(t_{i-1}^n)]^4\,du} =$$

$$= \sum_{i=1}^{n} C_9\sqrt{3\int_{t_{i-1}^n}^{t_i^n}(u-t_{i-1}^n)^3\,du} = \sum_{i=1}^{n} C_9\sqrt{\frac{1}{4}(t_i^n-t_{i-1}^n)^4} =$$

$$= C_9\sum_{i=1}^{n}(t_i^n-t_{i-1}^n)^2 \to 0.$$

Finally, we get

$$\|K_{223}\|_{L^2} \leq \sum_{j=1}^{d}\sum_{i=1}^{n}\left[E\Big[\int_{t_{i-1}^n}^{t_i^n}(\sigma^{jp}(u,X_u)-\sigma^{jp}(t_{i-1}^n,X_{t_{i-1}^n}))\,du\Big]^2\right]^{\frac{1}{2}} \leq$$

$$\leq \sum_{j=1}^{d}\sum_{i=1}^{n}\left[E(t_i^n-t_{i-1}^n)\int_{t_{i-1}^n}^{t_i^n}(\sigma^{jp}(u,X_u)-\sigma^{jp}(t_{i-1}^n,X_{t_{i-1}^n}))^2\,du\right]^{\frac{1}{2}} \leq$$

$$\leq \sum_{j=1}^{d}\sum_{i=1}^{n}\left[C_1(t_i^n-t_{i-1}^n)\int_{t_{i-1}^n}^{t_i^n}(u-t_{i-1}^n)\,du\right]^{\frac{1}{2}} =$$

$$= C_{10}\sum_{j=1}^{d}\sum_{i=1}^{n}\sqrt{(t_i^n-t_{i-1}^n)^3} \to 0.$$

Therefore, there exists $\lim_{n\to\infty} S_n$ and moreover, it is independent of the choice of the partition of the interval. Hence, we have also shown (8). The proof of the theorem is complete.

Remark 3. In particular, if $f(t,X_t)=\sigma(X_t)$ then the correction term in (8) has the form

$$\frac{1}{2}\sum_{p=1}^{m}\sum_{j=1}^{d}\int_{0}^{t}(D_j\sigma^{ip}(s,X_s))\sigma^{jp}(s,X_s)\,ds.$$

It is the same correction term as in the approximation theorem of the Wong-Zakai type in [8] for a one-dimensional Wiener process. The formula is still valid in the multi-dimensional case because the integral is a linear transformation.

References

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications.* J.Wiley & Sons, New York, 1974.

[2] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes.* North-Holland Publ. Co., Amsterdam, 1981.

[3] K. Itô and M. Nisio, *On stationary solutions of a stochastic differential equation*, J. Math. Kyoto Univ. 4 (1964), 1-75.

[4] R. Liptser and A. Shiryaev, *Statistics of Random Processes*, Springer, Berlin, 1977.

[5] D. Nualart and M. Zakai, *On the relation between the Stratonovich and Ogawa integrals*, Ann. Probability 17 (1989), 1536–1540.

[6] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1974.

[7] R. L. Stratonovich, *A new representation for stochastic integrals and equations*, SIAM J. Control Optim. 4 (1966), 362–371.

[8] K. Twardowska, *On the approximation theorem of the Wong-Zakai type for the functional stochastic differential equations*, Probab. Math. Statist. 12 (1991), 319–334.

[9] K. Twardowska, *Approximation theorems of Wong-Zakai type for stochastic differential equations in infinite dimensions*, Dissertationes Math. No. 325 (1993), 1–54.

[10] E. Wong and M. Zakai, *On the convergence of ordinary integrals to stochastic integral*, Ann. Math. Statist. 36 (1965), 1560–1564.

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