DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 2 1995

D. Przeworska-Rolewicz

FUNCTIONAL STABILITY OF LINEAR SYSTEMS

In the author's paper PR [3] there was introduced and studied a notion of stability of solutions to linear systems with right invertible operators in the homogeneous case. The non-homogeneous case has been examined in PR [4].

The purpose of the present paper is a generalization of the mentioned results for stability induced by functional shifts introduced by Z. Binderman (cf. B[1]–B[4]).

1. Preliminaries

We shall recall some definitions and theorems (without proofs) which will be used in our subsequent considerations.

Let X be a linear space over a field $\mathcal{F}$ of scalars. $L(X)$ will stand for the set of all linear operators with domains and ranges in X and $L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}$. By $\mathcal{F}[t]$ is denoted the set of all polynomials in the variable t with coefficients belonging to $\mathcal{F}$. Write

$$v_{\mathcal{F}}A = \{0 \neq \lambda \in \mathcal{F} : I - \lambda A \text{ is invertible}\} \quad \text{for } A \in L(X).$$

If $\lambda \in v_{\mathcal{F}_A}$ then $1/\lambda$ is a regular value of A.

Denote by $R(X)$ the set of all right invertible operators belonging to $L(X)$, by $\mathcal{R}_D$ - the set of all right inverses of a $D \in R(X)$ and by $\mathcal{F}_D$ - the set of all *initial* operators for D, i.e.

$$\mathcal{R}_D = \{R \in L_0(X) : DR = I\},$$
$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D \quad \text{and} \quad \exists_{R \in \mathcal{R}_D} FR = 0\}.$$

In the sequel we shall assume that $\ker D \neq \{0\}$, i.e. D is right invertible but not invertible. The theory of right invertible operators and its applications can be found in PR [1].

We admit that $0^0 = 1$. We also write: $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$. For a given operator $D \in R(X)$ write

$$S = \bigcup_{i=1}^{\infty} \ker D^i. \tag{1.1}$$

The set S is equal to the linear span $P(R)$ of all *D-monomials*:

$$S = P(R) = \operatorname{lin}\{R^k z : z \in \ker D, k \in \mathbf{N}_0\}$$

independently of the choice of a right inverse R of D (cf. PR [1]).

For a given $D \in R(X)$ we shall consider the space of *smooth* elements

$$D_\infty = \bigcap_{k \in \mathbf{N}} \operatorname{dom} D^k.$$

Clearly, $S \subset D_\infty \subset \operatorname{dom} D$ (cf. PR [2]).

Let $\mathcal{F} = \mathbf{C}$. In the sequel, K will stand either for a disk $K_\rho = \{h \in \mathbf{C} : |h| < \rho, 0 < \rho < \infty\}$ or for the complex plane $\mathbf{C}$. Denote by $H(K)$ the space of all functions analytic on the set K. Suppose that a function $f \in H(K)$ has the following expansion

$$f(h) = \sum_{k=0}^{\infty} a_k h^k \quad \text{for all } h \in K. \tag{1.2}$$

DEFINITION 1 .1. (cf. B [1]) Suppose that $D \in R(X)$ and $\ker D \neq \{0\}$. A family $T_K = \{T_h\}_{h \in K} \subset L_0(X)$ is said to be a family of *functional shifts* for the operator D induced by a function $f \in H(K)$ (i.e. of the form (1.2)) if

$$T_h x = [f(hD)]x = \sum_{k=0}^{\infty} a_k h^k D^k x \quad \text{for all } h \in K,\ x \in S, \tag{1.3}$$

where S is defined by Formula (1.1).

We should point out that, by definition of S, the last sum has only a finite number of members different than zero.

PROPOSITION 1.1 (cf. B [2]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $T_K = \{T_h\}_{h \in K} \subset L_0(X)$. Let $f \in H(K)$. Then the following conditions are equivalent*:

(i) *T_K is a family of functional shifts for the operator D induced by the function f*;

$$\text{(ii)} \qquad T_h R^k F = \sum_{j=0}^{k} a_j h^j R^{k-j} F \quad \textit{for all } h \in K,\ k \in \mathbf{N}_0.$$

PROPOSITION 1.2 (cf. B [2]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ is a family of functional shifts for the operator D induced by a function $f \in H(K)$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then*

(i) *For all $h \in K$, $z \in \ker D$, $k \in \mathbf{N}_0$*

$$T_{f,h}R^k z = \sum_{j=0}^{k} a_j h^j R^{k-j} z; \tag{1.4}$$

(ii) *The operators $T_{f,h}$ $(h \in K)$ are uniquely determined on the set S;*

(iii) *If X is a complete linear metric space, $\overline{S} = X$ and $T_{f,h}$ are continuous for $h \in K$ then $T_{f,h}$ are uniquely determined on the whole space;*

(iv) *For all $h \in K$ the operators $T_{f,h}$ commute on the set S with the operator D.*

The listed properties and other informations about functional shifts for right invertible operators can be found in B[1]–B[4] (cf. PR [1], PR [3], PR [4] for shifts induced by the function $f(h) = e^h$) .

Proposition 1.2 of B [3] implies

PROPOSITION 1.3. *Suppose that all assumptions of Proposition 1.2 are satisfied and $f(0) = a_0 \neq 0$. Let*

$$F_h = [f(0)]^{-1} F T_{f,h} \quad \textit{for } h \in K. \tag{1.5}$$

Then F_h is an initial operator for D corresponding to the right inverse

$$R_h = R - F_h R \quad (h \in K). \tag{1.6}$$

It is well-known that $H(K)$ is a commutative ring with the following algebraic operations

$$(f+g)(h) = f(h) + g(h), \quad (\alpha g)(h) = \alpha g(h), \quad (fg)(h) = f(h)g(h),$$

where $f, g \in H(K)$, $\alpha \in \mathbf{C}$, $h \in K$.

Let $T(K)$ be the set of all families of functional shifts for an operator $D \in R(X)$ induced by the members of $H(K)$, i.e.

$$T(K) = \{T_{f,K} : f \in H(K)\}. \tag{1.7}$$

Define the following operations

$$T_{f,K} + T_{g,K} = T_{f+g,K}, \quad \alpha T_{g,K} = T_{\alpha g,K}, \quad T_{f,K}T_{g,K} = T_{fg,K}, \tag{1.8}$$

where $f, g \in H(K)$, $\alpha \in \mathbf{C}$.

THEOREM 1.1 (cf. B [2]). *Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $T(K)$ is defined by Formula (1.7). Then*

(i) *The set $T(K)|_S = \{T_{f,K}|_S : f \in H(K)\}$ is a commutative ring with the operations defined by Formulae (1.8);*

(ii) *The rings* $H(K)$ *and* $T(K)|_S$ *are isomorphic. The mapping*

$$T : f \Longrightarrow T_{f,K}|_S$$

is a ring isomorphism of $H(K)$ *onto* $T(K)|_S$.

THEOREM 1.2 (cf. B&PR [1]). *Suppose that* $D \in R(X)$, $\ker D \neq \{0\}$ *and* $T_{f,K} = \{T_{f,h}\}_{h\in K} \in T(K)$. *Let* F *be an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$. *Suppose, moreover, that* $1/f \in H(K)$, *i.e.* $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K} \in T(K)$. *Then*

$$R_h^n z = f(0) T_{1/f,h} R^n z \quad \text{for all } n \in \mathbf{N},\ h \in K,\ z \in \ker D, \tag{1.9}$$

where the operators R_h $(h \in K)$ *are defined by Formula* (1.6).

Assume that X is a complete linear metric space over $\mathbf{C}$ and the function $f \in H(K)$ has the expansion (1.2). Write for an operator $D \in R(X)$ (cf. B [4])

$$S_f(D) = \Big\{ x \in X : \sum_{k=0}^{\infty} a_k h^k D^k x \text{ is convergent for all } h \in K \Big\}. \tag{1.10}$$

PROPOSITION 1.4 (cf. B [4]). *Suppose that* X *is a complete linear metric space over* $\mathbf{C}$, $D \in R(X)$, $\ker D \neq \{0\}$ *and* $f \in H(K)$. *Then* $S \subset S_f(D) \subset \operatorname{dom} D$ *for all* $n \in \mathbf{N}_0$.

Similarly, as Definition 1.1, we have

DEFINITION 1.2 (cf. B [2]). Suppose that X is a complete linear metric space over $\mathbf{C}$, $D \in R(X)$ and $\ker D \neq \{0\}$. A family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$ is said to be a family of *functional shifts* for D induced by a function $f \in H(K)$ if

$$T_{f,h} x = f(hD)x \quad \text{for all } h \in K,\ x \in S_f(D), \tag{1.11}$$

where the operator $f(hD)$ is defined by Formula (1.3) and the set $S_f(D)$ is defined by Formula (1.10).

NOTE 1.1. Let X be a complete linear metric space $X(\mathcal{F} = \mathbf{C}$ or $\mathcal{F} = \mathbf{R})$. Let be given $D \in R(X)$ and an initial operator F for D corresponding to an $R \in \mathcal{R}_D$. Consider the space of *D-analytic* elements

$$A_R(D) = \Big\{ x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x \Big\} = \{ x \in D_\infty : \lim_{n\to\infty} R^n D^n x = 0 \},$$

Clearly, $S \subset A_R(D) \subset D_\infty \subset \operatorname{dom} D$ (cf. PR [1], PR [2]). If $x \in A_R(D)$, then for an arbitrary family $T_{f,K}$ of continuous functional shifts induced by a function $f \in H(K)$ we have $T_{f,h}x = f(hD)x$ for all $h \in K$ (cf. B [4]).

DEFINITION 1.3 (cf. PR [3]). Let X be a complete linear metric space over $\mathcal{F}$ ($\mathcal{F} = \mathbf{C}$ or $\mathcal{F} = \mathbf{R}$). Let E be a subspace of X. A continuous operator

$A \in L(X)$ is said to be *almost quasinilpotent on* E if

$$\lim_{n\to\infty} \lambda^n A^n x = 0 \quad \text{for every } x \in E,\ \lambda \in v_{\mathcal{F}}A.$$

The set of all operators almost quasinilpotent on E will be denoted by $AQN(E)$.

THEOREM 1.3 (cf. PR [3]). *Let E be a subspace of a complete linear metric space (over $\mathcal{F}$). If $A \in L(X)$, $E \subset \operatorname{dom} A$ and $\lambda \in v_{\mathcal{F}}A$ then the following conditions are equivalent:*

(i) *A is almost quasinilpotent on E;*

(ii) *for every $x \in X$ the series $\sum_{n=0}^{\infty} \lambda^n A^n x$ is convergent and $(I - \lambda A)^{-1}x = \sum_{n=0}^{\infty} \lambda^n A^n x$;*

(iii) *for every $x \in X$ the series $\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x$ is convergent and $(I - \lambda A)^{-m}x = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \lambda^n A^n x$.*

THEOREM 1.4 (cf. B&PR [1]). *Suppose that X is a complete linear metric space over $\mathbf{C}$, $D \in R(X)$, $\ker D \neq \{0\}$, $\lambda \in \mathbf{C}$, $\ker(D - \lambda I) \neq \{0\}$, $\lambda K \subset K$, $R \in \mathcal{R}_D \cap AQN[\ker(D - \lambda I)]$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$, $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K}$ are families of functional shifts for D induced by the functions $f, 1/f \in H(K)$, respectively. If the operator $T_{f,h}$ is continuous for some $h \in K$ then*

$$R_h \in \mathcal{R}_D \cap AQN[\ker(D - \lambda I)],$$

where the operators $R_h (h \in K)$ are defined by Formula (1.6).

2. Homogeneous case

Let now $K = \mathbf{C}$. Consider the space $H(\mathbf{C})$ of entire functions. We begin with

DEFINITION 2.1. Suppose that X is a complete linear metric space over $\mathbf{C}$, $D \in R(X)$, $\ker D \neq 0$, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $T_{f,\mathbf{C}} = \{T_{f,h}\}_{h\in\mathbf{C}} \in T(\mathbf{C})$. Write

$$x^\wedge(h) = F_h x = F T_{f,h} x \quad \text{for } x \in S_f(D),\ h \in \mathbf{C}. \tag{2.1}$$

If
$$\lim_{h\in\mathbf{R}_+,\, h\to+\infty} T_{f,h}x = 0 \quad \text{for an } x \in S_f(D)$$

then x is said to be *f-stable*, or *functionally stable*. If

$$\lim_{h\in\mathbf{R}_+,\, h\to+\infty} F T_{f,h} x = 0, \text{ i.e. } \lim_{h\in\mathbf{R}_+,\, h\to+\infty} x^\wedge(h) = 0 \quad \text{for an } x \in S_f(D),$$

then x is said to be *(F, f)-stable*, or *functionally F-stable.*

The stability introduced in PR [3] is a functional stability induced by the function $f(h) = e^h$. It should be pointed out that the functional F-stability is, in a sense, a *local* functional stability, as a stability introduced by an initial operator. Clearly, if x is f-stable and F is continuous, then x is (F, f)-stable. The zero element is f-stable for every $f \in H(\mathbf{C})$.

Let $R \in \mathcal{R}_D$ be a VOLTERRA right inverse, i.e. $v_{\mathcal{F}}R = \mathcal{F} \setminus \{0\}$. Consider the space of *exponentials*:

$$E(R) = \bigcup_{\lambda \in \mathcal{F}} \ker(D - \lambda I) = \operatorname{lin}\{(I - \lambda R)^{-1}z : z \in \ker D,\ \lambda \in \mathcal{F}\} \subset D_\infty .$$

Note that $E(R)$ is independent of the choice of R (cf. PR [1], PR [2]).

Write for $A \in L_0(X)$:

$$(2.2) \quad \mathcal{F}_A[t] = \\ = \{P(t) = t^M Q(t) : Q(t) \in \mathcal{F}[t] \text{ and } Q(\lambda) = 0 \text{ implies } \lambda \in v_{\mathcal{F}}A;\ M \in \mathbf{N}_0\}.$$

THEOREM 2.1 (cf. PR [3]). *Suppose that X is a complete linear metric locally convex space (over $\mathbf{R}$), $D \in R(X)$ is closed, $\ker D \neq \{0\}$, F is a continuous initial operator corresponding to an $R \in \mathcal{R}_D \cap AQN(\ker D)$, $f(h) = e^h$, $A(\mathbf{R})$ is either $\mathbf{R}_+$ or $\mathbf{R}$, $\{T_{f,h}\}_{h \in A(\mathbf{R})}$ is a strongly continuous semigroup (group) of functional shifts induced by the function f and either $\overline{S} = X$ or R is a VOLTERRA right inverse and $\overline{E(R)} = X$. Then*

(i) *D is an infinitesimal generator for $\{T_{f,h}\}_{h \in A(\mathbf{R})}$, hence $\overline{\operatorname{dom} D} = X$ and $T_{f,h}D = DT_{f,h}$ on dom D. Moreover, for the canonical mapping κ defined as*

$$\kappa x = \{x^\wedge(t)\}_{t \in A(\mathbf{R})}, \quad \text{where} \quad x^\wedge(t) = FT_{f,t}x \ (x \in X)$$

we have

$$\kappa D = \frac{\mathrm{d}}{\mathrm{d}t}\kappa, \quad \kappa R = \int_0^t , \quad \kappa F x = \kappa x|_{t=0},$$

and $(\kappa T_{f,h})(t) = (\kappa x)(t + h)$ for $x \in X$, $t, h \in A(\mathbf{R})$;

(ii) *x is stable is and only if is F-stable;*

(iii) *$x \in \ker P(D)$ is stable if and only if all roots of the polynomials $P(t) \in \mathcal{F}_R[t]$ have negative real parts.*

Points (i) and (ii) of Theorem 2.1 are proved by Corollaries 2.1 and 3.1 of PR [3]. In order to prove Point (iii) of Theorem 2.1 we had to use Theorem 3.1 of PR [3] with the following

LEMMA 2.1. *Write*

$$p^\circ(t) = e^{-t} \sum_{j=0}^{\infty} p(j) \frac{t^j}{j!} \quad \text{for } p(t) \in \mathcal{F}[t].$$

Then $p^\circ(t) \in \mathcal{F}[t]$.

Theorem 2.1 shows that the following (non-trivial) elements are not stable (hence not F-stable):

D-polynomials, i.e elements of the set S;

exponentials, i.e. elements of $\ker(D - \lambda I)$, provided that $\operatorname{Re}\lambda \geq 0$ ($\lambda \in v_{\mathcal{F}}R$);

$T_{f,\omega}$*-periodic* elements, i.e. such elements x that $T_{f,\omega}x = x$ for an $\omega \in \mathbf{R}$ (cf. PR [3]).

Write

$$(2.3)\quad St(f) = \\ = \Big\{\lambda \in \mathbf{C} : \bigwedge_{p(t)\in\mathcal{F}[t]} \lim_{h\in\mathbf{R}_+,\, h\to+\infty} p(\lambda h)f(\lambda h) = 0\Big\} \quad \text{for } f \in H(\mathbf{C});$$

$$(2.4)\qquad H_0 = \Big\{f \in H(\mathbf{C}) : \bigwedge_{h\in\mathbf{C}} f(h) \neq 0\Big\}.$$

Clearly, if $f \in H_0$ then $1/f \in H_0$ and

$$(2.5)\qquad f = e^g, \quad \text{where} \quad g \in H(\mathbf{C}),$$

since $\mathbf{C}$ is simply connected.

PROPOSITION 2.1. *Suppose that all assumptions of Definition* 2.1 *are satisfied. If* $x \in S \setminus \{0\}$ *then* x *is not* f*-stable for every* $f \in H(\mathbf{C})$.

Proof. Take any element of the set S defined by Formula (1.1) of the form: $R^n z$, where $z \in \ker D \setminus \{0\}$. Then, by Proposition 1.4, for all $h \in \mathbf{C}$, $n \in \mathbf{N}_0$ we have

$$T_{f,h}R^n z = f(hD)R^n z = \sum_{k=0}^{n} a_k h^k R^{n-k} z \in S.$$

Hence $T_{f,h}R^n z \not\to 0)$ as $h \to +\infty (h \in \mathbf{R}_+)$. ∎

NOTE 2.1. Let R_h be defined by Formula (1.6). Suppose that $f, 1/f \in H(\mathbf{C})$. This means that $f(0) \neq 0$. Let $z \in \ker D \setminus \{0\}$, $n \in \mathbf{N}_0$ be arbitrarily fixed. Proposition 2.1 and Theorem 1.2 together imply that $T_{f,h}R_h^n z = f(0)R^n z \not\to 0$ as $h \to +\infty$ ($h \in \mathbf{R}_+$).

PROPOSITION 2.2. *Suppose that all assumptions of Definition* 2.1 *are satisfied. An element* $x \in \ker(D - \lambda I)$ *is* f*-stable if and only if* $\lim_{h\in\mathbf{R}_+,\, h\to+\infty} f(\lambda h) = 0$.

Proof. By our assumption, $T_{f,h}x = f(hD)x = f(\lambda h)x \to 0$ as $h \to +\infty (h \in \mathbf{R}_+)$ if and only if $\lim_{h\in\mathbf{R}_+,h\to+\infty} f(\lambda h) = 0$ (cf. B [4]). ∎

LEMMA 2.2. *Let $f \in H_0$ (i.e. f is of the form (2.5)). Write*

$$p^f(t) = e^{-g(t)} \sum_{j=0}^{\infty} p(j) \frac{g^j(t)}{j!} \quad \text{for } p(t) \in \mathcal{F}[t]. \tag{2.6}$$

Then $p^f(t) \in \mathcal{F}[t]$ if and only if $g \in \mathcal{F}[t]$.

Proof. Write

$$\alpha_n(t) = t^n \quad \text{for } n \in \mathbf{N}_0. \tag{2.7}$$

Then

$$\alpha_{n+1}^f(t) = g(t) \sum_{j=0}^{n} \binom{n}{j} \alpha_j^f(t) \quad \text{for } n \in \mathbf{N}_0 \text{ (we admit: } \alpha_0^f = 1). \tag{2.8}$$

Indeed,

$$\begin{aligned}
\alpha_{n+1}^f(t) &= e^{-g(t)} \sum_{j=0}^{\infty} j^{n+1} \frac{g^j(t)}{j!} = e^{-g(t)} \sum_{j=1}^{\infty} j^n \frac{g^j(t)}{(j-1)!} \\
&= e^{-g(t)} \sum_{k=0}^{\infty} (k+1)^n \frac{g^{k+1}(t)}{k!} = e^{-g(t)} g(t) \sum_{k=0}^{\infty} (k+1)^n \frac{g^k(t)}{k!} \\
&= g(t) e^{-g(t)} \sum_{k=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} k^j \frac{g^k(t)}{k!} \\
&= g(t) \sum_{j=0}^{n} \binom{n}{j} e^{-g(t)} \sum_{k=0}^{\infty} k^j \frac{g^k(t)}{k!} = g(t) \sum_{j=0}^{n} \binom{n}{j} \alpha_j^f(t).
\end{aligned}$$

We therefore conclude that all α_n^f are polynomials if and only if g is a polynomial. Suppose then that $g(t) \in \mathcal{F}[t]$. Let $p(t) \in \mathcal{F}[t]$. By (2.7), $p(t)$ is of the form

$$p(t) = \sum_{n=0}^{N} p_n t^n = \sum_{n=0}^{N} p_n \alpha_n(t).$$

This, and Formula (2.8) together imply that

$$\begin{aligned}
p^f(t) &= \sum_{n=0}^{N} p_n \alpha_n^f(t) = p_0 + \sum_{k=0}^{N+1} p_{k+1} \alpha_{k+1}^f(t) \\
&= p_0 \alpha_0(t) + g(t) \sum_{k=0}^{N+1} p_{k+1} \sum_{j=0}^{k} \binom{k}{j} \alpha_j^f(t) \in \mathcal{F}[t]. \quad \blacksquare
\end{aligned}$$

Write

$$H_1 = \{e^g : g(t) \in \mathcal{F}[t]\} \subset H_0 \subset H(\mathbf{C}). \tag{2.9}$$

Then Lemma 2.2 can be formulated in the following manner:

COROLLARY 2.1. *Suppose that* $f \in H_0$. *Then* $p^f(t) \in \mathcal{F}[t]$ *if and only if* $f \in H_1$, *where* H_1 *is defined by Formula* (2.9).

THEOREM 2.2. *Suppose that* X *is a complete linear metric space over* $\mathbf{C}$, $D \in R(X)$, $\ker D \neq \{0\}$, F *is a continuous initial operator for* D *corresponding to an* $R \in \mathcal{R}_D \cap AQN(\ker D)$, $T_{f,K} = \{T_{f,h}\}_{h \in K}$ *is a family of continuous functional shifts for* D *induced by a function* $f \in H_1$ *and* $P(t) \in \mathcal{F}_R[t]$. *Then* $x \in \ker P(D) \setminus \{0\}$ *is* f*-stable if and only if all roots of* $P(t)$ *belong to the set* $St(f)$ *defined by Formula* (2.3).

Proof. Suppose that $P(t) \in \mathcal{F}_R[t]$ (cf. Formula (2.2)). Then $P(t)$ is of the form

$$P(t) = t^M Q(t), \quad \text{where} \quad Q(t) = \prod_{j=1}^{n} (t - \lambda_j)^{r_j}, \tag{2.10}$$

$$r_1 + \ldots + r_n = N, \; \lambda_j \in v_{\mathcal{F}} R \quad (j = 1, \ldots, n).$$

If $x \in \ker P(D)$ then

$$x = \sum_{j=1}^{n} \sum_{m=1}^{r_j} (I - \lambda_j R)^{-m} z_{jm} + P_{M-1},$$

where $z_{jm} \in \ker D$ and P_{M-1} is an arbitrary D-polynomial of degree at most $M-1$. We admit: $P_{M-1} = 0$ if $M = 0$, i.e. if $P(0) \neq 0$ (cf. PR [1], PR [2]). By Proposition 2.1, non-trivial D-polynomials are not f-stable. Thus 0 cannot be a root of the polynomial $P(t)$. Hence we put $P_{M-1} = 0$. We have to examine elements of the form: $(I - \lambda R)^{-(m+1)} z$, where $z \in \ker D, m = 0, 1, \ldots, r_j$ $(j = 1, \ldots, n)$ and $\lambda \in \mathbf{C} \setminus \{0\}$ is not yet determined. For $k, m \in \mathbf{N}_0$ write

$$q_{mk}(t) = \prod_{j=1}^{m} (k + j + t) \in \mathcal{F}[t]; \tag{2.11}$$

$$A_{mk}(t) = -\frac{1}{m!} e^{g(t)} q^f_{mk}(t). \tag{2.12}$$

We will show that

$$A_{mk}(t) = -\frac{1}{m!} \sum_{j=0}^{\infty} q_{mk}(j) a_j t^j \quad \text{for } k, m \in \mathbf{N}_0. \tag{2.13}$$

Indeed, by our assumptions, $f = e^g$ where $g \in \mathcal{F}[t]$. Thus g can be represented in the form

$$g(t) = \prod_{j=1}^{m}(t - t_j)^{\alpha_j}, \quad \text{where} \quad t_i \neq t_j \text{ if } i \neq j, \alpha_1 + \ldots + \alpha_m = M = \deg g,$$

and $t_j \in \mathbf{C}$, $\alpha_j \in \mathbf{N}$ $(j = 1, \ldots, m)$. Hence for every $n \in \mathbf{N}$ we have

$$g^n(t) = \Big[\prod_{j=1}^{m}(t - t_j)^{\alpha_j}\Big]^n = \prod_{j=1}^{m}(t - t_j)^{\alpha_j n} = \sum_{k=0}^{Mn} g_{kn}t^k,$$

where the coefficients g_{kn} are well-determined. Observe that, by definition, we also have

$$\sum_{n=0}^{\infty} a_n t^n = f(t) = e^{g(t)} = \sum_{n=0}^{\infty} \frac{g^n(t)}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{Mn} g_{kn}t^k = \sum_{n=0}^{\infty} \frac{1}{n!} b_n t^n,$$

where the scalar coefficients b_j $(j \in \mathbf{N}_0)$ are well-determined by coefficients g_{kn}(hence by g), but there is no neediness to determine relations between their numerical values. Clearly, $a_n = \frac{1}{n!} b_n$ $(n \in \mathbf{N}_0)$. This implies that

$$p^f(t) = e^{-g(t)} \sum_{j=0}^{\infty} p(j) \frac{g^j(t)}{j!} = e^{-g(t)} \sum_{j=0}^{\infty} \frac{p(j)}{j!} b_j t^j.$$

We therefore conclude that for all $k, m \in \mathbf{N}_0$

$$-\sum_{j=0}^{\infty} \binom{m+k+j}{m} a_j t^j = -\frac{1}{m!} \sum_{j=0}^{\infty} \frac{(m+k+j)!}{(k+j)!} a_j t^j = -\frac{1}{m!} \sum_{j=0}^{\infty} q_{mk}(j) a_j t^j$$
$$= -\frac{1}{m!} e^{g(t)} e^{-g(t)} \sum_{j=0}^{\infty} \frac{p(j)}{j!} b_j t^j = -\frac{1}{m!} e^{g(t)} q^f_{mk}(t) = A_{mk}(t).$$

Formulae (2.12) and (2.13) together imply that for all $h \in \mathbf{C}$

$$\begin{aligned} T_{f,h}(I - \lambda R)^{-(m+1)} z &= -T_{f,h} \sum_{n=0}^{\infty} \binom{n+m}{m} \lambda^n R^n z \\ &= -\sum_{n=0}^{\infty} \binom{n+m}{m} \lambda^n T_{f,h} R^n z \\ &= -\sum_{n=0}^{\infty} \binom{n+m}{m} \lambda^n \sum_{k=0}^{n} a_{n-k} h^{n-k} R^k z \\ &= -\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n+m}{m} \lambda^n a_{n-k} h^{n-k} R^k z \end{aligned}$$

$$= -\sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}\binom{m+k+j}{m}a_j h^j \lambda^j\right]\lambda^k R^k z$$

$$= \sum_{k=0}^{\infty} A_{mk}(\lambda h)\lambda^k R^k z = -\frac{1}{m!}e^{g(\lambda h)}\sum_{k=0}^{\infty} q_{mk}^f(\lambda h)\lambda^k R^k z.$$

By Lemma 2.2, all $q_{mk}^f(t) \in \mathcal{F}[t]$, because $f = e^g$ and $g \in \mathcal{F}[t]$. Hence

$$\lim_{h\in\mathbf{R}_+,\, h\to+\infty} T_{f,h}(I-\lambda R)^{-(m+1)}z = 0,$$

if and only if $\lambda \in St(f) = St(e^g)$. ∎

COROLLARY 2.2. *Suppose that all assumptions of Theorem 2.2 are satisfied and $P(t) = (t-\lambda)^m(\lambda \in \mathbf{C},\ m \in \mathbf{N}$ are arbitrarily fixed). If $\lambda \in St(f)$ then for an arbitrary $z \in \ker D$ we have $\lim_{h\in\mathbf{R}_+,\, h\to+\infty} T_{f,h}(I-\lambda R)^m z = 0$, i.e. the element $(I-\lambda R)^m z$ is f-stable.*

COROLLARY 2.3. *Suppose that all assumptions of Theorem 2.2 are satisfied. Let $g \in \mathcal{F}[t]$ and let $M = \deg g$. Then*

$$St(f) = St(e^g) = \{\lambda \in \mathbf{C} : \mathrm{Re}[g^{(M)}(0)\lambda^M] < 0\}.$$

Proof. Let $g \in \mathcal{F}[t]$, i.e. g is of the form: $g(t) = \sum_{m=0}^{M} g_m t^m$, $g_0, \ldots, g_m \in \mathbf{C}$. Clearly, $g_m = \frac{g^{(m)}(0)}{m!}$ for $m = 0, 1, \ldots, M$. Hence

$$f(t) = e^{g(t)} = \exp\Big(\sum_{m=0}^{M} g_m t^m\Big) = \prod_{m=0}^{M}\exp(g_m t^m)$$

$$= g(0)\prod_{m=0}^{M}\exp\left(\frac{g^{(m)}(0)}{m!}t^m\right),$$

i.e. $\lim_{h\in\mathbf{R}_+,\, h\to+\infty} f(\lambda h) = 0$ if and only if $\mathrm{Re}[g^{(M)}\lambda^M] < 0$. ∎

In particular, we obtain the condition for the stability studied in PR [3]:

COROLLARY 2.4. *Suppose that all assumptions of Theorem 2.2 are satisfied and $g(t) \equiv t$. Then $St(f) = St(e^t) = \{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda < 0\}$.*

3. Non-homogeneous case

Let $D \in R(X)$ and $R \in \mathcal{R}_D$. Let $\mathcal{F}_R[t]$ be defined by Formula (2.2). If $P(t) \in \mathcal{F}_R[t]$ is of the form (2.10). Write

$$Q(t,s) = \prod_{j=1}^{n}(t-\lambda_j s)^{r_j}. \tag{3.1}$$

Decompose the rational function $1/Q(1,s)$ onto vulgar fractions

$$1/Q(1,s) = \sum_{j=1}^{n}\sum_{m=1}^{r_j} d_{jm}(1-\lambda_j s)^{-m}, \tag{3.2}$$

where d_{jm} are well-determined scalar coefficients.

Recall that the general solution of the equation

$$P(D)x = y \quad \text{where} \quad y \in X,\ P(t) \in \mathcal{F}_R[t] \tag{3.3}$$

is of the form

$$x = [Q(I,R)]^{-1}R^{M+N}y + x^o, \quad \text{where} \quad x^o \in \ker P(D). \tag{3.4}$$

(cf. PR [1], also PR [2]).

The f-stability of the element $x^o \in \ker P(D)$ is established by Theorem 2.2. Thus it is sufficient to study the f-stability of the element

$$y^o = [Q(I,R)]^{-1}R^{M+N}y = \sum_{j=1}^{n}\sum_{m=1}^{r_j} d_{jm}(I-\lambda_j R)^{-1}y. \tag{3.5}$$

(cf. Formulae (2.10), (3.1)–(3.4)).

DEFINITION 3.1. Suppose that all conditions of Definition 2.1 are satisfied. An element $x \in X$ is said to be *completely f-stable with respect to an $R \in \mathcal{R}_D$* if the elements $R^n x$ are f-stable for every $n \in \mathbb{N}_0$. The set of all elements in X which are completely f-stable with respect to R will be denoted by $S^f_\infty(X;R)$, i.e.

$$S^f_\infty(X;R) = \{x \in X : \lim_{h\in\mathbb{R}_+, h\to+\infty} T_{f,h}R^n x = 0 \quad \text{for all } n \in \mathbb{N}_0\}.$$

If $x \notin S^f_\infty(X;R)$ for any $R \in \mathcal{R}_D$ then x is said to be *not completely f-stable.*

PROPOSITION 3.1. *Suppose that all conditions of Definition* 1.2 *are satisfied. Then D-polynomials are not completely f-stable independently of the choice of an $f \in H(\mathbb{C})$. In particular, constants different than zero (i.e. elements of* $\ker D \setminus \{0\}$) *are not completely f-stable.*

Proof. By Proposition 2.1, D-polynomials are not f-stable. If $z \in \ker D \subset S$ then $T_{f,h}z = z$ for all $h \in (\mathbb{C})$, hence constants are not f-stable. If $u \in S$ then $R^n u \in S$ for all $n \in \mathbb{N}$. ∎

PROPOSITION 3.2. *Suppose that all conditions of Definition* 2.1 *are satisfied, $R \in \mathcal{R}_D$ is arbitrarily fixed and $f(h) = e^h$. Even if* $\operatorname{Re}\lambda < 0$ *for $\lambda \in v_{\mathcal{F}}R$, exponentials $(I-\lambda R)^{-1}z$ ($z \in \ker D$) are not completely f-stable with respect to R, provided that $z \in \ker D$ is not equal to zero.*

Proof. If $\lambda = 0$ then the corresponding exponential is a constant. Hence, by Proposition 3.1, is not completely f-stable independently of R. Suppose that $\lambda \neq 0$. Then the element $R(I-\lambda R)^{-1}z = \frac{1}{\lambda}[(I-\lambda R)^{-1}z - z]$ is not f-stable, even if $\mathrm{Re}\lambda < 0$, since constants different than zero are not f-stable. By an easy induction, we prove that elements $R^n(I-\lambda R)^{-1}z$ are not f-stable for all $z \in \ker D \setminus \{0\}$, $n \in \mathbf{N}$ ■

A consequence of Proposition 3.2 and Theorem 1.3(iii) is the following

PROPOSITION 3.3. *Suppose that all conditions of Definition 2.1 are satisfied, $f(h) = e^h$ and $R \in AQN(\ker D)$. Then elements $(I-\lambda R)^{-m}z$ are not completely f-stable with respect to R for all $z \in \ker D \setminus \{0\}$, $\lambda \in v_{\mathcal{F}}R$, $m \in \mathbf{N}$.*

The proof is going on similar lines as that of Theorem 3.1 in PR [3], by an application of Lemma 2.1.

THEOREM 3.1. *Suppose that all conditions of Definition 2.1 are satisfied, $R \in \mathcal{R}_D$ and $f \in H_1$. Then the following conditions are equivalent:*

(i) *the element y^o defined by Formula (3.5) is f-stable;*
(ii) $R^{M+N}y \in S^f_\infty(X;R)$;
(iii) *the characteristic roots $\lambda_1,\ldots,\lambda_n$ of the polynomial $Q(t)$ belong to $St(f)$.*

Proof. By definition, we have

$$y^o = [Q(I,R)]^{-1}R^{M+N}y = \sum_{j=1}^{n}\sum_{m=1}^{r_j} d_{jm}(I-\lambda_j R)^{-m}R^{M+N}y$$
$$= \sum_{j=1}^{n}\sum_{m=1}^{r_j} d_{jm}\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}\lambda_j^k R^{M+N+k}y.$$

Hence for $h \in \mathbf{C}$

$$T_{f,h}y^o = \sum_{j=1}^{n}\sum_{m=1}^{r_j} e^{\lambda_j h}\sum_{k=0}^{\infty} q^f_{jmk}(\lambda_h)\lambda_j^k R^{M+N+k}y,$$

where, by Lemma 2.2, $q^f_{jmk}(t)$ are polynomials. We therefore conclude that $R^{M+N}y \in S^f_\infty(X;R)$ if and only if $\lambda_1,\ldots,\lambda_n \in St(f)$. On the other hand, y^o is f-stable if and only if $R^{M+N}y \in S^f_\infty(X;R)$. ■

Theorems 2.2 and 3.1 together imply

COROLLARY 3.1. *Suppose that all conditions of Definition 2.1 are satisfied, $R \in \mathcal{R}_D$ and $f \in H_1$. Then the following conditions are equivalent:*

(i) *all solutions of Equation (3.3) are f-stable;*
(ii) $R^{M+N}y \in S^f_\infty(X;R)$;

(iii) *the characteristic roots* $\lambda_1, \ldots, \lambda_n$ *of the polynomial* $Q(t)$ *belong to* $St(f)$.

Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Consider Equation (3.3) together with the *initial conditions*:

$$FD^k x = x_k, \quad \text{where} \quad x_k \in \ker D \quad (k = 0, 1, \ldots, M+N-1). \tag{3.6}$$

It is well-known that by the substitution

$$u = x - \sum_{k=0}^{M+N-1} R^k F D^k x = x - \sum_{k=0}^{M+N-1} R^k x_k \tag{3.7}$$

we reduce the problem (3.3)-(3.6) to the initial value problem with *homogeneous* initial conditions

$$FD^k u = 0 \quad (k = 0, 1, \ldots, M+N-1). \tag{3.8}$$

(cf. PR [1]).

Observe that solutions x of the problem (3.3)–(3.6) and u of the problem (3.3)–(3.8) differs each from another by a D-polynomial. This means that f'stability of solutions to one of that problems does not necessarily imply the f-stability of solutions to another problem.

In a similar manner we can consider systems of linear equations with scalar coefficients

$$Dx_j - \sum_{k=1}^{K} a_{jk} x_k = y_j \quad (j = 1, \ldots, K), \; D \in R(X), \tag{3.9}$$

where $y_j \in X$, $a_{kj} \in \mathcal{F}$ $(j, k = 1, \ldots, K)$. Write

$$X^o = X^K = \underbrace{X \times \ldots \times X}_{K \text{ times}}, \; x = (x_1, \ldots, x_K), \; y = (y_1, \ldots, y_K) \in X^o,$$

where $x_1, \ldots, x_K, y_1, \ldots, y_K \in X$,

$$A^o = (a_{jk})_{j,k=1,\ldots,K}; \qquad I^o = (\delta_{jk} I)_{j,k=1,\ldots,K}, \tag{3.10}$$

$$D^o = (\delta_{jk} D)_{j,k=1,\ldots,K}; \quad R^o = (\delta_{jk} R)_{j,k=1,\ldots,K}), \tag{3.11}$$

$R \in \mathcal{R}_D$ and δ_{jk} denotes the KRONECKER *symbol*, i.e. $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Then the system (3.9) can be written as follows

$$P(D^o)x = A_1^o y, \tag{3.12}$$

where

$$A_1^o = \sum_{k=1}^{M+N} p_k \sum_{m=0}^{k-1} (A^o)^{k-1-m} (D^o)^m, \tag{3.13}$$

$P(t) = \sum_{k=0}^{M+N} p_k t^k = t^M Q(t)$ is the characteristic (minimal) polynomial of the matrix A^o, $D^o \in R(X^o)$, $R^o \in \mathcal{R}_{D^o}$ (cf. PR [1], Section 3.5). As a matter of fact, Equation (3.12) is of the same form, as (3.3). We therefore conclude that every solution of (3.12) is of the form

$$x = [Q(I^o, R^o)]^{-1}(R^o)^{M+N} A_1^o y + x^o, \quad \text{where} \quad x^o \in \ker P(D^o),$$

$$x^o = \sum_{j=1}^{n} \sum_{m=1}^{r_j} (I^o - \lambda R^o)^{-1} z_{jm} + \sum_{m=1}^{M} (R^o)^{m-1} z_{0m}$$

with the additional condition that

$$(3.14) \qquad (A^o - \lambda_j I^o)^m z_{jm} = 0, z_{jm} \in \ker D^o$$
$$(m = 1, \ldots, r_j; j = 0, 1, \ldots, n; r_0 = M).$$

(cf. PR [1], Theorem 3.5.1). This, and Theorem 3.1 together imply

COROLLARY 3.2. *Suppose that all conditions of Definition 2.1 are satisfied by the operators D and R, hence also by the operators D^o and R^o defined by Formulae* (3.11). *Let $\{T_{f,h}\}_{h\in\mathbb{C}}$ be a family of functional shifts for D induced by a function $f \in H_1$. Let Condition* (3.14) *be satisfied. Let A_1^o be defined by Formula* (3.13). *Then the following conditions are equivalent:*

(i) *all solutions of the system* (3.8) *are f-stable*;

(ii) $(R^o)^{M+N} A_1^o y \in S_\infty^f(X^K; R)$;

(iii) *all roots of the characteristic polynomial of the matrix $(a_{jk})_{j,k=1,\ldots,K}$ belong to the set $St(f)$.*

Acknowledgment: The author is very grateful to Professor Jan ROGULSKI for his valuable and helpful comments and suggestions concerning the presented results.

References

B[1] Z. Binderman, *Cauchy integral formula induced by right invertible operators*, Demonstratio Math., 25 (1992), 671–690.

B[2] Z. Binderman, *Functional shifts induced by right invertible operators*, Math. Nachrichten, 157 (1992), 211–214.

B[3] Z. Binderman, *Applications of sequential shifts to an interpolation problem*, Collect. Math., 44 (1993), 49–59.

B[4] Z. Binderman, *A unified approach to shifts induced by right invertible operators*, Math. Nachrichten, 161 (1993), 239–252.

B&PR[1] Z. Binderman and D. Przeworska-Rolewicz, *Almost quasinilpotent operators and functional shifts*, Math. Nachrichten (submitted to print).

B&PR[2] Z. Binderman and D. Przeworska-Rolewicz, *Limit property and perturbations of functional shifts*, (submitted to print).

PR[1] D. Przeworska-Rolewicz, *Algebraic Analysis*, PWN-Polish Scientific Publishers and D. Reidel, Warszawa–Dordrecht, 1988.

PR[2] D. Przeworska-Rolewicz, *Spaces of D-paraanalytic elements*, Dissertationes Math. 302, Warszawa, 1990.

PR[3] D. Przeworska-Rolewicz, *True shifts*, J. Math. Anal. Appl. 170 (1992), 27–48.

PR[4] D. Przeworska-Rolewicz, *Stability of non-homogeneous linear systems*, In: *Parametric Optimization and Related Topics III.*, Lang, Frankfurt/Main, 1993, 437–445.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Śniadeckich 8,
00-950 WARSZAWA, POLAND
E-mail: ROLEWICZ@IMPAN.IMPAN.GOV.PL

Received April 18, 1994.