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NONRESONANCE FOR ELLIPTIC EQUATIONS UNDER NONLINEAR BOUNDARY CONDITIONS

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$ and let $\partial/\partial n$ denote the outward normal derivative to $\partial\Omega$. We are interested in studying nonresonance conditions, in the spirit of the pioneering works of Dolph [3] and Landesman & Lazer [5], for the existence of a weak solution to problems of the form

(BP) $\qquad \Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n - a(x,u)u = f(x,u) \quad \text{on } \partial\Omega.$

It will be proved that if

$$\liminf_{|\xi|\to\infty} a(x,\xi), \qquad \limsup_{|\xi|\to\infty} a(x,\xi)$$

lie strictly between two consecutive eigenvalues of the associated eigenvalue problem

(EP) $\qquad \Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n = \lambda u \quad \text{on } \partial\Omega$

then (BP) is solvable for functions f which are either sublinear or have linear growth with sufficiently small slope. Our results extend and complement earlier results for (BP) due to Chmaj & Majchrowski [2] and Klingelhofer [4].

Our method of proof is based upon Schauder's inversion method together with the a-priori bound principle.

2. Preliminaries

Thoughout this note, c denotes a generic constant.

Let $W^k = W^{k,2}(\Omega)(k = 1,2)$ denote the usual Sobolev spaces with norms $\|\cdot\|_{k,2}$ defined by

$$\|u\|_{k,2} = \Big(\int_\Omega \sum_{|a|\leq k} |D^a u(x)|^2\, dx\Big)^{1/2}$$

and let $\gamma_0 : W^1 \to L^2(\partial\Omega)$ denote the trace operator. It is well known that there exists a constant c such that

(1) $$\|\gamma_0 u\|_{2,\partial\Omega} \le c\|u\|_{1,2} \quad \text{for all } u \in W^1.$$

Here $\|\cdot\|_{2,\partial\Omega}$ denotes the norm on $L^2(\partial\Omega)$. The corresponding inner product on $L^2(\partial\Omega)$ will be denoted by $(\cdot,\cdot)_{2,\partial\Omega}$.

Let $B[\cdot,\cdot]$ denote the symmetric bilinear form associated with Δ:

$$B[u,v] = \int_\Omega \sum_{i=1}^n D_i u D_i v\, dx \quad \text{for all } u, v \in W^1, \qquad (D_i = \partial/\partial x_i).$$

On W^1 the following equivalent norm and corresponding inner product will be convenient for our purposes:

$$\|u\|_{1,2,0}^2 = B[u,u] + \|\gamma_0 u\|_{2,\partial\Omega}^2 \quad \text{for all } u \in W^1$$

(2) $$(u,v)_{1,2,0} = B[u,v] + (\gamma_0 u, \gamma_0 v)_{2,\partial\Omega} \quad \text{for all } u, v \in W^1.$$

W_0^1 will denote the Hilbert consisting of the set of functions $u \in W^1$ with $\gamma_0 u = 0$. $(W_0^1)^\perp$ will denote the orthogonal complement of W_0^1 in W^1 with respect to the inner product (2).

We now recall some facts concerning eigenvalue problems of the form

(3) $$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n - a(x)u = \alpha u \quad \text{on } \partial\Omega$$

where $a \in L^\infty(\partial\Omega)$, $a(x) \le 0$ for $x \in \partial\Omega$.

A real number α is said to be an eigenvalue, with corresponding generalized eigenfunction $\phi \in (W_0^1)^\perp$, for the eigenvalue problem (3) if

$$B[\phi, v] - \int_{\partial\Omega} a(x)\gamma_0\phi \cdot \gamma_0 v\, dS = \alpha(\gamma_0\phi, \gamma_0 v)_{2,\partial\Omega} \quad \text{for all } v \in (W_0^1)^\perp.$$

These eigenvalues form a nondecreasing sequence $0 \le \alpha_1 \le \alpha_2 \le \ldots$, with $\alpha_n \to \infty$. The corresponding eigenfunctions $\{\phi_i\}$ are orthogonal and complete in $(W_0^1)^\perp$, whereas the corresponding traces $\{\gamma_0\phi_i\}$ form a complete orthonormal set for the space $L^2(\partial\Omega)$.

The smallest eigenvalue α_1 is characterized by

$$\alpha_1 = \min\Big(B[u,u] - \int_{\partial\Omega} a(x)(\gamma_0 u)^2\, dS\Big), \quad \|\gamma_0 u\|_{2,\partial\Omega} = 1, \quad u \in (W_0^1)^\perp$$

whereas for $n \ge 2$, the n-th eigenvalue α_n is characterized by the Courant maximum-minimum principle

$$\alpha_n = \min_{\mathcal{M}\in\mathcal{L}_n} \max_{u\in\mathcal{M}} \Big(B[u,u] - \int_{\partial\Omega} a(x)(\gamma_0 u)^2\, dS\Big).$$

Here $\mathcal{L}_n$ is the class of all the sets $S \cap L$, where $S = \{u \in (W_0^1)^\perp : \|\gamma_0 u\|_{2,\partial\Omega} = 1\}$ and L is an arbitrary n-dimensional subspace of $(W_0^1)^\perp$.

With respect to the linear inhomogeneous problem

(4) $$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n - a(x)u = f(x) \quad \text{on } \partial\Omega$$

we shall need the following existence result which is of independent interest.

LEMMA 2.1. *Let $a \in L^\infty(\partial\Omega)$, $f \in L^2(\partial\Omega)$ and suppose there exist real constants m, M and an integer N such that*

$$\lambda_N < m \le a(x) \le M < \lambda_{N+1} \quad \textit{for all } x \in \partial\Omega$$

where λ_N, λ_{N+1} are consecutive distinct eigenvalues of (EP). Then there exists a unique $u \in W^1$ such that

$$B[u,v] - \int_{\partial\Omega} a(x)\gamma_0 u \cdot \gamma_0 v \, dS = \int_{\partial\Omega} f(x)\gamma_0 v \, dS \quad \text{for all } v \in W^1$$

i.e., (4) *has a unique weak solution $u \in W^1$.*

Proof. Consider the bilinear form

$$A[u,v] \equiv B[u,v] - \int_{\partial\Omega} a(x)\gamma_0 u \cdot \gamma_0 v \, dS, \qquad u, v \in W^1.$$

By virtue of the Fredholm alternative for bilinear forms and the projection theorem, it suffices to show that $u \in (W_0^1)^\perp$, $A[u,v] = 0$ for all $v \in (W_0^1)^\perp$ implies $u = 0$.

In this direction, let $\{w_i\}$, with $w_i \in (W_0^1)^\perp$, be the generalized eigenfunctions corresponding to the eigenvalues $\{\mu_i\}$ of the eigenvalue problem

(5) $$\Delta w = 0 \quad \text{in } \Omega, \qquad \partial w/\partial n - (a(x) - M)w = \mu w \quad \text{on } \partial\Omega$$

and let $\{\gamma_i\}$ be the eigenvalues of the eigenvalue problem

$$\Delta z = 0 \quad \text{in } \Omega, \qquad \partial z/\partial n - (m - M)z = \gamma z \quad \text{on } \partial\Omega.$$

It follows, from the Courant maximum-minimum principle, that for each i

$$\lambda_i \le \mu_i \le \gamma_i = \lambda_i + M - m.$$

In particular,

$$\mu_N - M + m \le \lambda_N < m \le M < \lambda_{N+1} \le \mu_{N+1}$$

which shows that $M \in (\mu_N, \mu_{N+1})$, i.e., M cannot equal μ_i for any i.

Since the $\{w_i\}$ are generalized eigenfunctions of (5), we have

$$A[u, w_i] + M(\gamma_0 u, \gamma_0 w_i)_{2,\partial\Omega} = \mu_i(\gamma_o u, \gamma_o w_i)_{2,\partial\Omega} \quad \text{for all } u \in (W_0^1)^\perp.$$

Thus, for u satisfying $A[u,v] = 0$ for all $v \in (W_0^1)^\perp$ we see that

$$\mu_i(\gamma_0 u, \gamma_0 w_i)_{2,\partial\Omega} = M(\gamma_0 u, \gamma_0 w_i)_{2,\partial\Omega} \quad \text{for all } i$$

which, in view of the completeness of the $\{\gamma_0 w_i\}$ and the fact that $M \ne \mu_i$ for each i, implies $u \in W_0' \cap (W_0')^\perp$, i.e., $u = 0$. ∎

3. Main results

Formally, a function $u \in W^1$ is a weak solution of (BP) if

$$B[u,v] - \int_{\partial\Omega} a(x,\gamma_0 u)\gamma_0 u \cdot \gamma_0 v \, dS = \int_{\partial\Omega} f(x,\gamma_0 u)\gamma_0 v \, dS \quad \text{for all } v \in W^1.$$

LEMMA 3.1. *Let $a, f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then (BP) has at least one weak solution $u \in W^1$ provided*

(i) *There exists an integer N and real constants m, M such that*

$$\lambda_N < m \leq a(x,\xi) \leq M < \lambda_{N+1} \quad \textit{for all } x \in \partial\Omega,\ \xi \in \mathbb{R}$$

where λ_N, λ_{N+1} are consecutive distinct eigenvalues of (EP) and

(ii) $\lim_{|\xi|\to\infty} f(x,\xi)/\xi = 0$, *uniformly for $x \in \partial\Omega$*

or

(iii) *There exist constants $\alpha \geq 0, \beta \geq 0, R > 0$ such that*

$$|f(x,\xi)| \leq \alpha + \beta|\xi| \quad \textit{for all } x \in \partial\Omega,\ |\xi| > R$$

with β sufficiently small.

Proof. Let $w \in W^1$. By Lemma 2.1 and elliptic regularity theory, the linear problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n - a(x,w)u = f(x,w) \quad \text{on } \partial\Omega$$

has a unique weak solution $u \in W^2$. Thus, we can define a mapping $T : W^1 \to W^2$ by $Tw = u$. Since $a(x,\gamma_0 w)$ is uniformly bounded for all $w \in W^1$, the following standard a-priori estimatic holds:

$$\|Tw\|_{2,2} \leq c\|f(\cdot,\gamma_0 w)\|_{2,\partial\Omega} \quad \text{for all } w \in W^1 \tag{6}$$

where the constant c is independent of w.

To complete the proof, we show that T has a fixed point by using the a-priori bound principle. In this direction we first show that T is continuous. Thus, suppose $\|w_i - w_0\|_{1,2} \to 0$ as $i \to \infty$ and let $u_i = Tw_i$ for $i = 0, 1, 2, \ldots$ Clearly

$$\Delta(u_i - u_0) = 0 \quad \text{in } \Omega$$

$$\partial(u_i - u_0)/\partial n - a(x,w_i)(u_i - u_0) = f(x,w_i) - f(x,w_0) - (a(x,w_0) - a(x,w_i)u_0 \text{ on } \partial\Omega$$

in the weak sense. But this problem is of the same form as (BP) and hence, (6) implies

$$\begin{aligned}\|u_i - u_0\|_{2,2} &\leq c\|f_i - f_0 - (a_0 - a_i)\gamma_0 u_0\|_{2,\partial\Omega} \\ &\leq c(\|(a_0 - a_i)\gamma_0 u_0\|_{2,\partial\Omega} + \|f_i - f_0\|_{2,\partial\Omega})\end{aligned}$$

where we have set $a_i = a(x,w_i)$, $f_i = f(x,w_i)$ for $i = 0, 1, 2, \ldots$

Since a, f are continuous with respect to L^2 convergence and u_0 is finite a.e., it follows that $\|u_i - u_0\|_{2,2} \to 0$. Thus T is continuous. Furthermore, since W^2 is compactly imbedded in W^1, the mapping $T : W^1 \to W^1$ is compact.

Finally we need to show that every solution of $u = \sigma T u$ is uniformly bounded in $\|\cdot\|_{1,2}$, independently of $\sigma \in [0,1]$. To see this we first assume that (ii) holds. Then the continuity of f implies that for an arbitrary $\delta > 0$, there is a constant $m_\delta > 0$ such that $|f(x,\xi)| \le \delta|\xi| + m_\delta$ for all $\xi \in \mathbb{R}$. An application of Minkowski's inequality yields

$$\Big(\int_{\partial\Omega} |f(x,\gamma_0 u(x))|^2\, dS\Big)^{1/2} \le m_\delta |\partial\Omega|^{1/2} + \delta \|\gamma_0 u\|_{2,\partial\Omega}$$

where $|\partial\Omega|$ denotes the $n-1$ dimensional measure of $\partial\Omega$. Combining this with (1) and (6) we have

$$\|u\|_{1,2} = \|\sigma T u\|_{1,2} \le c_1 m_\delta + c_2 \delta \|u\|_{1,2}$$

where c_1, c_2 are positive constants independent of u. Now by choosing δ small enough so that $(1 - c_2\delta) > 0$ we arrive at the desired bound.

Clearly a similar analysis is valid if (iii) holds. ∎

Under hypothesis (ii) we can relax condition (i) as follows.

THEOREM 3.2. *Let $a, f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then (BP) has at least one weak solution $u \in W^1$ provided*

(i) *There exists an integer N and real constants m, M such that*

$$\lambda_N < m \le \liminf_{|\xi|\to\infty} a(x,\xi) \le \limsup_{|\xi|\to\infty} a(x,\xi) \le M < \lambda_{N+1}$$

for all $x \in \partial\Omega$ and

(ii) $\lim_{|\xi|\to\infty} f(x,\xi)/\xi = 0$, *uniformly for $x \in \partial\Omega$.*

Proof. By (i) there is a number $r > 0$ such that

$$m - \varepsilon \le a(x,\xi) \le M + \varepsilon \tag{7}$$

for $x \in \partial\Omega$, $|\xi| \ge r$, where ε is such that $\lambda_N < m - \varepsilon$, $M + \varepsilon < \lambda_{N+1}$. Following Ahmad and Salazar [1], we define functions $\widetilde{a}$ and $\widetilde{f}$ as follows: Let $\phi = \phi(\xi)$ be a continuous function on $\mathbb{R}$ satisfying $\phi(\xi) = 1$ if $|\xi| \le r$; $0 \le \phi(\xi) \le 1$ if $r \le |\xi| \le \partial r \le |\xi| \le 2r$; and $\phi(\xi) = 0$ if $|\xi| \ge 2r$. Set

$$\widetilde{a}(x,\xi) = \begin{cases} m & \text{if } |\xi| \le r \\ m\phi(\xi) + a(x,\xi)(1-\phi(\xi)) & \text{if } r \le |\xi| \le 2r \\ a(x,\xi) & \text{if } |\xi| \ge 2r \end{cases}$$

and

$$\widetilde{f}(x,\xi) = [a(x,\xi) - \widetilde{a}(x,\xi)]\xi + f(x,\xi).$$

Then $\tilde{a} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (7) for all $(x, \xi) \in (\partial\Omega, \mathbb{R})$, whereas $\tilde{f} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies condition (ii) of Lemma 3.1. Since (BP) is equivalent to the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial u/\partial n - \tilde{a}(x,u)u = \tilde{f}(x,u) \quad \text{on } \partial\Omega$$

the desired result follows from Lemma 3.1.

Remark. Let $\partial\Omega = S_0 \cup S_1 \cup S_2$ where $mes(S_0) = 0$, $mes(S_1) \geq 0$ and $mes(S_2) > 0$. Then analogous results hold for the more general mixed problem

$$\begin{cases} \sum_{|i|,|j|\leq 1} D_i(a_{ij}(x)D_j u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } S_1, \ \partial u/\partial\nu - a(x,u)u = f(x,u) \text{ on } S_2 \end{cases}$$

where $a_{ij} = a_{ji}$ are real-valued functions of class $C^{|j|}$, the matrix $[a_{ij}]$ is positive definite, $a_{0,0} \leq 0$ and $\partial/\partial\nu$ denotes the outward conormal derivative to S_2.

Remark. Analogous results also hold for corresponding higher order problems such as the biharmonic problem

$$\begin{cases} \Delta^2 u = 0 \text{ in } \Omega, \\ \partial u/\partial n = 0, \ \partial\Delta u/\partial n - a(x,u)u = f(x,u) \text{ on } \partial\Omega. \end{cases}$$

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