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ON ORDINARY LINEAR DIFFERENTIAL EQUATIONS IN $\mathbb{C}^J$

We consider the Cauchy problem for linear row-finite systems of ordinary differential equations. We discuss the connection between solvability and spectral properties of the related matrices for first and second order equations continuing the work of Ulm, Lemmert and Weckbach.

1. Introduction

Let J be a countably infinite set and let $\mathbb{C}^J$ be the Fréchet space of all mappings $x : J \to \mathbb{C}$ written as $x = (x_j)_{j\in J}$ provided with the product topology. The linear continuous mappings on $\mathbb{C}^J$ can be represented exactly by the row-finite matrices. Now let A be a row-finite matrix,

$$f \in C([0,T],\mathbb{C}^J) \quad \text{and} \quad y_0 \in \mathbb{C}^J.$$

We consider the initial value problem

$$\text{(1)} \qquad \begin{cases} y'(t) = Ay(t) + f(t), \\ y(0) = y_0. \end{cases}$$

The solvability of (1) depends on the spectrum of A. The spectrum of A is defined as the set of all $\mu \in \mathbb{C}$ such that $A - \mu I$ is not invertible. It turns out that for a row-finite matrix either $\sigma(A)$ or $\mathbb{C} \setminus \sigma(A)$ is at most countable (cf. [9] or [16]).

In 1984 Lemmert and Weckbach [10] proved the following theorem.

THEOREM 1. *The following assertions are equivalent:*

a) *$\sigma(A)$ is at most countable.*

b) *For every $f \in C([0,T],\mathbb{C}^J)$ and every $y_0 \in \mathbb{C}^J$ there is exactly one $y \in C^1([0,T],\mathbb{C}^J)$ that solves* (1).

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c) $\sum_{n=0}^{\infty} \frac{A^n x}{n!} t^n$ *is convergent in* $\mathbb{C}^J$ *for all* $x \in \mathbb{C}^J$ *and all* $t \in \mathbb{R}$.

Since linear initial value problems in Fréchet spaces may be locally not solvable even in Montel spaces (cf. [6]), it is not clear what happens if $\mathbb{C}\setminus\sigma(A)$ is at most countable. We will prove the following theorem.

THEOREM 2. *If* $\mathbb{C} \setminus \sigma(A)$ *is at most countable, then for every* $f \in C([0,T],\mathbb{C}^J)$ *and every* $y_0 \in \mathbb{C}^J$ *there are infinitely many* $y \in C^1([0,T],\mathbb{C}^J)$ *solving* (1).

Basing on these theorems we will survey row-finite initial value problems of second order and the uniqueness of positive solutions for initial value problems of the form (1) in the case that $\mathbb{C} \setminus \sigma(A)$ is at most countable for special sets of matrices that occur in applications, e.g., by semi-discretization of the heat equation on an infinite strip.

2. Notation and basic facts

In the following let J be a countably infinite set. The topological dual of $\mathbb{C}^J$ can be represented by $\mathbb{C}_J := \{(x_j)_{j\in J} : x_j \in \mathbb{C},\ \#\{j : x_j \neq 0\} < \infty\}$. In the following we will consider the duality $\langle x,y\rangle = \sum_{j\in J} x_j y_j$, $x = (x_j)_{j\in J} \in \mathbb{C}^J$, $y = (y_j)_{j\in J} \in \mathbb{C}_J$.

The continuous linear mappings of $\mathbb{C}^J$ can be represented exactly by the row-finite matrices in which a matrix $A = (a_{ij})_{i,j\in J}$ is called row-finite if $(a_{ij})_{j\in J} \in \mathbb{C}_J$ for all $i \in J$. The linear mappings of $\mathbb{C}_J$ can be represented exactly by the column-finite matrices where a matrix $A = (a_{ij})_{i,j\in J}$ is called column-finite if $(a_{ij})_{i\in J} \in \mathbb{C}_J$ for all $j \in J$. If $A = (a_{ij})_{i,j\in J}$ is a row-finite matrix, the dual mapping is represented by the transposed matrix ${}^{\mathsf{T}}\!A = (a_{ji})_{i,j\in J}$ which is column-finite; it holds that $\langle Ax,y\rangle = \langle x,{}^{\mathsf{T}}\!Ay\rangle$ for all $x \in \mathbb{C}^J$, $y \in \mathbb{C}_J$.

We indentify the set of all row-finite matrices with $L(\mathbb{C}^J)$, the set of all continuous linear mappings on $\mathbb{C}^J$, and the set of all column-finite matrices with $\mathrm{End}(\mathbb{C}_J)$.

DEFINITION 1. For $A \in \mathrm{End}(\mathbb{C}_J)$ resp. $A \in L(\mathbb{C}^J)$ we denote the spectrum of A by

$$\sigma(A) := \{\mu \in \mathbb{C} : A - \mu I \text{ is not invertible (in } \mathrm{End}(\mathbb{C}_J) \text{ resp. } L(\mathbb{C}^J))\}$$

and the point spectrum of A by

$$\sigma_p(A) := \{\mu \in \mathbb{C} : A - \mu I \text{ is not injective}\}.$$

For the following basic facts on row-finite and column-finite matrices, cf. Ulm [16], Körber [9] and Kaplansky [8].

PROPOSITION 1. *For $A \in L(\mathbb{C}^J)$ the following statements hold:*

a) $\sigma(A) = \sigma({}^{T}\!A) \neq \emptyset$.

b) *Either $\sigma(A)$ is at most countable or $\mathbb{C} \setminus \sigma(A)$ is at most countable.*

c) *$\sigma_p({}^{T}\!A)$ is at most countable ($\sigma_p(A)$ can be uncountable, cf. Körber* [9]).

d) *$\sigma(A)$ is at most countable if and only if one of the following statements holds:*

1) *$\{{}^{T}\!A^n x : n \in \mathbb{N}_0\}$ is linearly dependent for all $x \in \mathbb{C}_J$.*

2) *$\{{}^{T}\!A^n x : n \in \mathbb{N}_0\}$ is linearly dependent for all x in a basis of $\mathbb{C}_J$.*

3) $\sigma_p({}^{T}\!A) = \sigma({}^{T}\!A)$.

DEFINITION 2. A column-finite matrix with an at most countable spectrum is called *locally algebraic.*

We will now state and prove a theorem about a normal form of row-finite and column-finite matrices that is due to Ulm [16]; it will be the central tool for proving Theorem 2.

PROPOSITION 2. *For every $A \in \mathrm{End}(\mathbb{C}_J)$ with uncountable spectrum there is an invertible matrix $T \in \mathrm{End}(\mathbb{C}_J)$ and an order on J such that $C := T^{-1}AT$ has (with regard to this order) one of the following six forms. There S denotes the column-finite matrix*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

ordinal number of J	form of C	remarks
$n\omega,\ n \in \mathbb{N}$	$\begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & S_n \end{pmatrix}$	$S_l = S,\ 1 \leq l \leq n$
$n\omega + k,\ n, k \in \mathbb{N}$	$\begin{pmatrix} S_1 & 0 & \cdots & 0 & H_1 \\ 0 & S_2 & \cdots & 0 & H_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_n & H_n \\ 0 & 0 & \cdots & 0 & R \end{pmatrix}$	$S_l = S,\ 1 \leq l \leq n$; R is a $k \times k$ matrix and every $H_l,\ 1 \leq l \leq n$, has k columns and infinite many rows

ordinal number of J	form of C	remarks
$(n+1)\omega,\ n\in\mathbb{N}$	$\begin{pmatrix} S_1 & 0 & \cdots & 0 & H_1 \\ 0 & S_2 & \cdots & 0 & H_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & S_n & H_n \\ 0 & 0 & \cdots & 0 & R \end{pmatrix}$	$S_l = S,\ 1\le l\le n$; R is column-finite with $\sigma(R)$ at most countable and the $H_l,\ 1\le l\le n$, are column-finite
$n\omega^2,\ n\in\mathbb{N}$	$\begin{pmatrix} S_1 & 0 & 0 & \cdots \\ 0 & S_2 & 0 & \cdots \\ 0 & 0 & S_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	$S_l = S,\ l\in\mathbb{N}$
$\omega^2+k,\ k\in\mathbb{N}$	$\begin{pmatrix} S_1 & 0 & 0 & \cdots & H_1 \\ 0 & S_2 & 0 & \cdots & H_2 \\ 0 & 0 & S_3 & \cdots & H_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$	$S_l = S,\ l\in\mathbb{N}$; R and $H_l,\ l\in\mathbb{N}$, like in the second case
$\omega^2+\omega$	$\begin{pmatrix} S_1 & 0 & 0 & \cdots & H_1 \\ 0 & S_2 & 0 & \cdots & H_2 \\ 0 & 0 & S_3 & \cdots & H_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$	$S_l = S,\ l\in\mathbb{N}$; R and $H_l,\ l\in\mathbb{N}$, like in the third case

Proof. We consider $V := \mathbb{C}_J$ as $\mathbb{C}[\lambda]$-module generated by $P(\lambda).x = P(A)(x)$, $P\in\mathbb{C}[\lambda]$, $x\in V$. By Proposition 1 d), V is a torsion module if and only if $\sigma(A)$ is at most countable. So we get an nonempty, at most countable maximal $\mathbb{C}[\lambda]$-independent subset U of V, and we define W the $\mathbb{C}[\lambda]$-linear span of U. W is a submodule of V, and $\{\lambda^l.x : l\in\mathbb{N}_0,\ x\in U\}$ is a Hamel basis of W. Now we consider the factor module V/W, $P(\lambda).\tilde{x} = \widetilde{P(\lambda).x}$, $\tilde{x}\in V/W$, $P\in\mathbb{C}[\lambda]$. It holds that $\dim V/W$ is at most countable and that V/W is a torsion module. If there were an $\tilde{x}\in V/W$ with $P(\lambda).\tilde{x}\neq\tilde{0}$ for all $P\in\mathbb{C}[\lambda]$, $P\neq 0$, the set $U\cup\{x\}$ would be $\mathbb{C}[\lambda]$-independent in contradiction to the maximality of U. Now let $\widetilde{B}_1$ be a Hamel basis for V/W and B_1 a representative system of $\widetilde{B}_1$. Then $B = \{\lambda^l.x : l\in\mathbb{N}_0,\ x\in U\}\cup B_1$ is a Hamel basis for V. B is a countably infinite set, and we order B in the following way: U and B_1 are linearly ordered, $\{\lambda^l.x : l\in\mathbb{N}_0,\ x\in U\}$ is ordered lexicographically in (x,l) and $\{\lambda^l.x : l\in\mathbb{N}_0,\ x\in U\} < B_1$. So B looks like

$$\{x_1,\lambda.x_1,\lambda^2.x_1,\ldots,x_2,\lambda.x_2,\lambda^2.x_2,\ldots,x_l,\lambda.x_l,\lambda^2.x_l,\ldots,y_1,y_2,\ldots,y_l,\ldots\},$$

where $B_1 = \{y_1, y_2, \ldots, y_l, \ldots\}$. B is well-ordered, and the ordinal number of B is:

1) $n\omega$ in case that $\#U = n$, $B_1 = \emptyset$,
2) $n\omega + k$ in case that $\#U = n$, $\#B_1 = k$,
3) $(n+1)\omega$ in case that $\#U = n$, B_1 infinite,
4) ω^2 in case that U infinite, $B_1 = \emptyset$,
5) $\omega^2 + k$ in case that U infinite, $\#B_1 = k$,
6) $\omega^2 + \omega$ in case that U, B_1 infinite.

Now let φ be a bijective map between J and B and let J be ordered by the order induced by φ and the order of B.

Performing a basis transformation between $\{e_i : i \in J\}$, $e_i = (\delta_{ij})_{j \in J}$, and $B = \{b_i : i \in J\}$, we get a transformation matrix $T = (t_{ij})_{i,j \in J}$ such that $C = T^{-1}AT$ has exactly the form we claimed, depending on the order of J. In the case that B_1 is infinite, the matrix block R is a column-finite matrix with at most countable spectrum, since the module V/W is a torsion module. ∎

Remark. The ordinal number of J is not uniquely determined by the matrix A. We consider, for example, the matrix $A \in \mathrm{End}(\mathbb{C}_{\mathbb{N}})$ defined by $Ae_n = e_{n+1}$, $n \in \mathbb{N}$ $(e_n = (\delta_{nk})_{k \in \mathbb{N}})$, and so $\{e_1\}$ and $\{e_2\}$ are $\mathbb{C}[\lambda]$-independent maximal subsets of $\mathbb{C}_{\mathbb{N}}$. In the first case we get the ordinal number ω and in the second case the ordinal number $\omega + 1$.

3. Proof of Theorem 2

First we will prove Theorem 2 for the special case

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It holds that $\sigma(A) = \mathbb{C}$, and the linear homogeneous initial value problem $y' = Ay$, $y(0) = 0$, has the solution $y(t) = (h(t), h'(t), h''(t), \ldots, h^{(n)}(t), \ldots)$ for every $h \in C^\infty(\mathbb{R}, \mathbb{C})$ with $h^{(n)}(0) = 0$, $n \in \mathbb{N}_0$. So there is a lot of nontrivial solutions of this initial value problem. For example, take

$$h(t) = \begin{cases} e^{-1/t^2}, & t \neq 0, \\ 0, & t = 0 \end{cases}$$

(cf. Deimling [2]).

PROPOSITION 3. *Let* $J = \mathbb{N}$. *Theorem* 2 *holds for the matrix* A.

Proof. Since $f \in C([0,T],\mathbb{C}^{\mathbb{N}})$, f has the form $f = (f_k)_{k\in\mathbb{N}}$, $f_k \in C([0,T],\mathbb{C})$. To every $k \in \mathbb{N}$ there is a polynomial P_k such that $\|f_k - P_k\|_\infty \leq \left(\frac{1}{T}\right)^k$. We define

$$g_k(t) := \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_2} \int_0^{t_1} (f_k(t_0) - P_k(t_0))\, dt_0\, dt_1 \cdots dt_{k-2}\, dt_{k-1}$$

($k \in \mathbb{N}$, $t \in [0,T]$). It holds that $\|g_k\|_\infty \leq 1/k!$, $k \in \mathbb{N}$, and so $h(t) := \sum_{k=1}^{\infty} g_k(t)$, $t \in [0,T]$, converges uniformly on $[0,T]$, and we can define recursively $z_1 = h$, $z_{k+1} = z_k' - f_k$, $k \in \mathbb{N}$. Then $z := (z_k)_{k\in\mathbb{N}} \in C^1([0,T],\mathbb{C}^{\mathbb{N}})$ and $z'(t) = Az(t) + f(t)$, $t \in [0,T]$. Now we choose a $u \in C^1([0,T],\mathbb{C}^{\mathbb{N}})$ such that $u'(t) = Au(t)$, $t \in [0,T]$, and $u(0) = y_0 - z(0)$. Then $y = z + u$ is a solution of (1). It is always possible to find such a function u (cf. Deimling [2], Example 6.3). Since we have already seen that the initial value problem $y' = Ay$, $y(0) = 0$, has infinitely many solutions, Proposition 3 is proved. ∎

Now we are able to prove Theorem 2.

We will only consider the case of ordinal number $\omega^2 + \omega$, since the other cases can be proved in the same way.

According to Proposition 2, we may assume that A has the following form:

$$A = \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ 0 & L_2 & 0 & \cdots & 0 \\ 0 & 0 & L_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_1 & G_2 & G_3 & \cdots & Q \end{pmatrix}, \quad \text{where } L_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ n \in \mathbb{N},$$

Q is a row-finite matrix with $\sigma(Q)$ at most countable and G_n, $n \in \mathbb{N}$, are row-finite matrices. (A has the form of the transposed matrix in the sixth case of Propositon 2.) Remark that, since A itself is row-finite for every fixed row index j, the j-th row of G_n must be a zero-row for $n \geq n_0(j)$.

According to the order on J, we get $f = (f_1, f_2, \ldots, f_n, \ldots, f_\omega)$ with $f_n, f_\omega \in C([0,T],\mathbb{C}^{\mathbb{N}})$, $n \in \mathbb{N}$, and in the same manner $y_0 = (y_{01}, y_{02}, \ldots, y_{0n}, \ldots, y_{0\omega})$, $y_{0n}, y_{0\omega} \in \mathbb{C}^{\mathbb{N}}$, $n \in \mathbb{N}$. Thus the initial value problem (1) can be written as infinitely many initial value problems:

$$(1)_n \qquad \begin{cases} y_n'(t) = L_n y_n(t) + f_n(t), \\ y_n(0) = y_{0n}, \end{cases} \quad n \in \mathbb{N},$$

$$(1)_\omega \qquad \begin{cases} y_\omega(t) = Q y_\omega(t) + f_\omega(t) + \sum_{n=1}^{\infty} G_n y_n(t), \\ y_\omega(0) = y_{0\omega}. \end{cases}$$

By Proposition 3, the initial value problems $(1)_n$ are all solvable on $[0,T]$, and there are infinitely many solutions for every $n \in \mathbb{N}$. If we fix a solution of $(1)_n$ for every $n \in \mathbb{N}$, by Theorem 1 the initial value problem $(1)_\omega$ has a unique solution on $[0,T]$, since $\sigma(Q)$ is at most countable and since $f_\omega + \sum_{n=1}^{\infty} G_n y_n \in C([0,T],\mathbb{C}^{\mathbb{N}})$. Remark that $\sum_{n=1}^{\infty} G_n y_n$ is a finite sum in every coordinate. ∎

4. Linear row-finite differential equations of second order

Let $L, M \in L(\mathbb{C}^J)$, $f \in C([0,T],\mathbb{C}^J)$ and $y_0, y_1 \in \mathbb{C}^J$. We consider the initial value problem

$$(2) \qquad \begin{cases} y''(t) = Ly(t) + My'(t) + f(t), \\ y(0) = y_0, \quad y'(0) = y_1. \end{cases}$$

The initial value problem (2) is equivalent to the row-finite system of first order of the following form

$$(3) \qquad \begin{cases} \binom{v'}{w'} = \begin{pmatrix} 0 & I \\ L & M \end{pmatrix} \binom{v}{w} + \binom{0}{f(t)}, \\ \binom{v(0)}{w(0)} = \binom{y_0}{y_1}, \end{cases}$$

wherein we consider $\begin{pmatrix} 0 & I \\ L & M \end{pmatrix}$ as a row-finite matrix with index set $J_1 = J \times \{1,2\}$.

The initial value problem (2) has a solution $y \in C^2([0,T],\mathbb{C}^J)$ if and only if (3) has a solution $\binom{v}{w} \in C^1([0,T],\mathbb{C}^{J_1})$, and the solution of (2) is unique if and only if the solution of (3) is unique. Thus we get as a consequence of Theorems 1 and 2:

PROPOSITION 4. a) *The following assertions are equivalent:*

1) $\sigma\left(\begin{pmatrix} 0 & I \\ L & M \end{pmatrix}\right)$ *is at most countable.*

2) *For every* $f \in C([0,T],\mathbb{C}^J)$ *and every* $y_0, y_1 \in \mathbb{C}^J$ *there is exactly one* $y \in C^2([0,T],\mathbb{C}^J)$ *that solves* (2).

b) *If* $\mathbb{C}\backslash\sigma\left(\begin{pmatrix} 0 & I \\ L & M \end{pmatrix}\right)$ *is at most countable, then for every* $f \in C([0,T],\mathbb{C}^J)$ *and every* $y_0, y_1 \in \mathbb{C}^J$ *there are infinitely many* $y \in C^2([0,T],\mathbb{C}^J)$ *solving* (2).

Only using Proposition 1d), it is hard to see whether $\sigma\left(\begin{pmatrix} 0 & I \\ L & M \end{pmatrix}\right)$ is at most countable or not. We will now look for conditions on L and M such that (2) is uniquely solvable. Therefore we will consider $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ and give some criteria for this matrix to be locally algebraic.

Definition 3. For $A, B, C, D \in \mathrm{End}(\mathbb{C}_J)$ we define

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC.$$

In general, the invertibility of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is not equivalent to the invertibility of $AD - BC$. The following proposition is obvious:

Proposition 5. *Let $A, B \in \mathrm{End}(\mathbb{C}_J)$ and $\alpha \in \mathbb{C}$. Then*

$$\begin{pmatrix} \alpha I & A \\ I & B \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} A & B \\ -I & \alpha I \end{pmatrix}$$

are invertible if and only if their determinants are.

From Proposition 5 we get

Proposition 6. a) $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ *is locally algebraic if and only if* $\{\lambda \in \mathbb{C} : \lambda^2 I - \lambda {}^{\mathsf{T}}M - {}^{\mathsf{T}}L$ *is not invertible*$\}$ *is at most countable.*

b) $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ *is not locally algebraic if and only if* $\{\lambda \in \mathbb{C} : \lambda^2 I - \lambda {}^{\mathsf{T}}M - {}^{\mathsf{T}}L$ *is invertible*$\}$ *is at most countable.*

Proof. For every $\lambda \in \mathbb{C}$ it holds that

$$\det\left[\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] = \lambda^2 I - \lambda {}^{\mathsf{T}}M - {}^{\mathsf{T}}L.$$

So a) and b) follow from Propositions 5 and 1. ∎

With this result we get a sufficient condition for $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ to be locally algebraic.

Proposition 7. a) *The matrices $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$, $\lambda \in \mathbb{C}$, are locally algebraic either for all or for at most countably many $\lambda \in \mathbb{C}$.*

b) *If $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for all $\lambda \in \mathbb{C}$, then* $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ *is locally algebraic.*

Proof. a) Without loss of generality, we may assume $J = \mathbb{N}$. Let $x \in \mathbb{C}_{\mathbb{N}}$. Then there exist polynomials $P_{k,n,x} \in \mathbb{C}[\lambda]$, $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, such that $(\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n x = (P_{k,n,x}(\lambda))_{k \in \mathbb{N}}$, $n \in \mathbb{N}_0$. Since ${}^{\mathsf{T}}M$ and ${}^{\mathsf{T}}L$ are column-finite, there is a smallest index $k_0(n,x) \in \mathbb{N}$ such that $P_{k,n,x} = 0$ for all $k \geq k_0(n,x)$. If $k_0(n,x)$ is bounded in n for every $x \in \mathbb{C}_{\mathbb{N}}$, it holds that $\{(\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n x : n \in \mathbb{N}_0\}$ is linearly dependent for every $x \in \mathbb{C}_{\mathbb{N}}$ and so, by Proposition 1 d), $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for every $\lambda \in \mathbb{C}$.

If there is an $x \in \mathbb{C}_{\mathbb{N}}$ such that $k_0(n,x)$ is unbounded with respect to n, it holds that for every $\lambda \in \bigcap_{P_{k,n,x} \neq 0} \{\mu \in \mathbb{C} : P_{k,n,x}(\mu) \neq 0\}$ the set $\{(\lambda^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n x : n \in \mathbb{N}_0\}$ is linearly independent. Since $\bigcup_{P_{k,n,x} \neq 0} \{\mu \in \mathbb{C} : P_{k,n,x}(\mu) = 0\}$ is at most countable, $\lambda^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for at most countably many $\lambda \in \mathbb{C}$.

b) Assume $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ not to be locally algebraic. Then, by Proposition 6, $\lambda^2 I - \lambda^{\mathsf{T}}M - {}^{\mathsf{T}}L$ is invertible for at most countably many $\lambda \in \mathbb{C}$. Since $\lambda^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for all $\lambda \in \mathbb{C}$, $\lambda^2 I - \lambda^{\mathsf{T}}M - {}^{\mathsf{T}}L$ is locally algebraic for all $\lambda \in \mathbb{C}$. So $\sigma(\lambda^2 I - \lambda^{\mathsf{T}}M - {}^{\mathsf{T}}L) = \sigma_p(\lambda^2 I - \lambda^{\mathsf{T}}M - {}^{\mathsf{T}}L)$ by Proposition 1 d) and, therefore, there are uncountably many $\lambda \in \mathbb{C}$ such that there exists an $x_\lambda \in \mathbb{C}_{\mathbb{N}}$, $x_\lambda \neq 0$, with $(\lambda^2 I - \lambda^{\mathsf{T}}M - {}^{\mathsf{T}}L)x_\lambda = 0$. We define $l(z) = \max\{k \in \mathbb{N} : z_k \neq 0\} \cup \{0\}$, $z \in \mathbb{C}_{\mathbb{N}}$. Since $l(x_\lambda) \in \mathbb{N}_0$, there is an $n_0 \in \mathbb{N}$ and uncountably many $\lambda \in \mathbb{C}$ such that $l(x_\lambda) \leq n_0$ holds. For a column-finite matrix $A = (a_{ij})_{i,j \in \mathbb{N}}$ and $n \in \mathbb{N}$ we define $A_{[n]} = (a_{ij})_{i,j=1,\ldots,n}$, and for $z \in \mathbb{C}_{\mathbb{N}}$, $n \in \mathbb{N}$, we define $z_{[n]} = (z_1, \ldots, z_n)$. So we get

$$(\lambda^2 I_{[n_0]} - \lambda^{\mathsf{T}}M_{[n_0]} - {}^{\mathsf{T}}L_{[n_0]})(x_\lambda)_{[n_0]} = 0$$

for uncountably many $\lambda \in \mathbb{C}$. Since $l(x_\lambda) \leq n_0$, it holds that $(x_\lambda)_{[n_0]} \neq 0$.

Thus $\det(\lambda^2 I_{[n_0]} - \lambda^{\mathsf{T}}M_{[n_0]} - {}^{\mathsf{T}}L_{[n_0]}) = 0$ for uncountably many $\lambda \in \mathbb{C}$. But this is a contradiction, since $\det(\lambda^2 I_{[n_0]} - \lambda^{\mathsf{T}}M_{[n_0]} - {}^{\mathsf{T}}L_{[n_0]})$ is a polynomial of degree $2n_0$ and therefore has at most $2n_0$ zeros. So $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ is locally algebraic. ∎

We do not know whether $\lambda^{\mathsf{T}}M + {}^{\mathsf{T}}L$ locally algebraic for all $\lambda \in \mathbb{C}$ is necessary for $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ to be locally algebraic, but in some cases this is true as we will see now.

We first need the following proposition.

PROPOSITION 8. *Let A, B be column-finite locally algebraic matrices. If $A \cdot B = B \cdot A$, then $A \cdot B$ and $A + B$ are locally algebraic.*

Proof. Fix $x \in \mathbb{C}_J \setminus \{0\}$. Since B is locally algebraic, it holds that $1 \leq \dim \operatorname{span}\{B^n x : n \in \mathbb{N}_0\} = k < \infty$. We choose a basis $\{b_1, \ldots, b_k\}$ of $\operatorname{span}\{B^n x : n \in \mathbb{N}_0\}$. Then

$$\begin{aligned}\dim\operatorname{span}\{(AB)^n x : n\in\mathbb{N}_0\} &= \dim\operatorname{span}\{A^nB^nx : n\in\mathbb{N}_0\}\\ &\le \dim\operatorname{span}\{A^nb_i : n\in\mathbb{N}_0,\ i\in\{1,\dots,k\}\}\\ &\le \sum_{i=1}^{k}\dim\operatorname{span}\{A^nb_i : n\in\mathbb{N}_0\}<\infty\end{aligned}$$

and

$$\begin{aligned}&\dim\operatorname{span}\{(A+B)^n x : n\in\mathbb{N}_0\}\\ &\qquad = \dim\operatorname{span}\left\{\sum_{i=0}^{n}\binom{n}{i}A^iB^{n-i}x : n\in\mathbb{N}_0\right\}\\ &\qquad \le \dim\operatorname{span}\{A^nb_i : n\in\mathbb{N}_0,\ i\in\{1,\dots,k\}\}<\infty.\end{aligned}$$

So again by Proposition 1 d) it follows that $A\cdot B$ and $A+B$ are locally algebraic. ∎

Now we get the following Proposition.

PROPOSITION 9. *If the matrices L and M commute, the following assertions are equivalent:*

a) ${}^{\mathsf{T}}L$ *and* ${}^{\mathsf{T}}M$ *are locally algebraic.*
b) $\lambda\,{}^{\mathsf{T}}M+{}^{\mathsf{T}}L$ *is locally algebraic for all* $\lambda\in\mathbb{C}$.
c) $\begin{pmatrix}0 & {}^{\mathsf{T}}L\\ I & {}^{\mathsf{T}}M\end{pmatrix}$ *is locally algebraic.*

Proof. a)⇒b). For every $\lambda\in\mathbb{C}$ it holds that $\lambda\,{}^{\mathsf{T}}M$ and ${}^{\mathsf{T}}L$ are commuting and locally algebraic. Thus, by Proposition 8, it follows that $\lambda\,{}^{\mathsf{T}}M+{}^{\mathsf{T}}L$ is locally algebraic for all $\lambda\in\mathbb{C}$.

b)⇒c). This follows from Proposition 7b).

c)⇒a). It holds that

$$\det\begin{pmatrix}{}^{\mathsf{T}}M-\lambda I & -{}^{\mathsf{T}}L\\ -I & -\lambda I\end{pmatrix}=\det\begin{pmatrix}-\lambda I & {}^{\mathsf{T}}L\\ I & {}^{\mathsf{T}}M-\lambda I\end{pmatrix}$$

for all $\lambda\in\mathbb{C}$. By Propositions 5 and 6 it follows that also $\begin{pmatrix}{}^{\mathsf{T}}M & -{}^{\mathsf{T}}L\\ -I & 0\end{pmatrix}$ is locally algebraic. It holds that

$$\begin{pmatrix}{}^{\mathsf{T}}M & -{}^{\mathsf{T}}L\\ -I & 0\end{pmatrix}\begin{pmatrix}0 & {}^{\mathsf{T}}L\\ I & {}^{\mathsf{T}}M\end{pmatrix}=\begin{pmatrix}0 & {}^{\mathsf{T}}L\\ I & {}^{\mathsf{T}}M\end{pmatrix}\begin{pmatrix}{}^{\mathsf{T}}M & -{}^{\mathsf{T}}L\\ -I & 0\end{pmatrix}=\begin{pmatrix}-{}^{\mathsf{T}}L & 0\\ 0 & -{}^{\mathsf{T}}L\end{pmatrix}.$$

By Proposition 8, the matrices

$$\begin{pmatrix}-{}^{\mathsf{T}}L & 0\\ 0 & -{}^{\mathsf{T}}L\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0 & {}^{\mathsf{T}}L\\ I & {}^{\mathsf{T}}M\end{pmatrix}+\begin{pmatrix}{}^{\mathsf{T}}M & -{}^{\mathsf{T}}L\\ -I & 0\end{pmatrix}=\begin{pmatrix}{}^{\mathsf{T}}M & 0\\ 0 & {}^{\mathsf{T}}M\end{pmatrix}$$

are locally algebraic. Therefore ${}^{\mathsf{T}}L$ and ${}^{\mathsf{T}}M$ are locally algebraic. ∎

Another class of matrices, for which can be said more than in Proposition 7, is that of matrices with nonnegative entries:

PROPOSITION 10. *If the matrices L and M have nonnegative entries, the following statements are equivalent:*

a) ${}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ *is locally algebraic.*

b) $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ *is locally algebraic for all* $\lambda \in \mathbb{C}$.

c) $\begin{pmatrix} 0 & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M \end{pmatrix}$ *is locally algebraic.*

Proof. (The following inequalities between elements of $\mathbb{C}_J$ are meant elementwise.)

a)$\Rightarrow$b). For every $\lambda \in (0,1)$ and every $e_i = (\delta_{ij})_{j\in J}$, $i \in J$, it holds that $0 \le (\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n e_i \le ({}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n e_i$. It follows that $\dim \operatorname{span}\{(\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n e_i : n \in \mathbb{N}_0\} < \infty$ for all $i \in J$. So $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for all $\lambda \in (0,1)$. Since $(0,1)$ is an uncountable set, it follows by Proposition 7a) that $\lambda {}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic for all $\lambda \in \mathbb{C}$.

b)$\Rightarrow$c). This follows from Proposition 7b).

c)$\Rightarrow$a). $\begin{pmatrix} I & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M + I \end{pmatrix}$ is locally algebraic. It holds that

$$\begin{pmatrix} I & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M + I \end{pmatrix}^2 = \begin{pmatrix} I + {}^{\mathsf{T}}L & 2\,{}^{\mathsf{T}}L + {}^{\mathsf{T}}L\,{}^{\mathsf{T}}M \\ 2I + {}^{\mathsf{T}}M & {}^{\mathsf{T}}L + ({}^{\mathsf{T}}M + I)^2 \end{pmatrix}.$$

Fix $i \in J$. We define

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} I & {}^{\mathsf{T}}L \\ I & {}^{\mathsf{T}}M + I \end{pmatrix}^{2n} \begin{pmatrix} 0 \\ e_i \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Then it holds that

$$0 \le ({}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n e_i \le ({}^{\mathsf{T}}L + ({}^{\mathsf{T}}M + I)^2)^n e_i \le y_n, \quad n \in \mathbb{N}_0.$$

Since $\{y_n : n \in \mathbb{N}_0\}$ is linearly dependent because $\left\{ \binom{x_n}{y_n} : n \in \mathbb{N}_0 \right\}$ is linearly dependent, it follows that $\{({}^{\mathsf{T}}M + {}^{\mathsf{T}}L)^n e_i : n \in \mathbb{N}_0\}$ is linearly dependent and so ${}^{\mathsf{T}}M + {}^{\mathsf{T}}L$ is locally algebraic. ■

From Propositions 7, 9 and 10 we get

THEOREM 3. *Let $L, M \in L(\mathbb{C}^J)$. Then for every $f \in C([0,T], \mathbb{C}^J)$ and every $y_0, y_1 \in \mathbb{C}^J$ there is exactly one $y \in C^2([0,T], \mathbb{C}^J)$ solving* (2)

a) *if $\sigma(\lambda M + L)$ is at most countable for all $\lambda \in \mathbb{C}$,*

b) *on the condition that $LM = ML$ if and only if $\sigma(L)$ and $\sigma(M)$ are at most countable,*

c) *on the condition that L and M have only nonnegative entries if and only if $\sigma(L + M)$ is at most countable.*

EXAMPLE. Let $L, M \in L(\mathbb{C}^{\mathbb{N}})$ be defined by

$$Le_{2n} = e_{2n-1}, \quad Le_{2n-1} = 0, \quad Me_{2n+1} = e_{2n}, \quad Me_1 = Me_{2n} = 0, \quad n \in \mathbb{N}.$$

It holds that ${}^{\mathsf{T}}M^2 = {}^{\mathsf{T}}L^2 = 0$, so ${}^{\mathsf{T}}L$ and ${}^{\mathsf{T}}M$ are locally algebraic and therefore $\sigma(L)$ and $\sigma(M)$ are at most countable, but $L+M$ has uncountable spectrum.

Since L and M have only nonnegative entries, by Theorem 3 c) and Proposition 4 b) the initial value problem (2) is always solvable, but the solution is never unique.

5. On uniqueness of positive solutions of certain row-finite differential equations

We consider the following parabolic Cauchy problem:

$$\begin{cases} u_t = u_{xx}, \\ u(0,x) = \varphi(x), \end{cases} \tag{4}$$

where φ is a nonnegative continuous function on $\mathbb{R}$.

In [14] Pollard showed that for $T > 0$ there is at most one nonnegative solution $u \in C^2((0,T],\mathbb{R}) \cap C([0,T],\mathbb{R})$ of (4). In the longitudinal line method (cf., e.g., Walter [19]) the derivative u_{xx} is approximated by the symmetric difference quotient. This leads to the row-finite differential equation

$$\begin{cases} y' = A_h y, \\ y(0) = y_0, \end{cases} \tag{5}$$

where $h > 0$, $A_h = (S + S^{-1} - 2I)/h^2$, $S = (\delta_{i,j-1})_{i,j\in\mathbb{Z}}$ and $y_0 = (y_{0n})_{n\in\mathbb{Z}} = (\varphi(nh))_{n\in\mathbb{Z}}$. The matrix $A_h = (a_{ij})_{i,j\in\mathbb{Z}}$ is quasimonotone, which means that $a_{ij} \geq 0$ for $i \neq j$, and it holds that $\sigma(A_h)$ is uncountable.

In general, the initial value problems (4) and (5) can have more than one solution (cf. Gelfand [4], p. 60 for (4) and Theorem 2 for (5)). But we will see that, while looking only for positive solutions, also (5) has at most one solution.

We first need the following proposition due to Bernstein (cf., e.g., [1]).

PROPOSITION 11. *Let $f \in C^\infty([0,T],\mathbb{R})$ and $f^{(n)} \geq 0$ on $[0,T]$ for all $n \in \mathbb{N}_0$. Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad x \in [0,T].$$

Now we get the following theorem.

THEOREM 4. *Let $A \in L(\mathbb{C}^J)$ such that there exist matrices $B, C \in L(\mathbb{C}^J)$ having only nonnegative entries with the following three properties:*

a) $A = B - C$;
b) $BC = CB$;
c) $\sigma(C)$ *is at most countable.*

Then for every $y_0 \in [0, \infty)^J$ *the initial value problem*

$$\begin{cases} y'(t) = Ay(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{6}$$

has at most one nonnegative solution $y \in C^1([0,T], \mathbb{C}^J)$.

Proof. Let $y \in C^1([0,T],[0,\infty)^J)$ be a solution of (6). It holds that $y'(t) + Cy(t) = By(t)$, $t \in [0,T]$. Since B and C are commuting, $z(t) = e^{Ct}y(t)$, $t \in [0,T]$ is a solution of

$$\begin{cases} z' = Bz, \\ z(0) = y_0. \end{cases} \tag{7}$$

The exponential e^{Ct} exists, since $\sigma(C)$ is at most countable (cf. Theorem 1). Since B and C have only nonnegative entries, it holds that $z^{(n)} \geq 0$ on $[0,T]$ for all $n \in \mathbb{N}_0$. By Proposition 11, we have $z(t) = \sum_{n=0}^{\infty} \frac{z^{(n)}(0)}{n!} t^n$ in $\mathbb{C}^J$, $t \in [0,T]$, and since $y(t) = e^{-Ct}z(t)$, $t \in [0,T]$, it follows that $y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$, $t \in [0,T]$. Since $y^{(n)}(0)$, $n \in \mathbb{N}_0$, is uniquely determined by (6), y is the only nonnegative solution of (6). ∎

A special class of row-finite matrices which allow a decomposition as in Theorem 4 is that of quasimonotone matrices with diagonal bounded below. If $A = (a_{ij})_{i,j \in J}$ is quasimonotone and $a_{jj} \geq \beta$, $j \in J$, we choose $B = A - \beta I$ and $C = -\beta I$. The matrices A_h in (5) are in this class.

The following example shows that there are quasimonotone matrices with unbounded diagonal such that (6) has more than one nonnegative solution.

EXAMPLE. Let

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t = 0. \end{cases}$$

It holds that $f^{(n)}(0) = 0$, $n \in \mathbb{N}_0$, and $\lim_{t \to \infty} f^{(n)}(t) = 0$, $n \in \mathbb{N}$ (cf. [1, p. 180]).

We consider the function $g(t) = f(t) + e^t$, $t \geq 0$. It holds that $g(t) \geq 1$ on $[0, \infty)$ and therefore it is possible to define recursively:

$$y_1 = g$$
$$y_2 = g' + c_1 y_1, \; c_1 \geq 0 \text{ such that } y_2 \geq 1 \text{ on } [0,\infty)$$
$$y_3 = g'' + c_2 y_2 + c_1 y_1', \; c_2 \geq 0 \text{ such that } y_3 \geq 1 \text{ on } [0,\infty)$$
$$\vdots$$
$$y_n = g^{(n-1)} + \sum_{k=1}^{n-1} c_k y_k^{(n-k-1)}, \; c_{n-1} \geq 0 \text{ such that } y_n \geq 1 \text{ on } [0,\infty)$$
$$\vdots$$

It holds that $y := (y_n)_{n\in\mathbb{N}} \in C^1([0,\infty), \mathbb{C}^{\mathbb{N}})$ solves (6) with

$$A = \begin{pmatrix} -c_1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -c_2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -c_3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -c_4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $y_0 = y(0)$.

Now we define recursively

$$z_n(t) = e^t + \sum_{k=1}^{n-1} c_k z_k^{(n-k-1)}, \; n \in \mathbb{N}, \; t \geq 0.$$

Since $c_k \geq 0$, $k \in \mathbb{N}$, and $f^{(n)}(0) = 0$, $n \in \mathbb{N}_0$, the function $z = (z_n)_{n\in\mathbb{N}}$ is another nonnegative solution of (6) on $[0, \infty)$.

6. Final remarks

1) There is a lot of applications of row-finite matrices on initial value problems in Fréchet spaces. So, for example, Lemmert [11] showed the following theorem.

THEOREM 6. *Let F be a real Fréchet space, $f : [0,T] \times F \to F$ continuous and $||f(t,y_1) - f(t,y_2)|| \leq L||y_1 - y_2||$, where $||\cdot||$ denotes a polynorm $||y|| = (||y||_n)_{n\in\mathbb{N}}$ of seminorms $||\cdot||_n$, $n \in \mathbb{N}$, that induces the topology on F, and $L = (l_{ij})_{i,j\in\mathbb{N}}$ denotes a row-finite matrix with only nonnegative entries and $\sigma(L)$ at most countable. Then the initial value problem*

$$\begin{cases} y' = f(t,y), \\ y(0) = y_0 \end{cases} \tag{8}$$

is uniquely solvable on $[0,T]$ for every $y_0 \in F$.

For further applications cf. [6].

2) There is less known about linear row-finite differential equations with nonconstant coefficients. For some special systems see [3] and [18]. For linear differential equations with constant operator in Fréchet spaces see, e.g., [5], [12], [13] and [17].

3) Another proof that the initial value problem (1) always has a solution with completely different methods can be found in [15]. Theorem 6.2 in [2] treating row-finite systems is not correct.

4) Theorems 1 and 2 are in some way analytic characterizations of algebraic properties of column-finite matrices, since a matrix $A \in L(\mathbb{C}^J)$ with $\sigma(A)$ at most countable induces a torsion module by its transposed (cf. Propositions 1 and 2). In [7], there can be found an analytic characterization of those row-finite matrices whose transposed induces a torsion-free module.

5) This publication is part of the author's doctoral thesis *Über gewöhnliche Differentialgleichungen in Frécheträumen.* The author wishes to express his sincere gratitude to Dr. Roland Lemmert who initiated and took care of this work.

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