DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 2 1995

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ON SOME GENERAL PROPERTIES OF ORIENTED CONGRUENCE

1. Introduction

In this paper we are concerned with an oriented congruence, i.e. with an equivalence relation δ (or a halfequivalence δ), which induces an equidistance relation $\equiv$ with $\delta \subsetneq \equiv$.

For every halfequivalence $\delta \subseteq X \times X$ the relations $\delta \circ \delta$ and $\delta \cup \delta \circ \delta$ are equivalence relations in the set X (cf. [3]). In some earlier papers we have investigated situations in which $\delta \circ \delta$ or $\delta \cup \delta \circ \delta$ is an affine parallelity relation (or directed affine parallelity relation), see [5], [6]. In such a way we have obtained the systems which correspond (up to mutual definability) to metric affine, ordered affine, or ordered metric affine planes.

In this paper we consider the situation in which $\delta \circ \delta$ or $\delta \cup \delta \circ \delta$ coincides with the equidistance relation $\equiv$ of some metric affine plane (and $\delta \subsetneq \equiv$). It is not too difficult, to prove that for every equidistance relation $\equiv$ in the set X there exists a halfequivalence δ with $\delta \circ \delta = \equiv$, but in general the resulting structure $\langle X; \delta \rangle$ is not sufficiently homogeneous. We shall show that homogeneous structures $\langle X; \delta \rangle$ inducing an equidistance relation $\equiv$ really exist. Some general properties and characterizations of such structures are established.

2. Basic notions and definitions

Let $\mathfrak{F} = \langle F; +, \cdot, 0, 1 \rangle$ be a commutative field with $\operatorname{char} \mathfrak{F} \neq 2$ and let $\varepsilon \in F, \varepsilon \neq 0$. Then we set $\mathfrak{F}(\sqrt{\varepsilon}) := \langle F \times F; \oplus, \odot, 0, 1, ^{-} \rangle$, where

$$(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2),$$
$$(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1 \cdot \alpha_2 + \varepsilon \cdot \beta_1 \cdot \beta_2, \alpha_1 \cdot \beta_2 + \alpha_2 \cdot \beta_1),$$
$$\overline{(\alpha_1, \beta_1)} = (\alpha_1, -\beta_1) \quad \text{for all} \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \in F,$$

$$\text{and} \quad \underline{0} = (0,0),\ \underline{1} = (1,0).$$

For convenience we define the norm ν_ε in $F \times F$ by $\nu_\varepsilon(a) = a\bar{a}$. The structure $\mathfrak{F}(\sqrt{\varepsilon})$ is a commutative ring and the set $\mathbb{R}(\mathfrak{F}(\sqrt{\varepsilon})) = F \times \{0\}$ is a subring of $\mathfrak{F}(\sqrt{\varepsilon})$, isomorphic to $\mathfrak{F}$ under the map $\alpha \to (\alpha, 0)$. Every structure $\mathfrak{F}(\sqrt{\varepsilon})$ will be called a complex ring; thus complex ring is a special kind of commutative ring with the distinguished involutive automorphism. Whenever $\varepsilon \neq \alpha^2$ for all $\alpha \in F$, the ring $\mathfrak{F}(\sqrt{\varepsilon})$ is a commutative field.

The procedure which yields $\mathfrak{F}(\sqrt{\varepsilon})$ from $\mathfrak{F}$ and ε is a well known Cayley's Method generalizing the construction of complex numbers and dual numbers (cf. [1]).

Given any complex ring $\mathfrak{C} = \langle C; +, \cdot, 0, 1, ^{-} \rangle$ we define the equidistance relation $\equiv_\mathfrak{C}$, the orthogonality relation $\perp_\mathfrak{C}$, and the parallelity relation $\|_\mathfrak{C}$ as follows

$$ab \equiv_\mathfrak{C} cd :\Leftrightarrow (a-b)\overline{(a-b)} = (c-d)\overline{(c-d)},$$

$$ab \perp_\mathfrak{C} cd :\Leftrightarrow (a-b)\overline{(c-d)} + (c-d)\overline{(a-b)} = 0,$$

$$ab \|_\mathfrak{C} cd :\Leftrightarrow (a-b)\overline{(c-d)} = (c-d)\overline{(a-b)},$$

for all $a, b, c, d \in C$. Then we define the equidistance plane over $\mathfrak{C}$,

$$\mathbb{E}(\mathfrak{C}) := \langle C; \equiv_\mathfrak{C} \rangle,$$

and the affine plane over $\mathfrak{C}$,

$$\mathbf{A}(\mathfrak{C}) := \langle C; \|_\mathfrak{C} \rangle.$$

One can easily calculate that if $\varepsilon_1 \neq \varepsilon_2$, then $\mathbf{A}(\mathfrak{F}(\sqrt{\varepsilon_1})) = \mathbf{A}(\mathfrak{F}(\sqrt{\varepsilon_2}))$. In fact, the affine plane $\mathbf{A}(\mathfrak{F}(\sqrt{\varepsilon}))$ coincides with the usual affine plane coordinatized by $\mathfrak{F}$ (cf. [7]). For $a \neq b$ let $\mathbf{L}(a,b)$ be the line joining a and b and let $\mathcal{L}_\mathfrak{U}$ be the set of affine lines of an affine plane $\mathfrak{U}$. For every complex ring $\mathfrak{C}$ and any $a, b \in \mathfrak{C}$ with $a \neq b$ we set $S\binom{a}{b} := \{x \in C : ax \equiv_\mathfrak{C} xb\}$. Obviously, $\mathcal{L}_{\mathbb{A}(\mathfrak{C})} = \{S\binom{a}{b} : a, b \in C,\ a \neq b\}$ for every complex ring $\mathfrak{C}$ with $|\mathfrak{C}| = C$; therefore $\mathbf{A}(\mathfrak{C})$ is definable in $\mathbb{E}(\mathfrak{C})$. Moreover the structures $\mathbb{E}(\mathfrak{C})$ and $\langle C; \perp_\mathfrak{C} \rangle$ are mutually definable (cf. [2]).

If $\mathfrak{C} = \mathfrak{F}(\sqrt{\varepsilon})$ is simply a complex field, then $\mathbb{E}(\mathfrak{C})$ is a weak Euclidean plane (cf. [1], [2]). If not, i.e. if $\varepsilon = \lambda^2$ for some $\lambda \in F$, then $\mathbb{E}(\mathfrak{C})$ is a Minkowskian plane, and $\mathfrak{F}(\sqrt{\varepsilon}) \cong \mathfrak{F}(\sqrt{1})$ under the isomorphism $(\alpha, \beta) \to (\alpha, \lambda\beta)$.

Now let us consider an ordered commutative field $\mathfrak{F} = \langle F; +, \cdot, 0, 1, \leq \rangle$ and let $\varepsilon \in F$, $\varepsilon \neq 0$. In the set $C = F \times F$ we define the betweenness relation $\mathbf{B}_{\mathfrak{F}} \subseteq C^3$ as follows (cf. [7]):

$$\mathbf{B}_{\mathfrak{F}}((\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$$
$$\Leftrightarrow (\forall\lambda \in F)[0 \leq \lambda \leq 1 \wedge \alpha_2 = \lambda\alpha_1 + (1-\lambda)\alpha_3 \wedge \beta_2 = \lambda\beta_1 + (1-\lambda)\beta_3].$$

The ordered Gaussian plane over $(\mathfrak{F},\varepsilon)$ is finally defined to be the structure

$$\mathbf{G}(\mathfrak{F},\varepsilon) = \langle C; \equiv_{\mathfrak{C}}, \mathbf{B}_{\mathfrak{F}}\rangle.$$

An alternative approach which makes use of the notion of directed complex field is presented in [4]. Evidently, the Gaussian plane $\mathbf{G}(\mathfrak{F},\varepsilon)$ is an expansion of the structure $\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$. Finally, given any line K and points $a, b, \in C$ we set

$$\mathbf{B}_{\mathfrak{F}}(a,K,b) \Leftrightarrow (\exists p \in K)[\mathbf{B}_{\mathfrak{F}}(a,p,b) \wedge a,b \notin K].$$

3. Converse and oriented congruence

A relation $\triangle \subseteq X \times X$ is called a halfequivalence iff $\triangle$ is symmetric, $\triangle \circ \triangle \circ \triangle = \triangle$, and for every $a \in X$ there is $b \in X$ with $(a,b) \in \triangle$. If $\triangle$ is a halfequivalence, then the relation $\triangle \cup \triangle \circ \triangle$ is an equivalence relation (cf. [3]). We are going to investigate such relations $\triangle$, that $\triangle \cup \triangle \circ \triangle = \equiv_{\mathfrak{C}}$ for some complex field $\mathfrak{C}$ with ordered real part $\mathbb{R}(\mathfrak{C})$. Any such a relation $\triangle$ satisfying the additional assumption

OC1 $\qquad ab \,\triangle\, cd \wedge ab \,\triangle\, dc \Rightarrow a = b \vee c = d$

will be called a *converse congruence* (or a *converse equidistance relation*). Let $\mathfrak{F}$ be an ordered field and let $\varepsilon \in F$, $\varepsilon \neq \lambda^2$ for all $\lambda \in F$. We begin with a geometric definition of the relation $\equiv_{(\mathfrak{F},\varepsilon)}$. First we define auxiliary

$$a \rightleftharpoons_{(\mathfrak{F},\varepsilon)} b :\Leftrightarrow (\underline{0}a \equiv_{\mathfrak{F}(\sqrt{\varepsilon})} \underline{0}b \wedge$$
$$(\exists p)[p \neq \underline{0} \wedge \underline{0}p \perp_{\mathfrak{F}(\sqrt{\varepsilon})} \underline{0}a \wedge \mathbf{B}_{\mathfrak{F}}(b, \mathbf{L}(o,p), a)]) \vee \underline{0} = a = b$$

and next we put

$$ab \rightleftharpoons_{(\mathfrak{F},\varepsilon)} cd :\Leftrightarrow b - a \rightleftharpoons_{(\mathfrak{F},\varepsilon)} d - c.$$

Clearly, the relation $\rightleftharpoons_{(\mathfrak{F},\varepsilon)}$ is definable in $\mathbb{G}(\mathfrak{F},\varepsilon)$ and thus it is invariant under automorphisms of $\mathbb{G}(\mathfrak{F},\varepsilon)$, in particular under isometries and similarities.

PROPOSITION 1. *If* $a = (\alpha_1,\beta_1)$ *and* $b = (\alpha_2,\beta_2)$, *then* $\underline{0}a \rightleftharpoons_{(\mathfrak{F},\varepsilon)} \underline{0}b$ *is equivalent to*

$$a\bar{a} = b\bar{b} \;\&\; 0 < \frac{\alpha_1\alpha_2 - \varepsilon\beta_1\beta_2}{(\alpha_1\alpha_2 - \varepsilon\beta_1\beta_2) - (\alpha_1^2 - \varepsilon\beta_1^2)} < 1.$$

Proof. Proof is a straightforward calculation. We notice that

(i) $\underline{0}(\varepsilon\beta,\alpha)\perp_{\mathfrak{F}(\sqrt{\varepsilon})}\underline{0}(\alpha,\beta)$ for all $\alpha,\beta \in F$, and

(ii) for a, b as in Prop. 1 we have

$$(\gamma a + (1-\gamma)b) \in L(\underline{0}, (\varepsilon\beta_1, \alpha_1))$$

with $\gamma = \frac{\alpha_1\alpha_2 - \varepsilon\beta_1\beta_2}{(\alpha_1\alpha_2 - \varepsilon\beta_1\beta_2) - (\alpha_1^2 - \varepsilon\beta_1^2)}$. ∎

PROPOSITION 2. *The relation* $\underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$ *is symmetric, satisfies condition* OC1, *and* $ab \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)} ba$ *holds for all* a, b.

Proof. If $a = (\alpha,\beta)$, then $a\bar{a} = (\alpha^2 - \varepsilon\beta^2, 0)$. Thus the symmetry of $\underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$ is an immediate consequence of Proposition 1.

Next we prove that $a \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)} -a$ for every $a = (\alpha,\beta)$. Indeed, clearly $a\bar{a} = (-a)(\overline{-a})$ and in this case the coefficient γ given by Proposition 1 is

$$\gamma = \frac{-\alpha^2 + \varepsilon\beta^2}{-\alpha^2 + \varepsilon\beta^2 - (\alpha^2 - \varepsilon\beta^2} = \frac{1}{2},$$

so $0 < \gamma < 1$.

Finally we shall prove that $a \underleftarrow{\equiv} b$, $a \underleftarrow{\equiv} -b$ yields $a = \underline{0}$. Let $a \neq \underline{0}$ and $\underline{0}p\perp_{\mathfrak{F}(\sqrt{\varepsilon})}\underline{0}a$. Then also $\underline{0}p\perp_{\mathfrak{F}(\sqrt{\varepsilon})}\underline{0}(-a)$ and $\mathbf{B}_{\mathfrak{F}}(a, \mathbf{L}(\underline{0},p), -a)$. Therefore there is no b such that $a \equiv b$, $\mathbf{B}_{\mathfrak{F}}(a, \mathbf{L}(\underline{0},p), b)$, and $\mathbf{B}_{\mathfrak{F}}(-a, \mathbf{L}(\underline{0},p), b)$. ∎

Next we define

$$ab \underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)} cd :\Leftrightarrow ab \equiv cd \wedge (\neg ab \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)} cd \vee a = b).$$

Clearly, we have now

PROPOSITION 3. *The relation* $\underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$ *is symmetric and reflexive.* ∎

In every equidistance plane as above we have: $ab \equiv cc \Leftrightarrow a = b$. This yields

LEMMA 4. *If* $b \neq \underline{0}$, *then*

(i) $a \underrightarrow{\equiv} b \Leftrightarrow ab^{-1} \underleftarrow{\equiv} \underline{1}$;

(ii) $a \underleftarrow{\equiv} b \Leftrightarrow ab^{-1} \underleftarrow{\equiv} \underline{1}$.

Proof. The map $f : x \mapsto xb^{-1}$ is an automorphism of $\mathbb{G}(\mathfrak{F},\varepsilon)$ and preserves $\underline{0}$. Therefore f preserves $\underrightarrow{\equiv}$ and $\underleftarrow{\equiv}$, which proves (i) and (ii). ∎

LEMMA 5. *If* $a = (\alpha,\beta)$, *then* $a \underleftarrow{\equiv} 1$ *iff* $a\bar{a} = 1$ *and* $\alpha < 0$.

Proof. The condition $a\bar{a} = 1$ is equivalent to $\underline{0}a \equiv \underline{01}$. Next we note that $\underline{0}(0,1)\perp_{\mathfrak{F}(\sqrt{\varepsilon})}\underline{01}$ and $K = \mathbf{L}(\underline{0},(0,1)) = \{(\gamma,\delta) : \gamma = 0\}$; clearly for every $a = (\alpha,\beta)$

$$\mathbf{B}_F(a, K, \underline{1}) \quad \text{iff} \quad \alpha < 0. \ ∎$$

PROPOSITION 6. *The relation* $\underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$ *is a halfequivalence iff* $\varepsilon > 0$.

Proof. We prove the following conditions:

(*) If $\varepsilon > 0$, then

(i) $a, b \underleftarrow{\equiv} \underline{1} \Rightarrow ab \underrightarrow{\equiv} \underline{1}$;
(ii) $a, b \underrightarrow{\equiv} \underline{1} \Rightarrow ab \underrightarrow{\equiv} \underline{1}$, and
(iii) $a \underrightarrow{\equiv} \underline{1} \wedge b \underleftarrow{\equiv} \underline{1} \Rightarrow ab \underleftarrow{\equiv} \underline{1}$.

(**) If $\varepsilon < 0$, then there exists a with $a \underleftarrow{\equiv} 1$, $\overline{a} \underleftarrow{\equiv} \underline{1}$, and $a \underleftarrow{\equiv} \overline{a}$.

Clearly, $a, b \equiv \underline{1}$ yields $a \cdot b \equiv \underline{1}$. Assume first that $\varepsilon < 0$.

(ii) Let $a = (\alpha_1, \alpha_2)$, $b = (\beta_1, \beta_2)$, and $a, b \underrightarrow{\equiv} \underline{1}$. Then $\alpha_1, \beta_1 > 0$ and $\alpha_1^2 - \varepsilon\alpha_2^2 = 1 = \beta_1^2 - \varepsilon\beta_2^2$. We have $ab = (\alpha_1\beta_1 + \varepsilon\alpha_2\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1)$ and thus, to get $ab \underrightarrow{\equiv} \underline{1}$, we must prove $\alpha_1\beta_1 + \varepsilon\alpha_2\beta_2 > 0$.

Assume to the contrary $\alpha_1\beta_1 + \varepsilon\alpha_2\beta_2 < 0$. Then $\varepsilon\alpha_2\beta_2 < -\alpha_1\beta_1 < 0$, and, since $\varepsilon > 0$, we get $\alpha_2\beta_2 < 0$. Thus $\alpha_2 < 0, \beta_2 > 0$ or $\alpha_2 > 0$, $\beta_2 < 0$. Assume $\alpha_2 > 0$ and $\beta_2 < 0$; then $\alpha_2^2 = \frac{\alpha_1^2 - 1}{\varepsilon}$ and $\beta_2^2 = \frac{\beta_1^2-1}{\varepsilon}$ imply $\alpha_2 = \sqrt{\frac{\alpha_1^2-1}{\varepsilon}}$, $\beta_2 = -\sqrt{\frac{\beta_1^2-1}{\varepsilon}}$. We have

$$\varepsilon\alpha_2\beta_2 = -\varepsilon\sqrt{\frac{\alpha_1^2 - 1}{\varepsilon}}\sqrt{\frac{\beta_1^2 - 1}{\varepsilon}} = -\sqrt{(\alpha_1^2 - 1)(\beta_1^2 - 1)} < -\alpha_1\beta_1,$$

so $\alpha_1\beta_1 > -\sqrt{(\alpha_1^2 - 1)(\beta_1^2 - 1)}$ and thus $\alpha_1^2\beta_1^2 < (\alpha_1^2 - 1)(\beta_1^2 - 1)$. From this we get $0 < -\alpha_1^2 - \beta_1^2 + 1$ i.e. $\alpha_1^2 + \beta_1^2 < 1$. But then $\alpha_1^2 + \beta_1^2 = (1 + \varepsilon\alpha_2^2) + (1 + \varepsilon\beta_2^2) = 2 + \varepsilon(\alpha_2^2 + \beta_2^2) < 1$ leads to inconsistency.

(i) and (iii) are proved analogously.

(**) Now assume $\varepsilon < 0$. Let $a = (\alpha, \beta)$ and $a \underleftarrow{\equiv} \underline{1}$. Then $\alpha < 0$ and, clearly, $\overline{a} = (\alpha, -\beta) \underleftarrow{\equiv} \underline{1}$. We have $a\overline{a}^{-1} = (\alpha^2 + \varepsilon\beta^2, 2\alpha\beta)$. To get $a \underleftarrow{\equiv} \overline{a}$ it suffices to find $\alpha, \beta \in F$ with $\alpha^2 + \varepsilon\beta^2 < 0, \alpha^2 - \varepsilon\beta^2 = 1$. Then $\varepsilon\beta^2 = \alpha^2 - 1$ and $2\alpha^2 < 1$ i.e. $\alpha^2 < \frac{1}{2}, \beta^2 = \frac{\alpha^2-1}{\varepsilon}$. Clearly, one can find such α, β whenever $\varepsilon < 0$. ∎

As an immediate consequence we obtain the following

PROPOSITION 7. *If* $\varepsilon > 0$, *then* $\underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$ *is an equivalence relation,* $\underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)} = (\underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)})^2$, $\equiv_{(\mathfrak{F}\sqrt{\varepsilon})} = \underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)} \cup \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)}$, *and the condition* $ab \underrightarrow{\equiv}_{(F,\varepsilon)} cd \Leftrightarrow ab \underleftarrow{\equiv}_{(F,\varepsilon)} dc$ *holds for all* $a, b, c, d \in F \times F$. ∎

If $\varepsilon = \lambda^2$ for some $\lambda \in F$, then we can simply assume $\varepsilon = 1$. It is known that $\mathbf{E}(\mathfrak{F}(\sqrt{1}))$ contains isotropic directions. In this case we modify our definition of the relations $\underleftarrow{\equiv}$ and $\underrightarrow{\equiv}$ as follows:

$$a \underleftarrow{\equiv}_{(F,1)} b :\Leftrightarrow (\neg\underline{0}a\perp\underline{0}a \wedge \neg\underline{0}b\perp\underline{0}b \wedge \underline{0}a \equiv_{\mathfrak{F}(\sqrt{1})} \underline{0}b\wedge$$
$$\wedge(\exists p)[p \neq \underline{0} \wedge \underline{0}p\perp\underline{0}a \wedge \mathbf{B}(a, \mathbf{L}(\underline{0}, p), b)])\vee$$

$$\vee(\underline{0}a\perp\underline{0}a \wedge \underline{0}b\perp\underline{0}b \wedge a,b \neq \underline{0} \wedge \underline{0}a \upharpoonleft\!\downharpoonright \underline{0}b) \vee a = \underline{0} = b;$$

$$ab \underleftarrow{\equiv}_{(\mathfrak{F},1)} cd :\Leftrightarrow b - a \equiv_{(\mathfrak{F},1)} d - c;$$

$$a \underrightarrow{\equiv}_{(\mathfrak{F},1)} c :\Leftrightarrow (\neg\underline{0}a\perp\underline{0}a \wedge \neg\underline{0}b\perp\underline{0}b \wedge \underline{0}a \equiv_{\mathfrak{F}(\sqrt{1})} \underline{0}b \wedge \neg\underline{0}a \underleftarrow{\equiv}_{(\mathfrak{F},1)} \underline{0}b)\vee$$

$$\vee(\underline{0}a\perp\underline{0}a \wedge \underline{0}b\perp\underline{0}b \wedge a,b \neq 0 \wedge 0a \upharpoonleft\!\upharpoonright 0b) \vee a = \underline{0} = b;$$

$$ab \underrightarrow{\equiv}_{(\mathfrak{F},1)} cd :\Leftrightarrow b - a \underrightarrow{\equiv}_{(\mathfrak{F},1)} d - c.$$

One can easily prove now

PROPOSITION 8. *The relation* $\underleftarrow{\equiv}_{(\mathfrak{F},1)}$ *is a halfequivalence,* $\underrightarrow{\equiv}_{(\mathfrak{F},1)}$ *is an equivalence relation, and* $\underrightarrow{\equiv}_{(\mathfrak{F},1)} = (\underleftarrow{\equiv}_{(\mathfrak{F},1)})^2$.

In the sequel we shall establish some more important properties of the relations $\underrightarrow{\equiv}$ and $\underleftarrow{\equiv}$. To this aim we shall define some classes of transformations of complex fields. For any complex ring $\mathfrak{C} = \mathfrak{F}(\sqrt{\varepsilon})$ we set

$$\mathrm{Is}(\mathbb{E}(\mathfrak{C})) := \{f : |\mathfrak{C}| \to |\mathfrak{C}| : (\forall a,b)[ab \equiv_{\mathfrak{C}} f(a)f(b)]\}.$$

The following characterization of the class $\mathrm{Is}(\mathbb{E}(\mathfrak{C}))$ is known (cf. [2]):

$f \in \mathrm{Is}(\mathbb{E}(\mathfrak{C}))$ iff for some $a \in |\mathfrak{C}|$ with $a\overline{a} = 1$ and for some b

(i) $(\forall x \in |\mathfrak{C}|)[f(x) = ax + b]$, or
(ii) $(\forall x \in |\mathfrak{C}|)[f(x) = a\overline{x} + b]$.

Elements of $\mathrm{Is}(\mathbb{E}(\mathfrak{C}))$ are called *isometries* of $\mathbb{E}(\mathfrak{C})$, clearly they form a transformation group. Isometries characterized by the condition (i) form a subgroup denoted by $\mathrm{Is}^+(\mathbb{E}(\mathfrak{C}))$; such transformations are called *direct isometries*. Let

$$\mathrm{Is}^-(\mathbb{E}(\mathfrak{C})) = \mathrm{Is}(\mathbb{E}(\mathfrak{C})) \setminus \mathrm{Is}^+(\mathbb{E}(\mathfrak{C})).$$

Now we set

$$\mathbb{CE}(\mathfrak{F},\varepsilon) := \langle F \times F : \underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)}\rangle,$$
$$\mathbb{SE}(\mathfrak{F},\varepsilon) := \langle F \times F : \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)}\rangle,$$

and

$$\mathrm{CIs}(\mathbb{CE}(\mathfrak{F},\varepsilon)) := \{f : F \times F \to F \times F : (\forall a,b \in F \times F)[ab \underrightarrow{\equiv}_{(\mathfrak{F},\varepsilon)} f(a)f(b)]\}$$
$$\mathrm{SIs}(\mathbb{SE}(F,\varepsilon)) := \{f : F \times F \to F \times F : (\forall a,b \in F \times F)[ab \underleftarrow{\equiv}_{(\mathfrak{F},\varepsilon)} f(a)f(b)]\}.$$

Finally $\mathrm{Tr}(\mathbb{E}\mathfrak{C}))$ will be the group of *translations* and $\Pi(\mathbb{E}(\mathfrak{C}))$ — the set of *central symmetries* of the plane $\mathbb{E}\mathfrak{C})$.

Evidently $\mathbb{G}(\mathfrak{F},\varepsilon)$ is an expansion of the structures $\mathbb{CE}(\mathfrak{F},\varepsilon)$, $\mathbb{SE}(\mathfrak{F},\varepsilon)$, and $\mathbb{E}(\mathfrak{F},\varepsilon)$ and the structures $\mathbb{CE}(\mathfrak{F},\varepsilon)$ and $\mathbb{SE}(\mathfrak{F},\varepsilon)$ are definable in $\mathbb{G}(\mathfrak{F},\varepsilon)$;

thus we can use symbols $\mathrm{Is}(\mathbb{G}(\mathfrak{F},\varepsilon))$, $\mathrm{Is}(\mathbb{SE}(\mathfrak{F},\varepsilon))$, and $\mathrm{Is}(\mathbb{CE}(\mathfrak{F},\varepsilon))$ in their natural meaning.

LEMMA 9. *If a function f is defined by the formula $f(x) = ax + b$ with $a \neq \underline{0}$, then*

(i) $f \in \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \Leftrightarrow a \equiv \underline{1}$;
(ii) $f \in \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \Leftrightarrow a \equiv \underline{1}$.

Proof. (i) If $f \in \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$, then $\underline{01} \equiv f(\underline{0})f(\underline{1}) = b(b+a)$, so $a \equiv \underline{1}$. Conversely, if $a \equiv \underline{1}$, then for arbitrary x we get $ax \equiv x$, since for every x the transformation $y \mapsto yx$ is a similarity of $\mathbb{G}(\mathfrak{F},\varepsilon))$ and thus it preserves $\equiv$. Therefore $ax + b \equiv x + b$, so $f(x)f(y) \equiv xy$ for all x, y.

(ii) is proved analogously. ∎

THEOREM 10. *If $\varepsilon > 0$, then the following conditions hold*:

(i) $\mathrm{Tr}(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon})) \subseteq \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)))$;
(ii) $\Pi(\mathbb{G}(\mathfrak{F},\varepsilon)) \subseteq \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$;
(iii) $\mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$ *is a subgroup of* $\mathrm{Is}^{+}(\mathbb{G}(\mathfrak{F},\varepsilon))$;
(iv) $\mathrm{Is}^{+}(\mathbb{G}_{\mathfrak{F}},\varepsilon)) \setminus \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) = \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$;
(v) $[\mathrm{Is}^{+}(\mathbb{G}(\mathfrak{F},\varepsilon)) : \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))] = 2$.

Proof. (i) is a consequence of the definition and (ii) is a consequence of Proposition 2.

To prove (iii) we note first that by Proposition 6, $\mathrm{CIs}(\mathbb{G}(F,\varepsilon))$ is a transformation group. Evidently

$$\mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \cup \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \subseteq \mathrm{Is}(\mathbb{G}(\mathfrak{F},\varepsilon)).$$

Let $\Sigma(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$ be the set of axial symmetries of the plane $\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$. We shall prove that

$$(*) \qquad \Sigma(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon})) \cap (\mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \cup \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))) = \emptyset.$$

Assume that $f \in \Sigma(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$ and $f \in \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$. There are $a, b \in F \times F$ such that $a \neq b$ and $f(a) = a$, $f(b) = b$. By the definition, $ab \equiv f(a)f(b)$ which yields $\underline{1} \equiv \underline{1}$, or $a = b$. Now assume $f \in \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$ and consider any point a with $a \neq f(a)$; let $b = f(a)$. By the definition, $ab \equiv f(a)f(b) = ba$, so $-\underline{1} \equiv 1$ or $a = b$. This proves (*).

From this we get

$$(**) \qquad \mathrm{Is}^{-}(\mathbb{G}(\mathfrak{F},\varepsilon)) \cap (\mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \cup \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))) = \emptyset.$$

Indeed, every element f of $\mathrm{Is}^{-}(\mathbb{G}(\mathfrak{F},\varepsilon))$ can be written in the form $f = g \circ h$ with $h \in \Sigma(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$, $g \in \mathrm{Tr}(\mathbb{G}(\mathfrak{F},\varepsilon))$ and in the form $f = g' \circ h'$, where

$h' \in \Sigma(\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$, $g' \in \Pi(\mathbb{G}(\mathfrak{F},\varepsilon))$ (cf. [FB]). Now (**) follows from (i), (ii), and (*).

The condition (**) immediately yields (iii) and the condition $\mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon)) \subseteq \mathrm{Is}^+(\mathbb{G}(\mathfrak{F},\varepsilon))$.

(iv) & (v) is a consequence of Lemma 9 and (i), (ii) and (iii) of the above. ∎

As a consequence of Lemma 9 and Lemma 4 we get the following natural properties of the introduced classes of maps.

THEOREM 11 [Homogeneity].

(i) *If* $ab \rightrightarrows cd$, *then there is* $f \in \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$ *with* $f(a)=c$ *and* $f(b)=d$.

(ii) *If* $ab \leftrightarrows cd$, *then there is* $f \in \mathrm{SIs}(G(\mathfrak{F},\varepsilon))$ *with* $f(a) = c$ *and* $f(b) = d$.

Proof. If $a = b$, then $c = d$; in such a case in (i) we set f to be a translation which maps a onto c, in (ii) f is a central symmetry which maps a onto c. If $a \neq b$, then we consider f defined by

$$f(x) = \frac{c-d}{a-b}x + \frac{da-cb}{a-b},$$

which has the desired properties. ∎

THEOREM 12 [Separation principle].

(i) *If* $f \in \mathrm{Is}^+(\mathbb{G}(\mathfrak{F},\varepsilon))$ *and* $ab \rightrightarrows f(a)f(b)$ *for some* a,b *with* $a \neq b$, *then* $f \in \mathrm{CIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$.

(ii) *If* $f \in \mathrm{Is}^+(\mathbb{G}(\mathfrak{F},\varepsilon))$ *and* $ab \leftrightarrows f(a)f(b)$ *for some* a,b *with* $a \neq b$, *then* $f \in \mathrm{SIs}(\mathbb{G}(\mathfrak{F},\varepsilon))$. ∎

For the relation $\rightrightarrows$ we use the term oriented congruence. This term can be justified by the following theorem.

THEOREM 13. *Let* M *be a line in* $\mathbb{G}(\mathfrak{F},\varepsilon)$. *Then for any* $a,b,c,d \in M$ *the following conditions are equivalent*:

(i) $ab \rightrightarrows_\varepsilon cd$;

(ii) *there is a translation* f *with* $f(a) = c, f(b) = d$;

(iii) $ab \equiv cd \wedge ab \upuparrows cd$.

Proof. Without loss of generality we can assume $\underline{0} \in M$. Moreover we can assume $a = \underline{0} = c$, because $xy \rightrightarrows_\varepsilon \underline{0}(y-x)$ and $xy \upuparrows \underline{0}(y-x)$ for all x,y. Now the theorem is evident. ∎

Evidently, by Proposition 7, the structures $\mathbb{CE}(\mathfrak{F},\varepsilon)$ and $\mathbb{SE}(\mathfrak{F},\varepsilon)$ are mutually definable for any $\varepsilon > 0$ such that $\varepsilon \neq \lambda^2$ for all $\lambda \in F$; clearly

$\mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon}))$ is definable in $\mathbb{CE}(\mathfrak{F},\varepsilon)$. The same holds for $\varepsilon = 1$, in this case we have

$$ab \equiv cd \Leftrightarrow ab \Rrightarrow cd \vee ab \Lleftarrow cd(\mathbf{I}(ab) \wedge \mathbf{I}(cd)),$$

where $\mathbf{I}(ab)$ means that ab is an isotropic segment. The relation $\mathbf{I}$ can be defined in terms of $\Rrightarrow$ by the condition:

$$\mathbf{I}(ab) \Leftrightarrow (\forall x_1, x_2)\Big[\bigwedge_{i=1}^{2}(ax_i \Rrightarrow x_i b \vee ax_i \Lleftarrow x_i b) \wedge x_1 \neq x_2 \Rightarrow$$

$$x_1x_2 \Rrightarrow ab \vee x_1x_2 \Lleftarrow ab\Big] \vee a = b.$$

We are going to prove that $\mathbb{G}(\mathfrak{F},\varepsilon)$ is definable in $\mathbb{CE}(\mathfrak{F},\varepsilon)$ and thus ordered Euclidean geometry can be formulated in terms of one primitive notion $\Lleftarrow$ which is a halfequivalence, such that $\Lleftarrow \cup (\Lleftarrow)^2 = \equiv$, and satisfying conclusions of Theorems 11, 12 and 13.

Let us fix a commutative ordered field $\mathfrak{F}$ and $\varepsilon \in F = |\mathfrak{F}|$ with $\varepsilon > 0$. For any $a, b \in F^2$ we set

$$\mathbf{M}_\varepsilon(a,b) := \{x : ax \Rrightarrow_\varepsilon xb\}.$$

Evidently, $M_\varepsilon(p,q) \subseteq S\binom{p}{q}$ for all points p, q.

LEMMA 14. *Let* $p = (0,-\alpha)$, $q = (0,\alpha)$, $\alpha > 0$, *and* $x = (\gamma_1,\gamma_2)$. *The following conditions are equivalent*:

(i) $x \in \mathbf{M}_\varepsilon(p,q)$;
(ii) $\gamma_2 = 0$ *and* $\gamma_1^2 < \varepsilon\alpha^2$.

Proof. Clearly, $\gamma_2 = 0$ iff $x \in S\binom{p}{q}$. Thus, we can simply assume $x = (\gamma, 0)$. Set $c = (x-p)(q-x)^{-1} = (\beta_1,\beta_2)$. By Lemmas 4 and 5, $px \Rrightarrow xq$ iff $c \Rrightarrow \underline{1}$, i.e. iff $\beta_1 > 0$. By definitions, we calculate

$$\beta_1 = \frac{\gamma^2 + \varepsilon\alpha^2}{\varepsilon\alpha^2 - \gamma^2}.$$

Therefore, $x \in \mathbf{M}_\varepsilon(p,q)$ iff $\varepsilon\alpha^2 - \gamma^2 > 0$, as required. ∎

Note (cf. [2]) that for any a, b, a', b' with $a \neq b$ and $a' \neq b'$ there is a similarity f of $\mathbf{G}(\mathfrak{F},\varepsilon)$ such that $f(a) = a'$ and $f(b) = b'$; similarity f preserves $\Rrightarrow$, so $f(\mathbf{M}_\varepsilon(a,b)) = \mathbf{M}_\varepsilon(f(a), f(b))$. As a consequence of Lemma 14 we immediately conclude with

PROPOSITION 15. *Every set* $\mathbf{M}_\varepsilon(a,b)$ *with* $a \neq b$ *is an open segment in the plane* $\mathbb{G}(\mathfrak{F},\varepsilon)$. ∎

Any open segment is a convex set, which justifies the following definition:

$$\mathbf{B}_\varepsilon(abc) :\Leftrightarrow a = b = c \vee a \neq b \wedge (\forall p, q)[a, c \in \mathbf{M}_\varepsilon(p, q) \Rightarrow b \in \mathbf{M}_\varepsilon(p, q)].$$

We say that *squares are dense* in $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies the following condition:

$$0 < a < b \Rightarrow (\exists c)[a < c^2 < b].$$

PROPOSITION 16. *Let squares be dense in $\mathfrak{F}$ and let $\varepsilon > 0$ be arbitrary, or let ε be a square in $\mathfrak{F}$ and $\mathfrak{F}$ be arbitrary.*
Then $\mathbf{B}_\varepsilon = \mathbf{B}_\mathfrak{F}$.

Proof. Proposition 15 immediately yields that $\mathbf{B}_\mathfrak{F} \subseteq \mathbf{B}_\varepsilon$. Let a, c be points with $a \neq c$. Assume $\mathbf{B}_\varepsilon(abc)$. If $b = a$ or $b = c$, then the thesis is evident; thus we assume $b \neq a, c$. First we prove

(i) $a \neq c \Rightarrow (\exists p, q)[p \neq q \wedge a, c \in \mathbf{M}_\varepsilon(p, q)]$.

Without loss of generality we can assume $a = (-\gamma, 0)$, $c = (\gamma, 0)$, $\gamma > 0$. By Lemma 14, it suffices to find $\alpha > 0$ with $\gamma^2 < \varepsilon\alpha^2$ and set $p = (0, -\alpha), q = (0, \alpha)$.

Clearly, $\mathbf{B}_\varepsilon(abc)$ implies that a, b, c are collinear. To complete the proof it suffices to show the following:

(ii) $\mathbf{B}_\mathfrak{F}(abc) \wedge b \neq c \Rightarrow (\exists p, q)[a, b \in \mathbf{M}_\varepsilon(p, q) \wedge c \notin \mathbf{M}_\varepsilon(p, q)]$.

Again, without loss of generality we can assume $a = (-\gamma, 0)$, $b = (\gamma, 0)$, $c = (\eta, 0)$, and $0 < \gamma < \eta$. In view of Lemma 14, we need to find $\alpha > 0$ with $\gamma^2 < \varepsilon\alpha^2 < \eta^2$. Such α must exist, if squares are dense in $\mathfrak{F}$.

Note that if $\varepsilon = \lambda^2$, then both (i) and (ii) remain valid without assuming squares to form a dense set. Indeed, to prove (i) we set $\alpha = \frac{\gamma}{\lambda} + 1$ and to prove (ii) we set $\alpha = \frac{\gamma+\eta}{2\lambda}$. ■

Then we come to the main

THEOREM 17. *If squares are dense in $\mathfrak{F}$, then for arbitrary ε with $\varepsilon > 0$ the structure $\mathbb{G}(\mathfrak{F}, \varepsilon)$ is definable in $\mathbb{CE}(\mathfrak{F}, \varepsilon)$ and in $\mathbb{SE}(\mathfrak{F}, \varepsilon)$.*

Proof. Proof is immediate in view of Propositions 7 and 16. ■

As a consequence we get that the relation $\equiv\!\!\!\!\!\rightarrow_\varepsilon$ and the relation $\equiv\!\!\!\!\!\leftarrow_\varepsilon$ can be used as a primitive notion for the theory of the class

$$\{\mathbb{G}(\mathfrak{F}, \varepsilon) : \mathfrak{F} \text{ — an ordered field, } \varepsilon > 0, \text{ squares are dense in } \mathfrak{F}\}.$$

4. General investigations on oriented congruence

In this section we shall establish some properties of the general notion of oriented congruence. Let $\varepsilon \neq \lambda^2$ for all $\lambda \in F$ and let $\mathfrak{C} = \mathbb{E}(\mathfrak{F}(\sqrt{\varepsilon})) = \langle P; \equiv \rangle$

be a fixed equidistance plane; let $\Rrightarrow \subseteq P^2 \times P^2$ be a weak oriented congruence i.e. a binary relation on segments satisfying OC1 and the following conditions:

OC0 $\Rrightarrow$ is an equivalence relation.
OC2 for all $a, bc, d \in P$

$$ab \equiv cd \Leftrightarrow ab \Rrightarrow cd \vee ab \Rrightarrow dc.$$

From the assumptions the relation $\equiv$ defined by the condition

$$ab \Lleftarrow cd :\Leftrightarrow ab \Rrightarrow dc$$

is a halfquivalence satisfying $\Rrightarrow \cup \Lleftarrow = \equiv$ and $(\Lleftarrow)^2 = \Rrightarrow$ such that $ab \Lleftarrow ba$ implies $a = b$ for all $a, b \in P$.

Let $\mathfrak{M} = \langle P; \Rrightarrow \rangle$; as previously we define

$$\mathrm{CIs}(\mathfrak{M}) := \{f : P \to P : (\forall a, b \in P)[ab \Rrightarrow f(a)f(b)]\}$$

and

$$\mathrm{SIs}(\mathfrak{M}) := \{f : P \to P : (\forall a, b \in P)[ab \Lleftarrow f(a)f(b)]\}.$$

For every line K we consider the relation $\Rrightarrow_K = \Rrightarrow \cap K^4$ and for every two points a, q we consider the circle $c(a,q) = \{x : ax \equiv aq\}$. Let $\equiv_{\mathrm{Tr}}$ be the quaternary relation defined by

$$ab \equiv_{\mathrm{Tr}} cd :\Leftrightarrow (\exists f \in \mathrm{Tr}(\mathfrak{C}))[f(a) = c \wedge f(b) = d].$$

Next we consider the following conditions:

Cc $\mathrm{Tr}(C) \subseteq \mathrm{CIs}(\mathfrak{M})$;
CH $ab \Rrightarrow cd \Rightarrow (\exists f \in \mathrm{CIs}(\mathfrak{M})[f(a) = c \wedge f(b) = d]$;
Sc $\Pi(\mathfrak{C}) \subseteq \mathrm{SIs}(\mathfrak{M})$;
SH $ab \Lleftarrow cd \Rightarrow (\exists f \in \mathrm{SIs}(\mathfrak{M})[f(a) = c \wedge f(b) = d]$;
$\mathrm{D}_{\mathcal{G}}$ $f \in \mathcal{G}$, $a \neq b$, $ab \Rrightarrow f(a)f(b) \Rightarrow f \in \mathrm{CIs}(\mathfrak{M})$, for any $\mathcal{G} \subseteq \mathrm{Is}(\mathfrak{C})$.

We begin with some relationships between the above conditions.

PROPOSITION 18. *The conditions* Cc *and* Sc *are equivalent.*

Proof. We note the following fact:

$$(\exists f \in \mathrm{Tr}(\mathfrak{C}))[f(a) = b \wedge f(c) = d] \Leftrightarrow (\exists \sigma \in \Pi(C))[\sigma(a) = d \wedge \sigma(c) = b],$$

which immediately yields Cc$\Leftrightarrow$Sc. ■

We have also evident.

PROPOSITION 19. Cc *implies that* $\Rrightarrow_K = \equiv_{\mathrm{Tr}|K}$ *for every line* K. ■

Let σ_a be the central symmetry with center a. The set $\sigma_{o|c(o,q)}$ is an equivalence relation; in every its (two element) equivalence class we choose one element. Thus for every $x \in c(o,q) = c$ we have an assignment $\pi_c : x \mapsto \pi_c(x)$ such that $\pi_c(x) = \pi_c(y) \Leftrightarrow x = y \vee x = \sigma_o(y)$ and $oq \equiv ox \equiv oy$. Using the function π we define

$$ab \rightrightarrows_\pi cd :\Leftrightarrow (\exists \tau_1, \tau_2 \in Tr)[\tau_1(a) = o = \tau_2(c) \wedge \\ \pi_{c(0,\tau_1(b))}(b) = \tau_1(b) \Leftrightarrow \pi_{c(o,\tau_2(d))}(d) = \tau_2(d)].$$

Immediately we get.

PROPOSITION 20. *The relation* $\rightrightarrows_\pi$ *is an ordered congruence satisfying the following condition*:

$$\text{if } ab \,||\, cd, \text{ then } ab \equiv_\pi cd \Leftrightarrow ab \equiv_{\mathrm{Tr}} cd.$$

Hence $\rightrightarrows_\pi$ *satisfies* Cc. ∎

It is easy to show a function π, such that $\rightrightarrows_\pi$ does not satisfy CH nor SH. We also have an evident converse of Proposition 20.

PROPOSITION 21. *If* $\rightrightarrows$ *is an ordered congruence satisfying* Cc, *then* $\rightrightarrows = \rightrightarrows_\pi$ *some function* π.

Proof. We define

$$\pi_{c(o,q)}(x) = y :\Leftrightarrow ox \rightrightarrows oy \wedge ox \,||\, oy \wedge oq \equiv ox. \ \blacksquare$$

It is also possible to give an example of an oriented congruence not satisfying Cc such that $\rightrightarrows_K = \equiv_{\mathrm{Tr}\,|K}$ for every line K; the construction however is rather artificial.

PROPOSITION 22. *No axial symmetry of* $\mathbb{E}(\mathfrak{C})$ *belongs to* $\mathrm{CIs}(\mathfrak{M}) \cup \mathrm{SIs}(\mathfrak{M})$.

Proof. If f is an axial symmetry, then there are points a, b, p, q with $a \neq b$ and $p \neq q$ such that $ab \rightrightarrows f(a)f(b) = ab$ and $pq \rightrightarrows f(p)f(q) = qp$. ∎

As a consequence we get.

PROPOSITION 23. *If an oriented congruence satisfies* Cc, *then*

$$\mathrm{CIs}(\mathfrak{M}) \cup \mathrm{SIs}(\mathfrak{M}) = \mathrm{Ist}^+(\mathfrak{C}).$$

Clearly, since $ab \not\rightrightarrows ba$, the condition SH implies CH. In the presence of Sc we have even more.

PROPOSITION 24. *Let* $\rightrightarrows$ *be an oriented congruence satisfying* Sc. *Then the following conditions are equivalent*: CH, SH, $\mathrm{D}_{\mathrm{Ist}^+(\mathfrak{C})}$.

Proof. The equivalence CH⇔SH is evident. The remaining equivalences follow from Proposition 22 and 2-rigidity of the group $\mathrm{Ist}^+(\mathfrak{C})$. ∎

PROPOSITION 25. *Let* $\rightrightarrows$ *be an oriented congruence such that* $\mathrm{CIs}(\mathfrak{M}) \cup \mathrm{SIs}(\mathfrak{M}) \subseteq \mathrm{Ist}^+(\mathfrak{C})$ *and let* $\rightrightarrows$ *satisfy* $\mathrm{D}_{\mathrm{Ist}^+(\mathfrak{C})}$. *Then* $\rightrightarrows$ *satisfies* Cc.

Proof. If $f \in \mathrm{Tr}(\mathfrak{C})$, then $f = g^2$ for some $g \in \mathrm{Tr}(\mathfrak{C})$. Clearly $g \in \mathrm{CIs}(\mathfrak{M}) \cup \mathrm{SIs}(\mathfrak{M})$ implies $g^2 \in \mathrm{CIs}(\mathfrak{M})$, which proves the thesis. ∎

We write for short D instead of $\mathrm{D}_{\mathrm{Ist}^+(C)}$.

The structure $\mathfrak{M} = \langle P; \rightrightarrows\rangle$ will be called a *homogeneous oriented congruence plane* iff $\rightrightarrows$ is an oriented congruence satisfying Cc and CH.

PROPOSITION 26. *If* $\mathfrak{M}$ *is a homogeneous oriented congruence plane, then* $[\mathrm{Ist}^+(\mathfrak{C}) : \mathrm{CIs}(\mathfrak{M})] = 2$ *and* $\mathrm{Ist}^+(\mathfrak{C}) \setminus \mathrm{CIs}(\mathfrak{M}) = \mathrm{SIs}(\mathfrak{M})$.

Proof. We immediately prove that $f \in \mathrm{CIs}(\mathfrak{M})$ and $g_1, g_2 \in \mathrm{SIs}(\mathfrak{M})$ implies $g_1^{-1}, fg_1, g_1 f \in \mathrm{SIs}(\mathfrak{M})$ & $g_1 g_2 \in \mathrm{CIs}(\mathfrak{M})$. ∎

Let $\mathfrak{C} = \mathrm{Ist}^+(\mathfrak{C})$ and let $\mathfrak{S}$ be a subgroup of $\mathfrak{C}$ with $[\mathfrak{C} : \mathfrak{S}] = 2$ and $\mathfrak{S} \cap \Pi(\mathfrak{C}) = \emptyset$. Let $\rightrightarrows_{\mathfrak{S}}$ be defined by

$$ab \rightrightarrows_{\mathfrak{S}} cd :\Leftrightarrow (\exists f \in \mathfrak{S})[f(a) = c \wedge f(b) = d].$$

PROPOSITION 27. *The structure* $\mathfrak{M} = \langle P; \rightrightarrows_{\mathfrak{S}}\rangle$ *is a homogeneous oriented congruence plane and* $\mathrm{CIs}(\mathfrak{M}) = \mathfrak{S}$.

Proof. By $[\mathfrak{C} : \mathfrak{S}] = 2$ we get $\mathrm{Tr} \subseteq \mathfrak{S}$. Since $\Pi \cap \mathfrak{S} = \emptyset$, the relation $\rightrightarrows_{\mathfrak{S}}$ satisfies OC1. The condition OC0 and OC2 are evident, so $\rightrightarrows_{\mathfrak{S}}$ is an oriented congruence. The equality $\mathrm{CIs}(\mathfrak{M}) = \mathfrak{S}$ follows from 2-rigidity of the group $\mathrm{Ist}^+(\mathfrak{C})$ and therefore $\mathfrak{M}$ is homogeneous. ∎

Thus one can see that homogeneous oriented congruences in $\mathbb{E}(\mathfrak{C})$ are in a one to one correspondence with sets $U \subseteq F \times F$ such that

(i) $-\underline{1} \notin U$, and

(ii) U is a subgroup of index 2 in the multiplicative group $\{a \in F \times F : \nu(a) = 1\}$.

For a given homogeneous congruence we have $U = \{x : \underline{0}x \rightrightarrows \underline{01}\}$; and given a set U with (i) and (ii) we define

$$\mathfrak{S} = \{f : F^2 \to F^2 : (\forall x)[f(x) = ax + b],\ a \in U,\ b \in F^2\}$$

and we put $\rightrightarrows = \rightrightarrows_{\mathfrak{S}}$.

If $\mathfrak{F}$ is ordered and $\varepsilon > 0$, then $\rightrightarrows_\varepsilon$ corresponds to the group

$$U = \{(\alpha, \beta) : \alpha^2 - \varepsilon\beta^2 = 1,\ \alpha > 0\}.$$

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Received May 31, 1993.